point arbitrarily and continue this vector into the neighborhood by means of (5.87).

Let us now summarize the results of this section. We can establish a vector field $\xi^\alpha(x)$ by parallel displacement of an arbitrary vector ξ^α from some initial point to all points in a neighborhood in the Riemann space (by an arbitrary route) if and only if the Riemann tensor of the space is identically zero. In other words, we have found that (5.87) is integrable if and only if the space is flat, that is, has a zero Riemann curvature tensor. We shall henceforth refer to such a space as an integrable space. In addition, note that the covariant derivative of a vector field formed by parallel displacement is clearly zero. Therefore such a vector field is a natural generalization of a constant vector field in Cartesian coordinates. The results of this section then indicate that such a generalized constant vector field can exist only in a flat space with a zero Riemann curvature tensor.

5.6 Pseudo-Euclidean and Flat Spaces

From the previous section we know that we can establish a vector field with a zero covariant derivative if and only if the Riemann tensor is everywhere zero. We shall show in this section that the existence of such a generalized constant vector field ensures the existence of a coordinate system where the metric tensor has constant components. If one can find such a coordinate system where the metric tensor has constant components, the space is termed by definition a pseudo-Euclidean space. Thus we can say that the goal of this section is to show that a flat space (a space with a null Riemann tensor) is also pseudo-Euclidean.

Let us then suppose that the Riemann tensor of a space is everywhere zero; in that case we can establish a generalized constant vector field by the parallel displacement of some arbitrary vector ξ_α from an initial point to any given point in space. Let us do this with the following set of four vectors $\xi_\alpha^{(\gamma)}$:

(5.88)
$$\xi_\alpha^{(0)} = (1,0,0,0) \qquad \xi_\alpha^{(1)} = (0,1,0,0)$$
$$\xi_\alpha^{(2)} = (0,0,1,0) \qquad \xi_\alpha^{(3)} = (0,0,0,1)$$

which we can write more simply as

(5.89)
$$\xi_\alpha^{(\gamma)} = \delta_\alpha^{(\gamma)} = \begin{cases} 1 & \text{for } \gamma = \alpha \\ 0 & \text{for } \gamma \neq \alpha \end{cases}$$

(Note that γ is *not* a tensor index.) These vectors then represent the value of four generalized constant vector fields at an initial point, say, P_0, in some fixed but arbitrary coordinate system x^α:

(5.90)
$$\xi_\alpha^{(\gamma)}(P_0) = \delta_\alpha^{(\gamma)}$$

Now since each vector field $\xi_\alpha^{(\gamma)}(x)$ has a zero covariant derivative,

(5.91)
$$\xi_{\alpha\|\beta}^{(\gamma)} = 0$$

(we omit the argument x for clarity), it must also have a zero curl:

(5.92)
$$\xi_{\alpha\|\beta}^{(\gamma)} - \xi_{\beta\|\alpha}^{(\gamma)} = \xi_{\alpha|\beta}^{(\gamma)} - \xi_{\beta|\alpha}^{(\gamma)} = 0$$

The zero curl implies that each $\xi_\alpha^{(\gamma)}$ then has a scalar potential $\varphi^{(\gamma)}(x^\mu)$ such that

(5.93)
$$\xi_\alpha^{(\gamma)} = \frac{\partial \varphi^{(\gamma)}}{\partial x^\alpha}$$

Let us now use these scalar potential functions to define a transformation to a new coordinate system $\bar{x}^\gamma$:

(5.94)
$$\bar{x}^\gamma = \varphi^{(\gamma)}(x^\mu)$$

Such a transformation is permissible in *some neighborhood* of P_0 since the Jacobian of the transformation at P_0 is

(5.95)
$$\left\| \frac{\partial \varphi^{(\gamma)}}{\partial x^\alpha} \right\| = \|\xi_\alpha^{(\gamma)}\| = 1$$

In the following development we shall consider only the neighborhood of P_0 where the Jacobian (5.95) remains positive until we note otherwise. In the barred system Eq. (5.93) takes the form

(5.96)
$$\bar{\xi}_\alpha^{(\gamma)} = \frac{\partial \varphi^{(\gamma)}}{\partial \bar{x}^\alpha}$$

which, by virtue of the transformation (5.94), gives

(5.97)
$$\bar{\xi}_\alpha^{(0)} = (1,0,0,0) \qquad \bar{\xi}_\alpha^{(1)} = (0,1,0,0)$$
$$\bar{\xi}_\alpha^{(2)} = (0,0,1,0) \qquad \bar{\xi}_\alpha^{(3)} = (0,0,0,1)$$

so each of the vectors $\bar{\xi}_\alpha^{(\gamma)}$ has *constant components everywhere* in the barred coordinate system $\bar{x}^\alpha$.

Consider next the set of 16 scalar inner products

$$\bar{\xi}_\alpha^{(\gamma)}\bar{\xi}_\beta^{(\delta)}\bar{g}^{\alpha\beta} = \xi_\alpha^{(\gamma)}\xi_\beta^{(\delta)}g^{\alpha\beta} \tag{5.98}$$

Since the vectors $\bar{\xi}_\alpha^{(\gamma)}$ and the metric tensor $\bar{g}^{\alpha\beta}$ have zero covariant derivatives, each of these scalars has a zero derivative:

$$\begin{aligned}(\bar{\xi}_\alpha^{(\gamma)}\bar{\xi}_\beta^{(\delta)}\bar{g}^{\alpha\beta})_{|\lambda} &= (\bar{\xi}_\alpha^{(\gamma)}\bar{\xi}_\beta^{(\delta)}\bar{g}^{\alpha\beta})_{\|\lambda}\\ &= \bar{\xi}_\alpha^{(\gamma)}\bar{\xi}_\beta^{(\delta)}\bar{g}^{\alpha\beta}{}_{\|\lambda} + \bar{\xi}_\alpha^{(\gamma)}\bar{\xi}_{\beta\|\lambda}^{(\delta)}\bar{g}^{\alpha\beta} + \bar{\xi}_{\alpha\|\lambda}^{(\gamma)}\bar{\xi}_\beta^{(\delta)}\bar{g}^{\alpha\beta}\\ &= 0 \qquad \text{for all pairs } \gamma \text{ and } \delta\end{aligned} \tag{5.99}$$

and is therefore a constant:

$$(\bar{\xi}_\alpha^{(\gamma)}\bar{\xi}_\beta^{(\delta)}\bar{g}^{\alpha\beta}) = \text{const} \tag{5.100}$$

From the fact that this inner product is a constant and from the explicit form of the constant vectors $\bar{\xi}_\alpha^{(\gamma)}$ given in (5.97), we see that each component of the metric tensor must be a constant:

$$\bar{\xi}_\alpha^{(\gamma)}\bar{\xi}_\beta^{(\delta)}\bar{g}^{\alpha\beta} = \bar{g}^{\gamma\delta} = \text{const} \tag{5.101}$$

We have therefore shown that, in the neighborhood of P_0, where the Jacobian (5.95) is nonzero, there is a coordinate system in which the metric tensor has constant components; the space is by definition pseudo-Euclidean.

The problem now remains to extend our analysis to include all space instead of the neighborhood of P_0, which we considered in the preceding paragraphs. As an aid in this extension, let us recall a few facts concerning the theory of symmetric matrices:

1. For any symmetric matrix G there is a nonsingular matrix A (whose transpose is denoted by A^T) which will transform G by a *congruence transformation* AGA^T to a diagonal matrix of the general form that is, with l diagonal elements equal to 1, m elements equal to -1, and n elements equal to zero. This is the well-known *Sylvester canonical form* for congruence transformations.

2. Although the matrix A in the transformation (5.102) is not unique, the set of diagonal elements in the Sylvester canonical form for any given

$$\begin{pmatrix}1 &&&&&&&&& \\ & \ddots &&&&&&&& \\ && 1 &&&&&&& \\ &&& -1 &&&&&& \\ &&&& \ddots &&&&& \\ &&&&& -1 &&&& \\ &&&&&& 0 &&& \\ &&&&&&& \ddots && \\ &&&&&&&& 0 \end{pmatrix} = \begin{pmatrix}I_l && \\ & -I_m & \\ && 0_n\end{pmatrix} = AGA^T \tag{5.102}$$

matrix G is unique. This set of elements, composed of $+1$, -1, and 0, is termed the *signature* of the matrix G.

3. Any two matrices, say, G and H, which have the same signature and thereby the same canonical form, are related by a congruence transformation; that is, there exists a nonsingular matrix B such that $G = BHB^T$. (This last fact is indeed evident from facts 1 and 2.)

The use of these ideas in extending our preceding analysis over all space is quite straightforward, and we shall only briefly sketch the procedure. We have shown that in *some* neighborhood of *any* point P_0 there exists a transformation to a barred coordinate system in which the metric tensor is constant:

$$\bar{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}g_{\mu\nu} = \text{const} \tag{5.103}$$

By defining the matrices $(A)_{\alpha\mu} = \partial x^\mu/\partial\bar{x}^\alpha$, $(G)_{\mu\nu} = g_{\mu\nu}$, and $(\bar{G})_{\alpha\beta} = \bar{g}_{\alpha\beta}$, we can write this very simply in matrix notation as

$$\bar{G} = AGA^T = \text{const} \tag{5.104}$$

As we noted above, both $\bar{G}$ and G are assumed to have signature

$$(1, -1, -1, -1)$$

Indeed, it is evident that, without loss of generality, we may suppose that $\bar{G}$ is itself the Sylvester canonical form identical to the Lorentz metric.

Consider now a nearby point P_1 such that the neighborhoods of P_1 and P_0 overlap where the transformation (5.93) has a nonzero Jacobian

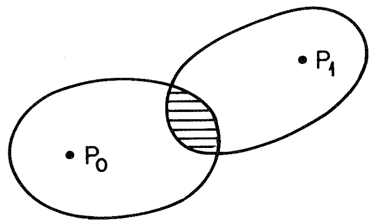

Fig. 5.2

(Fig. 5.2). The relevant φ functions for the neighborhoods need not necessarily agree, but we shall have a matrix equation similar to (5.104) in the neighborhood of P_1

(5.105) $$\bar{G} = BGB^T = \text{const}$$

where $\bar{G}$ is again the Lorentz canonical form. In the *overlap* region we then have

(5.106) $$\bar{G} = BGB^T = AGA^T = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Since A and B are nonsingular matrices, this gives

(5.107) $$G = B^{-1}\bar{G}(B^T)^{-1} = A^{-1}\bar{G}(A^T)^{-1}$$

Thus

(5.108) $$\bar{G} = AB^{-1}\bar{G}(B^T)^{-1}A^T = (AB^{-1})\bar{G}(AB^{-1})^T$$

that is, AB^{-1} *transforms the Lorentz metric into itself* at every point in the overlap region. All matrices L such that

(5.109) $$L\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}L^T = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

form a group, the so-called *Lorentz group* of matrices or of linear transformations, which is well known in the special theory of relativity. By definition, AB^{-1} then belongs to the Lorentz group at each point in the overlap region and we write:

(5.110) $$AB^{-1} = L$$

where L is a Lorentz rotation matrix. We thus see that A and B can differ only by a Lorentz rotation at each point of the overlapping region,

(5.111) $$A = LB$$

However, from the beginning the matrix A is arbitrary up to a constant Lorentz rotation, so we can just as well absorb an appropriate L into the matrix A to give

(5.112) $$A = B$$

We then have one transformation which puts the metric in the Lorentz form in the *combined* neighborhoods of P_0 and P_1. This process can then be continued to any point nearby P_0 or P_1 just as one analytically continues a function in complex analysis. Eventually, any point of space can be included, so we have indeed extended our analysis to all space and shown that the entire Riemann space is pseudo-Euclidean.

Combined with the results of Sec. 5.2 and 5.5, the result of this section allows us to construct in summary the following list of *equivalent statements* about a given Riemann space.

1. The space is flat; that is, $R_{\alpha\beta\gamma\delta} = 0$.
2. The space is integrable, and parallel displacement is path-independent.
3. The space is pseudo-Euclidean, so there exists a coordinate system where the metric is everywhere constant.

Furthermore, for physical reasons, we shall restrict ourselves to considering the special case of a pseudo-Euclidean space in which the metric has signature $(1, -1, -1, -1)$.

5.7 The Einstein Field Equations for Free Space

We wish to obtain in this section an acceptable set of differential equations in tensor form to describe the gravitational field in free space; these equations must satisfy the four criteria which we stated in Sec. 5.1. The developments of the preceding sections lead us to expect that these equations should in some way involve the Riemann tensor $R_{\alpha\beta\gamma\delta}$ since this tensor appears to contain a great deal of information about the geometric structure of space. Note, for instance, the role played by the Riemann tensor in the parallel displacement of a vector in Eq. (5.82). Furthermore, we already know from the results of Sec. 5.2 that the special case of a gravity-free space with a Lorentz metric is correctly described by the equation $R_{\alpha\beta\gamma\delta} = 0$. Thus we expect the complete field equations to be some generalization of $R_{\alpha\beta\gamma\delta} = 0$, which, as is demanded by the third criterion of Sec. 5.1, still admits the Lorentz metric as one solution. In short, we wish in some way to *weaken* the above flat-space equation to admit more general solutions.

To obtain a clue as to how we might appropriately weaken this equation, let us consider Laplace's equation for the classical gravitational potential φ:

(5.113) $$\sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial x^{i^2}} = \sum_{i=1}^{3} \varphi_{|i|i} = 0$$

From Sec. 4.3, we know that, if Newton's second law of motion and the geodesic equation of motion are to yield approximately the same trajectories for slowly moving particles in a weak gravitational field, the g_{00} component of the metric tensor must be approximately given by

(5.114) $$g_{00} = 1 + \frac{2\varphi}{c^2}$$

Therefore φ is given by

(5.115) $$\varphi = \frac{c^2}{2}(g_{00} - 1)$$

and Laplace's equation may be written in terms of g_{00} as

(5.116) $$\sum_{i=1}^{3} g_{00|i|i} = 0$$

This equation, which must be an approximate form of the relativistic field equations (Sec. 7.2), involves second derivatives of the metric tensor with a summation over the repeated index i. In a covariant tensor equation the analogue of such a summation is a contraction, so we are led to expect a contraction to occur in the relativistic field equations. This observation suggests that we try weakening the equation $R_{\alpha\beta\gamma\delta} = 0$ by a contraction of the Riemann tensor. Fortunately, there is only one

meaningful contraction of the Riemann tensor $R_{\alpha\beta\gamma\delta}$. Observe that a contraction between α and β or between γ and δ yields a null tensor since $R_{\alpha\beta\gamma\delta}$ is antisymmetric in these index pairs. Similarly, we see that contractions between α and γ, between α and δ, and between β and δ differ only in sign. Thus the only meaningful contraction which we may perform on $R_{\alpha\beta\gamma\delta} = 0$ yields the equation

(5.117) $$R^{\alpha}{}_{\beta\alpha\delta} = R_{\beta\delta} = 0$$

where $R_{\beta\delta}$ is termed the *contracted Riemann tensor* or *Ricci tensor*. Following Einstein, this is the equation we shall adopt to describe the gravitational field in free space.

Let us observe that the Ricci tensor is symmetric by virtue of the last symmetry relation in (5.51) for the full Riemann tensor. Indeed, using the symmetry of $g^{\alpha\beta}$, we obtain the chain of equations

(5.117′) $$R_{\eta\gamma} = R^{\alpha}{}_{\eta\alpha\gamma} = g^{\alpha\beta}R_{\beta\eta\alpha\gamma} = g^{\alpha\beta}R_{\alpha\gamma\beta\eta} = R_{\gamma\eta}$$

Thus the Ricci tensor has 10 independent components.

Equation (5.117) clearly satisfies criteria (*a*) and (*b*) of Sec. 5.1: it is a tensor equation and was explicitly constructed so as to have the Lorentz metric as one solution. That it is second-order and quasi-linear in the components of $g_{\mu\nu}$, and thereby satisfies criteria (*b*) and (*d*), can be seen by writing out $R_{\beta\delta}$ in terms of the metric tensor in the form

(5.118) $$R_{\beta\delta} = \left\{ {\alpha \atop \beta\ \alpha} \right\}_{|\delta} - \left\{ {\alpha \atop \beta\ \delta} \right\}_{|\alpha} + \left\{ {\alpha \atop \tau\ \delta} \right\}\left\{ {\tau \atop \beta\ \alpha} \right\} - \left\{ {\alpha \atop \tau\ \alpha} \right\}\left\{ {\tau \atop \beta\ \delta} \right\}$$
$$= \tfrac{1}{2}g^{\alpha\lambda}(g_{\beta\lambda|\alpha} + g_{\lambda\alpha|\beta} - g_{\beta\alpha|\lambda})_{|\delta} - \tfrac{1}{2}g^{\alpha\lambda}(g_{\beta\lambda|\delta} + g_{\lambda\delta|\beta} - g_{\beta\delta|\lambda})_{|\alpha}$$
$$+ \text{(terms involving } \textit{first} \text{ derivatives of } g_{\mu\nu})$$

Thus the four criteria of Sec. 5.1 are indeed satisfied by (5.117). Let us repeat in summary that the Einstein free-space field equations which we shall use in the following chapters are

(5.119) $$R_{\beta\delta} = \left\{ {\alpha \atop \beta\ \alpha} \right\}_{|\delta} - \left\{ {\alpha \atop \beta\ \delta} \right\}_{|\alpha} + \left\{ {\alpha \atop \tau\ \delta} \right\}\left\{ {\tau \atop \beta\ \alpha} \right\} - \left\{ {\alpha \atop \tau\ \alpha} \right\}\left\{ {\tau \atop \beta\ \delta} \right\} = 0$$

In Chap. 10 we shall also consider more general field equations for the interior of a distribution of matter and for space on a cosmic scale. Before doing this, however, we shall investigate the above system, which is the most important case.

5.8 The Divergenceless Form of the Einstein Field Equations

In special relativity we usually associate a vector or tensor of zero four-divergence with some conserved quantity; for instance, the zero divergence of the electromagnetic four-current j^μ is directly associated with the conservation of electric charge. It is thus useful in many branches of physics to write as many equations as possible in terms of divergenceless vectors or tensors. Equations written in this form deal directly with *persistent phenomena* instead of *transient events;* such a situation is certainly desirable whenever possible. We shall see in this section that it is possible to express the free-space gravitational field equations (5.119) quite simply in terms of a single divergenceless tensor. Let us begin by obtaining the divergence of $R_{\beta\delta}$. Raising the first two indices in the Bianchi identities (5.62) gives

$$\{R^{\alpha\eta}{}_{\beta\gamma\|\delta}\}_{(\beta,\gamma,\delta)} = R^{\alpha\eta}{}_{\beta\gamma\|\delta} + R^{\alpha\eta}{}_{\gamma\delta\|\beta} + R^{\alpha\eta}{}_{\delta\beta\|\gamma} = 0 \qquad (5.120)$$

Contracting α with β and η with γ gives

$$R^{\alpha\eta}{}_{\alpha\eta\|\delta} + R^{\alpha\eta}{}_{\eta\delta\|\alpha} + R^{\alpha\eta}{}_{\delta\alpha\|\eta} = 0 \qquad (5.121)$$

By definition of the Ricci tensor and by the symmetry properties of the Riemann tensor (5.51), we find

$$R^{\eta}{}_{\eta\|\delta} - R^{\alpha}{}_{\delta\|\alpha} - R^{\eta}{}_{\delta\|\eta} = 0 \qquad (5.122)$$

Relabeling indices and rearranging terms, we then obtain

$$R^{\eta}{}_{\eta\|\delta} = 2R^{\beta}{}_{\delta\|\beta} \qquad (5.123)$$

Denoting the doubly contracted Riemann tensor $R^{\eta}{}_{\eta}$ by R, the *Riemann scalar,* we can write this divergence in the form

$$\tfrac{1}{2}R_{\|\delta} = \tfrac{1}{2}g^{\beta}{}_{\delta}R_{\|\beta} = R^{\beta}{}_{\delta\|\beta} \qquad (5.124)$$

For both indices in contravariant position, we have, then,

$$R^{\beta\delta}{}_{\|\beta} = \tfrac{1}{2}(g^{\beta\delta}R)_{\|\beta} \qquad (5.125)$$

since the metric tensor has a zero covariant derivative. Thus the *Einstein tensor,* which we define as

$$G^{\beta\delta} = R^{\beta\delta} - \tfrac{1}{2}g^{\beta\delta}R \qquad (5.126)$$

has zero divergence:

$$G^{\beta\delta}{}_{\|\beta} = 0 \qquad (5.127)$$

Suppose, now, that the Riemann curvature tensor satisfies the free-space equations $R^{\beta\delta} = 0$. Then $R = 0$ also, and

$$G^{\beta\delta} = R^{\beta\delta} - \tfrac{1}{2}g^{\beta\delta}R = 0 \qquad (5.128)$$

and the Einstein tensor is also zero. Conversely, if $G^{\beta\delta}$ is zero, then

$$G^{\beta}{}_{\delta} = R^{\beta}{}_{\delta} - \tfrac{1}{2}g^{\beta}{}_{\delta}R = 0 \qquad (5.129)$$

Contracting this we see that R, the Riemann scalar, is zero:

$$G^{\beta}{}_{\beta} = 0 = R - \tfrac{1}{2}g^{\beta}{}_{\beta}R = R - 2R \qquad (5.130)$$

Thus the Ricci tensor is also zero:

$$R^{\beta}{}_{\delta} = G^{\beta}{}_{\delta} + \tfrac{1}{2}g^{\beta}{}_{\delta}R = 0 \qquad (5.131)$$

We conclude that *$G^{\beta}{}_{\delta}$ is zero if and only if $R^{\beta}{}_{\delta}$ is zero.* This allows us to write the Einstein field equations entirely in terms of the zero-divergence Einstein tensor:

$$G^{\beta}{}_{\delta} = R^{\beta}{}_{\delta} - \tfrac{1}{2}g^{\beta}{}_{\delta}R = 0 \qquad (5.132)$$

This form of the equations will be extremely useful in the mathematical investigations of Chap. 8 and again in the physical developments of Chap. 10. Also in Chap. 10 the nature of the conservation law associated with the zero-divergence property of the Einstein tensor will become apparent.

5.9 The Riemann Tensor and Fields of Geodesics

We already know the role of geodesics in relativistic mechanics and in the theory of light rays. Hence the significance of the Riemann tensor in physical applications will be well illustrated by a formula which relates fields of geodesic curves in a Riemann space to the theory of the Riemann tensor. We consider a one-parameter family of geodesics $\Gamma(v)$

which are described by the system of equations

$$x^\alpha = x^\alpha(u,v) \tag{5.133}$$

where we suppose x^α to be twice continuously differentiable functions of u and v. The parameter v distinguishes between the different geodesics of the family, while the parameter u is the curve parameter on each $\Gamma(v)$. We have, for fixed v, the differential equation of the geodesics

$$\frac{\partial^2 x^\alpha}{\partial u^2} = -\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \frac{\partial x^\beta}{\partial u}\frac{\partial x^\gamma}{\partial u} \qquad x^\alpha = x^\alpha(u,v) \tag{5.134}$$

We might in general identify u with the arc length on $\Gamma(v)$; however, we prefer to leave u to be defined just by (5.134) so that our reasoning remains valid also for null geodesics.

The family of geodesics gives rise to the field of tangent vectors

$$t^\alpha(u,v) = \frac{\partial x^\alpha(u,v)}{\partial u} \tag{5.135}$$

Let us also introduce the vector field

$$w^\alpha(u,v) = \frac{\partial x^\alpha(u,v)}{\partial v} \tag{5.136}$$

which describes the deviation of two points on two infinitesimally near geodesics which have the same parameter value u. We call w^α the vector of geodesic deviation in the geodesic field. The law of interchange of partial differentiation leads to the identity

$$\frac{\partial t^\alpha(u,v)}{\partial v} = \frac{\partial^2 x^\alpha}{\partial u\,\partial v} = \frac{\partial w^\alpha(u,v)}{\partial u} \tag{5.137}$$

We wish now to calculate the absolute derivative of the vector field $w^\alpha(u,v)$ on the geodesic $\Gamma(v)$. Let us use the definition (3.34) of this derivative and find by use of the Christoffel symbols instead of connections Γ

$$\begin{aligned} \frac{Dw^\alpha}{Du} &= \frac{\partial w^\alpha}{\partial u} + \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \frac{\partial x^\beta}{\partial u} w^\gamma \\ &= \frac{\partial t^\alpha}{\partial v} + \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} t^\beta w^\gamma \end{aligned} \tag{5.138}$$

We have thus created a new vector field along each $\Gamma(v)$ and can therefore repeat the process of absolute differentiation. The remarkable fact appears that this second differentiation leads us directly to the Riemann curvature tensor. Indeed, we find

$$\frac{D^2 w^\alpha}{Du^2} = \frac{\partial}{\partial u}\left(\frac{Dw^\alpha}{Du}\right) + \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} t^\beta \frac{Dw^\gamma}{Du} \tag{5.139}$$

Inserting for Dw^α/Du from (5.138), we obtain by use of (5.135) and (5.137)

$$\begin{aligned} \frac{D^2 w^\alpha}{Du^2} = \frac{\partial}{\partial v}\left(\frac{\partial^2 x^\alpha}{\partial u^2}\right) &+ \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix}_{|\delta} \frac{\partial x^\beta}{\partial u}\frac{\partial x^\gamma}{\partial v}\frac{\partial x^\delta}{\partial u} + \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \frac{\partial^2 x^\beta}{\partial u^2}\frac{\partial x^\gamma}{\partial v} \\ &+ 2\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \frac{\partial x^\beta}{\partial u}\frac{\partial^2 x^\gamma}{\partial u\,\partial v} + \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix}\begin{Bmatrix} \gamma \\ \mu\ \nu \end{Bmatrix} \frac{\partial x^\beta}{\partial u}\frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial v} \end{aligned} \tag{5.140}$$

We now apply the equation of the geodesics (5.134) in order to eliminate the terms $\partial^2 x^\alpha/\partial u^2$. A simple rearrangement and some obvious cancellations lead to the result

$$\begin{aligned} \frac{D^2 w^\alpha}{Du^2} = \Bigg(\begin{Bmatrix} \alpha \\ \beta\ \delta \end{Bmatrix}_{|\gamma} - \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix}_{|\delta} &+ \begin{Bmatrix} \alpha \\ \gamma\ \tau \end{Bmatrix}\begin{Bmatrix} \tau \\ \beta\ \delta \end{Bmatrix} \\ &- \begin{Bmatrix} \alpha \\ \tau\ \delta \end{Bmatrix}\begin{Bmatrix} \tau \\ \beta\ \gamma \end{Bmatrix}\Bigg) \frac{\partial x^\beta}{\partial u}\frac{\partial x^\gamma}{\partial u} w^\delta \end{aligned} \tag{5.141}$$

Comparing this result with the definition (5.10) of the Riemann curvature tensor, we can simplify (5.141) to

$$\frac{D^2 w^\alpha}{Du^2} = R^\alpha{}_{\beta\delta\gamma} \frac{\partial x^\beta}{\partial u} w^\delta \frac{\partial x^\gamma}{\partial u} \tag{5.142}$$

The equation system (5.142) is an ordinary second-order differential system for the geodesic deviation $w^\alpha(u,v)$ along a fixed geodesic $\Gamma(v)$. To illustrate the result, let us consider the case of Euclidean space in which the Riemann tensor vanishes identically. Hence the differential system (5.142) reduces to

$$\frac{D^2 w^\alpha}{Du^2} = 0 \tag{5.143}$$

We may choose a coordinate system in the large in which the Christoffel symbols vanish identically and use as curve parameter u the arc length s. We find, then,

(5.144) $$\frac{d^2w^\alpha}{ds^2} = 0 \qquad w^\alpha(s) = a^\alpha s + b^\alpha$$

where a^α and b^α are constant vectors. Since the geodesics in Euclidean geometry are straight lines, we find the elementary result that the distance between two points moving with speed 1 along two given lines which are infinitesimally near is a linear function of time. The Riemann tensor measures, by (5.142), the departure from this linear behavior.

To illustrate the physical meaning of (5.142), let us consider an observer moving on a timelike geodesic and observing an object which moves near him on its own geodesic. The observer may use his arc length $ds = c\,dt = dx^0$ as time measure and will interpret the geodesic deviation $z^\alpha = \epsilon w^\alpha(s)$ as the Euclidean distance vector of the object. Here ϵ is a small positive factor measuring the distance of the object at the first moment of observation. According to (5.142), an acceleration of the object relative to the observer will be seen as

(5.145) $$\frac{d^2z^\alpha}{dt^2} = R^\alpha{}_{0\delta 0}z^\delta$$

since in his coordinate system $t^\alpha \equiv (1,0,0,0)$.

Consider, on the other hand, the following problem in classical mechanics and Euclidean geometry. Suppose that observer and object move in a field of force which is mass-proportional and varies in space. If $F^i(x^k)$ is the vector of acceleration connected with this field, the object will be accelerated relative to the observer according to

(5.146) $$\frac{d^2z^i}{dt^2} = F^i(x^k + z^k) - F^i(x^k) = \frac{\partial F^i}{\partial x^k}z^k + O(z^k)$$

if z^i is the vector from observer to object. The analogy between (5.145) and (5.146) is evident. We are led to the intuitive interpretation

(5.147) $$R^i{}_{0k0} \leftrightarrow \frac{\partial F^i}{\partial x^k}$$

If the force field possesses a potential $\varphi(x^i)$ such that

(5.148) $$F^i = -F_i = -\frac{\partial\varphi}{\partial x^i}$$

we find the correspondence

(5.149) $$R^i{}_{0k0} \leftrightarrow -\frac{\partial^2\varphi}{\partial x^i\,\partial x^k}$$

Thus the condition for a Laplacian potential $\nabla^2\varphi = 0$ leads to

(5.150) $$R^i{}_{0i0} = 0$$

We come automatically to the Ricci tensor. To obtain an equation which is in tensor form and coordinate-invariant, we have to demand more generally that

(5.151) $$R^\mu{}_{\alpha\mu\beta} = R_{\alpha\beta} = 0$$

as a generalization of Laplace's equation. This consideration gives additional motivation to the choice of the field equations (5.117) to describe the gravitational field in empty space. The reader should also observe the analogy and differences between our present heuristic considerations and those of Sec. 5.7. While in the preceding section we considered the case of weak gravitational fields in the large, we dealt in this section with the approximate form of (5.117) for arbitrary gravitational fields, but in small distances. In both cases the analogy between the classical and relativistic formulas is striking.

To understand the physical significance of the equation of geodesic deviation (5.142) let us return to the Einstein box, discussed in the Introduction. We have seen that an observer in such a box could not decide whether the box was in a gravitational field or in accelerated motion in flat space by performing experiments with a single test body. This is due to the mathematical fact that at a given point in a Riemann space we can introduce geodesic coordinates; in this case the equation of motion of a freely falling body becomes $\ddot{x}^\alpha = 0$, which is precisely the same as the equation of force free motion in flat space, i.e. in special relativity. However, every laboratory has a finite size, and the Riemann tensor cannot be transformed away, so that we can, in principle, measure the inhomogeneities of the gravitational field by observing the *relative* motion of several freely falling bodies, using (5.142) or its classical analogue (5.146). For example, consider an experiment in an earth based laboratory in which two balls separated horizontally by 1 m are allowed to fall simultaneously a distance of 1 m. The two trajectories will converge by about 10^{-5} cm since the balls move toward the center of the earth. (To a first approximation they move on parallel trajectories, as they would in an accelerated laboratory in flat space.) Thus one determines that one is in a gravitational field and not in an accelerated laboratory. We emphasize that the measurement of a gravitational field, as manifested by a nonzero Riemann tensor, requires the observation of the trajectory of more than one freely falling body.

5.10 Algebraic Properties of the Riemann Tensor

The number of algebraically independent components of the Riemann tensor is reduced by symmetry; in dealing with solutions to the field equations the number of algebraically independent components is reduced yet further. In order to clarify this, Petrov has introduced a notation with indices that run over six values. With this notation we can also obtain an intrinsic classification of space-time geometries in terms of the algebraic properties of the Riemann tensor; this classification is analogous to the classification of electromagnetic fields as radiation and nonradiation fields.

The first set of two indices of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ assumes 16 values: 00, 01, . . . , 33. However, from (5.51) the tensor is antisymmetric in these two indices, and so it is clear that we need the values of the tensor for only six pairs of indices. The same is true of the second set of two indices. We are therefore led to introduce the Petrov mapping which associates pairs of tensor indices with a single index as follows:

$$\begin{array}{lll} \text{Tensor indices: } \alpha\beta = 23,\ 31,\ 12,\ 10,\ 20,\ 30; & R_{\alpha\beta\gamma\delta} \\ \qquad\qquad\quad\ \updownarrow & \updownarrow \\ \text{Petrov index*: } A = 1,\ 2,\ 3,\ 4,\ 5,\ 6; & \mathbf{R}_{AB} \end{array} \tag{5.152}$$

The Riemann tensor is thus completely described by the 6×6 matrix $\mathbf{R}_{AB}$; all nonzero algebraically independent components of the tensor occur in the matrix $\mathbf{R}_{AB}$. Moreover, the symmetries in the index pairs expressed by Eqs. (5.51) are now embodied in the very simple statement that $\mathbf{R}_{AB}$ is symmetric, $\mathbf{R}_{AB} = \mathbf{R}_{BA}$. We next write $\mathbf{R}_{AB}$ in terms of 3×3 submatrices, two of which are symmetric by virtue of the symmetry of $\mathbf{R}_{AB}$:

$$\mathbf{R}_{AB} = \begin{pmatrix} M & N \\ N^T & Q \end{pmatrix} \qquad M = M^T \qquad Q = Q^T \tag{5.153}$$

Then the cyclic symmetry (5.53) may conveniently be written as the condition that the trace of N vanish,

* We denote the Riemann tensor in the Petrov indication by $\mathbf{R}$ in order to avoid confusion with the contracted Riemann tensor. Observe that, for example, $\mathbf{R}_{11} = R_{2323}$, but $R_{11} = R^{\nu}{}_{1\nu1}$.

$$\mathrm{Tr}(N) = 0 \tag{5.154}$$

Thus the algebraic symmetries of the Riemann tensor are simply summarized by the statement that $\mathbf{R}_{AB}$ is a symmetric 6×6 matrix for which $\mathrm{Tr}(N) = 0$. From this we see in a very transparent way that it has 20 algebraically independent components.

If we work in a special coordinate system where the metric is Lorentzian at a given point, the field equations (5.117) yield further simplifications. For this purpose it is convenient to consider the mixed tensor $R^{\alpha\beta}{}_{\gamma\delta} \leftrightarrow \mathbf{R}^{A}{}_{B}$. To raise the Petrov index A we note from (5.152) that values of A from 1 to 3 correspond to two spatial tensor indices and therefore imply no sign change when raised. Values of A from 4 to 6 correspond to one space and one time tensor index and imply a sign change. Thus just as with tensor indices and the Lorentz metric we can raise a Petrov index by multiplication with a very simple matrix G^{AC}

$$G^{AC} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \mathbf{R}^{A}{}_{B} = G^{AC}\mathbf{R}_{CB} = \begin{pmatrix} M & N \\ -N^T & -Q \end{pmatrix} \tag{5.155}$$

Using the mixed Petrov matrix $\mathbf{R}^{A}{}_{B}$, we can write the field equations in elegant form

$$\begin{aligned} R^0{}_0 &= 0 \Rightarrow \mathrm{Tr}(Q) = 0 \\ R^0{}_i &= 0 \Rightarrow N^T = N \\ R^i{}_j &= 0 \Rightarrow Q = -M \end{aligned} \tag{5.156}$$

If we combine these relations, implied by the field equations, with (5.153) and (5.154), implied by the algebraic symmetries, we see that $\mathbf{R}^{A}{}_{B}$ can be characterized by a simple statement

$$\mathbf{R}^{A}{}_{B} = \begin{pmatrix} M & N \\ -N & M \end{pmatrix} \qquad M = M^T \quad N = N^T \quad \mathrm{Tr}(M) = \mathrm{Tr}(N) = 0 \tag{5.157}$$

This matrix has only 10 algebraically independent components, but we emphasize that this will be true *only* in a coordinate system in which the metric is Lorentzian at the point of interest.

The mixed matrix (5.157) is not only very simple but is well suited to the study of invariants. This follows from the fact that a coordinate transformation of the Riemann tensor corresponds to a similarity transformation of the matrix $\mathbf{R}^{A}{}_{B}$. To show this we write the transformation of the tensor as

(5.158) $$\bar{R}^{\alpha\beta}{}_{\gamma\delta} = \left(\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu}\right) R^{\mu\nu}{}_{\lambda\tau} \left(\frac{\partial x^\lambda}{\partial \bar{x}^\gamma}\frac{\partial x^\tau}{\partial \bar{x}^\delta}\right)$$

Thanks to the symmetry properties of the tensor, we need not sum over all the indices $\mu\nu$ and $\lambda\tau$ but only over those corresponding to a Petrov index in (5.152). Thus we may write two equations that correspond term by term (see Exercise 5.7):

(5.159) $$\bar{R}^{\alpha\beta}{}_{\gamma\delta} = \sum_{\binom{\mu\nu,\ \gamma\delta\ \leftrightarrow}{\text{Petrov}}} \left(2\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu}\right) R^{\mu\nu}{}_{\lambda\tau} \left(2\frac{\partial x^\lambda}{\partial \bar{x}^\gamma}\frac{\partial x^\tau}{\partial \bar{x}^\delta}\right)$$

$$\bar{\mathbf{R}}^A{}_B = S^A{}_C \mathbf{R}^C{}_D \tilde{S}^D{}_B$$

We easily verify that $S^A{}_C\tilde{S}^C{}_B = \delta^A{}_B$, that is, $\tilde{S}$ is the inverse matrix of S, and so we have established that $\bar{\mathbf{R}}^A{}_B$ and $\mathbf{R}^A{}_B$ are related by a similarity transformation. This result will be very useful since many algebraic properties of a matrix are invariant under similarity transformations.

To classify the space-time geometry, it is convenient to use not the Riemann tensor but its *self dual*, defined as

(5.160) $$R^{(+)\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} + i\,{}^*R^{\alpha\beta}{}_{\gamma\delta}$$

Here ${}^*R^{\alpha\beta}{}_{\gamma\delta}$ is the dual of the Riemann tensor (see Sec. 3.5)

(5.161*a*) $${}^*R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}e_{\alpha\beta}{}^{\sigma\tau}R_{\sigma\tau\gamma\delta}$$

(5.161*b*) $$R_{\alpha\beta\gamma\delta} = -\tfrac{1}{2}e_{\alpha\beta}{}^{\sigma\tau}\,{}^*R_{\sigma\tau\gamma\delta}$$

We shall show that ${}^*R_{\alpha\beta\gamma\delta}$ has the same symmetry properties as $R_{\alpha\beta\gamma\delta}$ and also that the contracted tensor ${}^*R^\alpha{}_{\beta\alpha\delta}$ is zero; then the same reasoning used for $R^{\alpha\beta}{}_{\gamma\delta}$ earlier implies that ${}^*R^{\alpha\beta}{}_{\gamma\delta}$ corresponds to a 6×6 matrix ${}^*\mathbf{R}^A{}_B$, analogous to $\mathbf{R}^A{}_B$ in Eq. (5.157). To show first that the contracted tensor ${}^*R^\alpha{}_{\beta\alpha\delta}$ is zero we use the antisymmetry of $e^{\alpha\beta\sigma\tau}$ in the definition (5.161*a*) to write

(5.162) $${}^*R^{\alpha\beta}{}_{\alpha\delta} = \tfrac{1}{2}e^{\alpha\beta\sigma\tau}R_{\sigma\tau\alpha\delta} = \tfrac{1}{2}e^{\alpha\beta\sigma\tau}\{R_{\sigma\tau\alpha\delta}\}_{(\sigma\tau\alpha)}$$

From the symmetry of the Riemann tensor expressed in (5.39) we see that this is indeed zero. To obtain the symmetries of the dual tensor we impose the field equations on (5.161*b*)

(5.163) $$R^{\alpha\beta}{}_{\alpha\delta} = 0 = -\tfrac{1}{2}e^{\alpha\beta\sigma\tau}\,{}^*R_{\sigma\tau\alpha\delta} = -\tfrac{1}{2}e^{\alpha\beta\sigma\tau}\{{}^*R_{\sigma\tau\alpha\delta}\}_{(\sigma\tau\alpha)}$$

It is clear from this that $\{{}^*R_{\alpha\beta\gamma\delta}\}_{(\alpha\beta\gamma)}$ is zero, which implies that ${}^*R_{\alpha\beta\gamma\delta}$ has the same symmetries as $R_{\alpha\beta\gamma\delta}$; that is, Eq. (5.39) or equivalently (5.51) and (5.53) hold for ${}^*R_{\alpha\beta\gamma\delta}$. Thus ${}^*R^{\alpha\beta}{}_{\gamma\delta}$ corresponds to a 6×6 matrix ${}^*\mathbf{R}^A{}_B$ in the same way as the Riemann tensor corresponds to $\mathbf{R}^A{}_B$.

We continue to work with a local Lorentz metric, so that $e_{\alpha\beta\gamma\delta}$ is equal to $\epsilon_{\alpha\beta\gamma\delta}$. It is then easy to show from the definition (5.161) that the matrix ${}^*\mathbf{R}^A{}_B$ is given in terms of the 3×3 matrices M and N as

(5.164) $${}^*\mathbf{R}^A{}_B = \begin{pmatrix} N & -M \\ M & N \end{pmatrix}$$

Thus corresponding to the self-dual tensor $R^{(+)\alpha\beta}{}_{\gamma\delta}$ is the matrix

(5.165) $$\mathbf{R}^{(+)A}{}_B = \begin{pmatrix} P & -iP \\ iP & P \end{pmatrix} = P \otimes J$$
$$P \equiv M + iN \qquad J \equiv \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

The matrix P will be referred to as the *Petrov matrix*. Since it is a complex 3×3 traceless symmetric matrix, like $\mathbf{R}^A{}_B$ and $\mathbf{R}^{(+)A}{}_B$ it has 10 algebraically independent components.

Our classification will be according to the eigenvalues and the multiplicity of the eigenvectors of $R^{(+)\alpha\beta}{}_{\gamma\delta}$, the self-dual Riemann tensor, or equivalently the matrix $\mathbf{R}^{(+)A}{}_B$. These satisfy the equation

(5.166) $$R^{(+)\alpha\beta}{}_{\gamma\delta}U^{\gamma\delta} = \lambda U^{\alpha\beta}$$
$$\updownarrow$$
$$\mathbf{R}^{(+)A}{}_B U^B = \lambda U^A$$

and are therefore defined invariantly even though our analysis will use the special form of $\mathbf{R}^{(+)A}{}_B$ given in (5.162), which holds only in a special coordinate system. Specifically, $\mathbf{R}^{(+)A}{}_B$ transforms as in (5.159), U^B transforms via $\bar{U}^C = S^C{}_D U^D$, and λ is invariant.

The direct-product relation (5.165) between $\mathbf{R}^{(+)A}{}_B$ and the 3×3 Petrov matrix P allows us to express the nonzero eigenvalues and eigenvectors of $\mathbf{R}^{(+)A}{}_B$ in terms of those of P, in effect reducing the problem to that of a complex 3×3 matrix instead of a real 6×6. Indeed the eigenvectors of $\mathbf{R}^{(+)A}{}_B$ will be the direct product of those of P and J. We define for P and deduce for J the following eigenvector relations

(5.167) $$P\zeta = \tau\zeta \qquad \zeta = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(5.167′)
$$J\eta = \sigma\eta \qquad \sigma = 0 \text{ and } \eta = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \text{or} \qquad \sigma = 2 \text{ and } \eta = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Thus $\mathbf{R}^{(+)A}{}_B$ has *at least* three zero eigenvalues, and so must the tensor $R^{(+)\alpha\beta}{}_{\gamma\delta}$. For our classification we consider only the remaining eigenvalues, which may or may not be zero; they and their corresponding eigenvectors are given explicitly by the direct products

(5.168)
$$\lambda = 2\tau \qquad U^A = \zeta \otimes \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ ia \\ ib \\ ic \end{pmatrix}$$

Moreover, since P is traceless by (5.157), the sum of the eigenvalues τ (and λ) must be zero, which leaves only two independent eigenvalues to consider.

To complete our classification of the interesting eigenvectors and eigenvalues of $\mathbf{R}^{(+)A}{}_B$ we consider the Jordan canonical form of P. According to Jordan, any $n \times n$ complex matrix is related by a similarity transformation to a matrix of the form

(5.169)
$$C = \begin{pmatrix} C_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & C_N \end{pmatrix} \qquad C_i = \begin{pmatrix} \tau_i & 1 & 0 & 0 & \cdots \\ 0 & \tau_i & 1 & 0 & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \tau_i \end{pmatrix}$$

That is, the only nonzero elements of C_i are τ_i's along the diagonal and 1's along the first superdiagonal. (Observe that complex symmetric matrices cannot necessarily be put into diagonal form by a similarity transformation and that similarity transformations do not in general preserve symmetry.) The eigenvalues and eigenvectors of C are very easy to obtain; we easily verify that each C_i submatrix has eigenvalues equal to τ_i and only one eigenvector, (1,0, . . . ,0). Thus C has at most N distinct eigenvalues and N distinct eigenvectors. In general, the number N may be less than the dimension of the matrix, and so C may not have a full complement of eigenvectors. Since the eigenvalues and the algebraic relations among the eigenvectors are invariant properties under similarity transformations, the Jordan canonical form of P provides a complete description of these properties and will serve there-

fore to classify the space-time geometry. There are only five interesting possibilities for the Jordan canonical form of a traceless 3×3 matrix. We list these in Table 5.1, along with the consequent eigenvector and eigenvalue complement, according to Petrov's naming scheme; flat space, $\mathbf{R}^{(+)A}{}_B = 0$, is not included.

TABLE 5.1 PETROV CLASSIFICATION

Petrov type	Jordan form of P	Number of distinct eigenvectors	Number of distinct eigenvalues
I	$\begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}$ $\tau_1 + \tau_2 + \tau_3 = 0$	3	3
ID	$\begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_1 & 0 \\ 0 & 0 & -2\tau_1 \end{pmatrix}$	3	2
II	$\begin{pmatrix} \tau_1 & 1 & 0 \\ 0 & \tau_1 & 0 \\ 0 & 0 & -2\tau_1 \end{pmatrix}$	2	2
II*N*	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2	All zero
III	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	1	All zero

In this section we have made frequent use of a local Lorentz coördinate system for convenience, but we again emphasize that the invariant nature of the eigenvalue problem (5.166) gives our classification scheme invariant meaning. One use of the scheme is thereby immediately evident; two solutions of different type cannot be transformed into each other by any coordinate transformation. Such intrinsic differences are very important in the physical interpretations of general relativity.

We have found that unlike the electromagnetic field, which may be classified simply as radiation or nonradiation, the gravitational field has a much richer fivefold classification structure. This stems from the nonlinearity of the field equations; the lack of superposition in the gravitational theory prevents us from decomposing the field into simpler structures and therefore demands the richer classification scheme. In the problems for the following chapters we shall illustrate the use of the Petrov classification and its physical meaning. For example, the Schwarzschild solution, the gravitational analogue of the Coulomb field in electromagnetism, is Petrov type ID, whereas plane gravitational waves are type IIN.

Exercises

5.1 How many algebraically independent components does the Riemann tensor $R_{\alpha\beta\gamma\delta}$ have in two dimensions? How many does it have in n dimensions?

5.2 Show that for a metric of a two-dimensional space, of the form, $ds^2 = (dx^1)^2 + G^2(x^1)(dx^2)^2$ one has

$$R^1{}_{212} = -G\frac{d^2G}{(dx^1)^2}$$

Obtain all nonzero components of the Riemann tensor from this.

5.3 What is the Riemann tensor $R^\alpha{}_{\beta\gamma\eta}$ for the two-dimensional surface of a sphere? What is it for the surface of a cylinder? What is the Riemann scalar $R = R^\mu{}_\mu$ for these surfaces?

5.4 Consider a three-space imbedded in four-space in the particularly simple way $g_{00} = 1$, $g_{0i} = 0$, with the other g_{ij} independent of the time coordinate. How are the four-space Christoffel symbols related to the three-space symbols? What is the relation between $R_{\alpha\beta\gamma\delta}$ in four-space and R_{ijkl} in three-space?

5.5 (*continued*) What is R_{00}? What is G_{00}? What is the Riemann scalar in four-space? What is the Riemann scalar ${}^{(3)}R$ in three-space? Relate G_{00} and ${}^{(3)}R$.

5.6 Consider a geodesic triangle drawn on a sphere as follows. One vertex is at the north pole and the two others are on the equator, separated by 90°. Parallel-displace a vector around this triangle using the geometric result of Exercise 3.1. How is the vector changed after a complete circuit? (You may wish to choose a convenient initial orientation for simplicity.) Interpret your result and compare with Prob. 5.1.

5.7 Verify explicitly that (5.159) is equivalent to (5.158); i.e., the factors of 2 in (5.159) are correct.

5.8 Prove that the right dual tensor and the left dual tensor of the Riemann tensor, as defined in (5.160), are equal. (Is this true if the Einstein equations do not hold?)

Problems

5.1 In the plane, the total angle Δ through which the tangent to a closed curve turns in one circuit is always 2π. On a curved surface the corresponding Δ is defined as the sum of angles being measured in the successive local tangent planes. Δ will generally differ from 2π. A beautiful theorem (Gauss-Bonnet) says that

$$2\pi - \Delta = \int R\,dA$$

the integral of the curvature scalar over the enclosed area. Consider a sphere of radius a and on it a "geodesic triangle" formed by three geodesics making a right angle at each vertex (see Exercise 5.6). Test the theorem for both the areas that can be considered enclosed by this triangle.

5.2 A conformally flat space is defined as one with a metric tensor of the form $g_{\mu\nu} = f(x^\alpha)\eta_{\mu\nu}$, where f is an arbitrary positive function and $\eta_{\mu\nu}$ is the Lorentz metric. Show that for such a metric the *Weyl tensor*, defined as

$$C^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} + g_{\nu\sigma}R^\mu{}_\rho - g_{\nu\rho}R^\mu{}_\sigma + R_{\nu\sigma}g^\mu{}_\rho - R_{\nu\rho}g^\mu{}_\sigma - \tfrac{1}{3}(g_{\nu\sigma}g^\mu{}_\rho - g_{\nu\rho}g^\mu{}_\sigma)R$$

is zero.

5.3 If the Einstein free-space field equations are satisfied, then the Weyl tensor is identical with the Riemann tensor. This implies that a conformally flat space for which $R_{\mu\nu} = 0$ is actually flat. Show this also by a coordinate transformation.

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CHAPTER

6

The Schwarzschild Solution and Its Consequences: Experimental Tests of General Relativity

The free-space field equations (5.119) are nonlinear and hence difficult to solve. However, by imposing *symmetry conditions* dictated by physical arguments on the line element, we can greatly simplify the field equations in special cases. One such case is the *time-independent* and *spherically symmetric* line element; the resultant field equations were solved exactly by Schwarzschild in 1916. This solution is of particular importance since it corresponds to the basic one-body problem of classical astronomy. Indeed, the only reliable experimental verifications of the field equations (5.119), which we shall treat in Secs. 6.3, 6.5, and 6.6, are based on the Schwarzschild line element. In this chapter we shall obtain Schwarzschild's solution and discuss its consequences.

6.1 The Schwarzschild Solution

Consider the free-space field equations that we obtained in Chap. 5,

$$(6.1)\quad R_{\mu\nu} = \begin{Bmatrix} \beta \\ \beta\ \nu \end{Bmatrix}_{|\mu} - \begin{Bmatrix} \beta \\ \mu\ \nu \end{Bmatrix}_{|\beta} + \begin{Bmatrix} \beta \\ \tau\ \mu \end{Bmatrix} \begin{Bmatrix} \tau \\ \beta\ \nu \end{Bmatrix} - \begin{Bmatrix} \beta \\ \tau\ \beta \end{Bmatrix} \begin{Bmatrix} \tau \\ \mu\ \nu \end{Bmatrix} = 0$$

We shall seek a solution which is time-independent and radially symmetric. By virtue of the requirement of radial symmetry, such a solution should represent the external field of a spherically symmetric body stationary at the origin. The limiting form of the line element at large distances from the origin may be expected to be Lorentzian and thus to be expressible in spherical coordinates r, θ, and φ as

$$ds^2 = c^2\,dt^2 - (dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2) \qquad c\,dt = dx^0 \tag{6.2}$$

Next let us consider the above symmetry requirements and try to form the simplest line element which meets the demands of time-independence and radial symmetry. The reasoning which follows is based on plausibility only, in order to guess a heuristically reasonable and convenient line element. We should expect the line element to be invariant under inversion of the coordinate interval dx^0 (representing time); that is, ds^2 should be invariant under the replacement of dx^0 by $-dx^0$. This dictates that we use Gaussian coordinates in which the off-diagonal elements g_{0i} of the metric tensor are zero and the line element has the form $g_{00}\,(dx^0)^2 + g_{ik}\,dx^i\,dx^k$ with the g_{ik} independent of x^0. This is referred to as a *static* metric; it is to be distinguished from a metric which is merely independent of time, or *stationary*, as discussed in Sec. 3.7. Second, if there is to be *no preferred angular direction* in space, the line element should be independent of a change of $d\theta$ to $-d\theta$ and a change of $d\varphi$ to $-d\varphi$. This requires that there be no terms of the form $dr\,d\theta$, $d\theta\,d\varphi$, etc., in the line element, so the metric tensor must be entirely diagonal for the type of solution we desire. Thus we may write ds^2 as

$$ds^2 = Ac^2\,dt^2 - (B\,dr^2 + Cr^2\,d\theta^2 + Dr^2\sin^2\theta\,d\varphi^2) \tag{6.3}$$

Furthermore, by our assumption of radial symmetry, the functions A, B, C, and D must be functions of r only. One more simplification of the form of the line element can be made on the basis of symmetry: we can suppose that the functions $C(r)$ and $D(r)$ which appear in (6.3) are equal. This can be seen as follows: A displacement by $\epsilon = r\,d\theta$ from the north pole ($\theta = 0$) corresponds to $ds^2 = -C\epsilon^2$, and a displacement by $\epsilon = r\,d\varphi$ along the equator ($\theta = \pi/2$) corresponds to $ds^2 = -D\epsilon^2$. If θ and φ are to represent angular coordinates, we should expect these quantities to be equal due to isotropy, which requires that $C \equiv D$. Then

$$ds^2 = Ac^2\,dt^2 - B\,dr^2 - C(r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2) \tag{6.4}$$

The above line element represents the simplest form which is dictated by the symmetry requirements; however, it is possible to obtain a further simplification by a judicious choice of a radial coordinate. Specifically, consider a radial coordinate defined by

$$\hat{r} = \sqrt{C(r)}\,r \tag{6.5}$$

It then follows that

$$Cr^2 = \hat{r}^2 \tag{6.6}$$

and

$$B\,dr^2 = \frac{B}{C}\left(1 + \frac{r}{2C}\frac{dC}{dr}\right)^{-2} d\hat{r}^2 \equiv \hat{B}\,d\hat{r}^2 \tag{6.7}$$

By means of (6.5) we can express $\hat{B}$ also as a function of the new radial coordinate $\hat{r}$. It is now clear that writing the line element (6.3) in terms of $\hat{r}$ by substituting from (6.6) and (6.7) yields a line element in which the coefficient of the angular term $d\theta^2 + \sin^2\theta\,d\varphi^2$ is 1. This, however, is equivalent to taking $C \equiv 1$ in the line element (6.4), so we conclude that, by a suitable choice of the radial coordinate, we can put the line element in the form

$$ds^2 = Ac^2\,dt^2 - B\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{6.8}$$

with only two unknown functions of r. In order to exhibit clearly the signature of $g_{\mu\nu}$ and the sign of the determinant $\|g_{\mu\nu}\| = g$, let us write $A(r)$ as the intrinsically positive function $e^{\nu(r)}$ and $B(r)$ as $e^{\lambda(r)}$. The line element accordingly is written as

$$ds^2 = e^{\nu(r)}c^2\,dt^2 - e^{\lambda(r)}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{6.9}$$

This equation represents the final form of the line element we shall use in obtaining the Schwarzschild solution; as we have constructed it, the demands of time-independence and radial symmetry are clearly met.

The coordinate r used in (6.9) has a clear physical meaning. Consider a spherical surface defined by a constant value of r, on which points are labeled by θ and φ. The line element on this surface is

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{6.10}$$

The physical length of the equator, defined as the line $\theta = \pi/2$, is obtained by integrating $\sqrt{-ds^2}$ from (6.10) from $\varphi = 0$ to 2π, which gives

$$l = \int_0^{2\pi} \sqrt{-ds^2} = \int_0^{2\pi} r\,d\varphi = 2\pi r \tag{6.11}$$

This is identical to the flat-space result for a spherical surface. Thus by a measurement of the physical length of a great circle we can determine the value of the coordinate r for the sphere considered. Similarly it is easy to show that the physical area of such a sphere is $4\pi r^2$, as in flat space, which again allows us to determine the value of r. It is thereby clear that r is geometrically distinguished, and, moreover, the three space coordinates r, θ, and φ correspond closely to the variables used by astronomers in actual observations.

Even with the simplified metric form of (6.9), the work of computing the 40 Christoffel symbols appearing in the field equations is rather tedious. There is, however, a simple and convenient artifice which we may use to obtain all the nonzero Christoffel symbols. The Euler-Lagrange equations of the geodesic lines in the form

$$\ddot{x}^\alpha + \begin{Bmatrix} \alpha \\ \beta\ \eta \end{Bmatrix} \dot{x}^\beta \dot{x}^\eta = 0 \qquad \dot{x}^r \equiv \frac{dx^r}{ds} \tag{6.12}$$

contain all the Christoffel symbols. Conversely, if we know the Euler-Lagrange equations for the geodesic lines, we can identify all the nonzero Christoffel symbols; the identification is especially simple for a diagonal metric tensor. The Euler-Lagrange equations can be obtained from the variational problem

$$\delta \int ds = \delta \int [e^\nu (\dot{x}^0)^2 - (e^\lambda \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2)]^{1/2}\, ds = 0 \tag{6.13}$$

However, since we are using s as the variable of integration, we can just as well consider the equivalent and somewhat simpler variational problem in which we square the integrand of (6.13), according to the results of Sec. 2.3. That is, instead of (6.13), we use

$$\delta \int [e^\nu (\dot{x}^0)^2 - (e^\lambda \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2)]\, ds = 0 \tag{6.14}$$

Using F to represent the integrand, we write the Euler-Lagrange equations for this variational problem in the form

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^\alpha}\right) = \frac{\partial F}{\partial x^\alpha} \tag{6.15}$$

We shall write out the four equations (6.15) explicitly and compare with the form in (6.12). We identify $x^1 = r$, $x^2 = \theta$, and $x^3 = \varphi$. The comparison will allow us to write out explicitly the non-vanishing Christoffel symbols.

The Euler-Lagrange equation for $x^\alpha = x^0$ is obtained from (6.14) as

$$\frac{d}{ds}(2e^\nu \dot{x}^0) = 0 \tag{6.16}$$

Denoting differentiation with respect to r by a prime, we then have

$$\ddot{x}^0 + \nu' \dot{r} \dot{x}^0 = 0 \tag{6.17}$$

This is a particularly simple equation because of the time-independence of the line element. Comparing (6.12) with (6.17), we obtain the following nonzero Christoffel symbols whose upper indices are zero:

$$\begin{Bmatrix} 0 \\ 1\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0\ 1 \end{Bmatrix} = \tfrac{1}{2}\nu' \tag{6.18}$$

Similarly, for the variable r, we have the Euler-Lagrange equation

$$\ddot{r} + \tfrac{1}{2}\lambda' \dot{r}^2 + \tfrac{1}{2}\nu' e^{\nu-\lambda}(\dot{x}^0)^2 - e^{-\lambda} r \dot{\theta}^2 - r \sin^2\theta\, \dot{\varphi}^2 e^{-\lambda} = 0 \tag{6.19}$$

The only nonzero Christoffel symbols with the upper indices 1 are therefore

$$\begin{Bmatrix} 1 \\ 0\ 0 \end{Bmatrix} = \tfrac{1}{2}\nu' e^{\nu-\lambda} \qquad \begin{Bmatrix} 1 \\ 1\ 1 \end{Bmatrix} = \tfrac{1}{2}\lambda'$$
$$\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = -e^{-\lambda} r \qquad \begin{Bmatrix} 1 \\ 3\ 3 \end{Bmatrix} = -r \sin^2\theta\, e^{-\lambda} \tag{6.20}$$

Continuing in this way, we obtain the Euler-Lagrange equation for the variable θ,

$$\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \sin\theta \cos\theta\, \dot{\varphi}^2 = 0 \tag{6.21}$$

so the corresponding nonzero Christoffel symbols are

$$\begin{Bmatrix} 2 \\ 2\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix} = \frac{1}{r} \qquad \begin{Bmatrix} 2 \\ 3\ 3 \end{Bmatrix} = -\sin\theta \cos\theta \tag{6.22}$$

The final Euler-Lagrange equation for the variable φ is

$$\ddot{\varphi} + 2\cot\theta\, \dot{\varphi}\dot{\theta} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \tag{6.23}$$

so the corresponding Christoffel symbols are

$$(6.24)\qquad \left\{ \begin{matrix} 3 \\ 2\;3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 3\;2 \end{matrix} \right\} = \cot\theta \qquad \left\{ \begin{matrix} 3 \\ 1\;3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 3\;1 \end{matrix} \right\} = \frac{1}{r}$$

We now have all the nonzero Christoffel symbols displayed in (6.18), (6.20), (6.22), and (6.24).

The field equations (6.1) contain contracted Christoffel symbols of the form $\left\{ \begin{matrix} \tau \\ \tau\;\beta \end{matrix} \right\}$; these may be written in the convenient form $(\log \sqrt{-g})_{|\beta}$ according to (3.11), which allows us to write the field equations as

$$(6.25)\qquad R_{\mu\nu} = (\log \sqrt{-g})_{|\mu|\nu} - \left\{ \begin{matrix} \alpha \\ \mu\;\nu \end{matrix} \right\}_{|\alpha} + \left\{ \begin{matrix} \beta \\ \tau\;\mu \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \beta\;\nu \end{matrix} \right\} - \left\{ \begin{matrix} \tau \\ \mu\;\nu \end{matrix} \right\} (\log \sqrt{-g})_{|\tau} = 0$$

Using the line element (6.9), we can write the expression $\log \sqrt{-g}$ explicitly in terms of the coordinates r, θ, φ, and t. The metric tensor is

$$(6.26)\qquad g_{\mu\nu} = \begin{pmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

so the determinant g is

$$(6.27)\qquad g = \|g_{\mu\nu}\| = -e^{\nu(r)+\lambda(r)} r^4 \sin^2\theta$$

Thus

$$(6.28)\qquad \log \sqrt{-g} = \frac{\nu+\lambda}{2} + 2\log r + \log|\sin\theta|$$

We are now ready to write out the field equations (6.25) in terms of r, θ, φ, and t.

First let us consider the $\mu = \nu = 0$ component of (6.25)

$$(6.29)\qquad R_{00} = (\log \sqrt{-g})_{|0|0} - \left\{ \begin{matrix} \alpha \\ 0\;0 \end{matrix} \right\}_{|\alpha} + \left\{ \begin{matrix} \beta \\ \tau\;0 \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \beta\;0 \end{matrix} \right\} - \left\{ \begin{matrix} \tau \\ 0\;0 \end{matrix} \right\} (\log \sqrt{-g})_{|\tau} = 0$$

Many of the terms appearing in this equation are identically zero; the nonzero terms are displayed in (6.18), (6.20), (6.22), (6.24), and (6.28), and we are left with

$$(6.30)\qquad R_{00} = -\left\{ \begin{matrix} 1 \\ 0\;0 \end{matrix} \right\}_{|1} + 2\left\{ \begin{matrix} 1 \\ 0\;0 \end{matrix} \right\} \cdot \left\{ \begin{matrix} 0 \\ 0\;1 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 0\;0 \end{matrix} \right\} (\log \sqrt{-g})_{|1}$$

$$= -(\tfrac{1}{2}\nu' e^{\nu-\lambda})' + (\tfrac{1}{2}\nu'^2 e^{\nu-\lambda}) - (\tfrac{1}{2}\nu' e^{\nu-\lambda}) \left(\frac{\nu'+\lambda'}{2} + \frac{2}{r} \right) = 0$$

This reduces to

$$R_{00} = \frac{-e^{\nu-\lambda}}{2} \left(\nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{r} \right)$$

$$(6.31)\qquad \nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\lambda'\nu' + \frac{2\nu'}{r} = 0$$

We proceed in similar fashion for the $\mu = \nu = 1$ component of Eq. (6.25)

$$(6.32)\qquad R_{11} = (\log \sqrt{-g})_{|1|1} - \left\{ \begin{matrix} \alpha \\ 1\;1 \end{matrix} \right\}_{|\alpha} + \left\{ \begin{matrix} \beta \\ \tau\;1 \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \beta\;1 \end{matrix} \right\} - \left\{ \begin{matrix} \tau \\ 1\;1 \end{matrix} \right\} (\log \sqrt{-g})_{|\tau} = 0$$

Discarding identically vanishing terms, we obtain

$$(6.33)\qquad R_{11} = (\log \sqrt{-g})_{|1|1} - \left\{ \begin{matrix} 1 \\ 1\;1 \end{matrix} \right\}_{|1} + \left\{ \begin{matrix} 0 \\ 1\;0 \end{matrix} \right\} \left\{ \begin{matrix} 0 \\ 1\;0 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ 1\;1 \end{matrix} \right\} \left\{ \begin{matrix} 1 \\ 1\;1 \end{matrix} \right\} + \left\{ \begin{matrix} 2 \\ 2\;1 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 2\;1 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 3\;1 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 3\;1 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 1\;1 \end{matrix} \right\} (\log \sqrt{-g})_{|1} = 0$$

that is,

$$(6.34)\qquad R_{11} = \left(\frac{\nu''+\lambda''}{2} - \frac{2}{r^2} \right) - \tfrac{1}{2}\lambda'' + \tfrac{1}{4}\nu'^2 + \tfrac{1}{4}\lambda'^2 + \frac{2}{r^2} - \tfrac{1}{2}\lambda' \left(\frac{\lambda'+\nu'}{2} + \frac{2}{r} \right) = 0$$

This reduces to

$$R_{11} = \tfrac{1}{2} \left(\nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\lambda'\nu' - \frac{2\lambda'}{r} \right)$$

$$(6.35)\qquad \nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\lambda'\nu' - \frac{2\lambda'}{r} = 0$$

Equations (6.31) and (6.35) now represent a system of two ordinary differential equations which we may solve for the functions $\nu(r)$ and $\lambda(r)$. Subtraction of (6.35) from (6.31) yields

$$\nu' + \lambda' = 0 \tag{6.36}$$

Thus

$$\nu + \lambda = \text{const} = k \tag{6.37}$$

We can choose the constant k to be zero by a simple device. Replace the time coordinate t by another coordinate $t \exp(k/2)$; from (6.9) it is clear that this is equivalent to replacing ν by $\nu + k$, so that (6.37) becomes

$$\lambda = -\nu \tag{6.38}$$

We shall see that this choice of time coordinate has the very desirable feature of making the line element asymptotically equal to the flat-space line element (6.2). The coordinate t which we select in this way will be seen to correspond to the physical time as measured by an observer at infinity.

Substituting $-\lambda$ for ν in (6.35), we obtain a second-order ordinary differential equation for $\lambda(r)$

$$\lambda'' - \lambda'^2 + \frac{2\lambda'}{r} = 0 \tag{6.39}$$

This can be more conveniently written as

$$(re^{-\lambda})'' = 0 \tag{6.40}$$

Integration is then trivial, and we have

$$(re^{-\lambda})' = \text{const} \tag{6.41}$$

It will be convenient to leave (6.41) as it stands (with an undetermined constant) and proceed to the component corresponding to R_{22} of the system of equations (6.25); the reason for this will be apparent in the next paragraph.

Proceeding as with the R_{00} and R_{11} equations, we have

$$R_{22} = (\log \sqrt{-g})_{|2|2} - \begin{Bmatrix} \alpha \\ 2\ 2 \end{Bmatrix}_{|\alpha} + \begin{Bmatrix} \beta \\ \tau\ 2 \end{Bmatrix}\begin{Bmatrix} \tau \\ \beta\ 2 \end{Bmatrix} - \begin{Bmatrix} \tau \\ 2\ 2 \end{Bmatrix}(\log \sqrt{-g})_{|\tau} \tag{6.42}$$

Substitution of the nonzero terms from (6.18), (6.20), (6.22), (6.24), and (6.28) gives

$$R_{22} = (\log \sqrt{-g})_{|2|2} - \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}_{|1} + \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}\begin{Bmatrix} 2 \\ 2\ 1 \end{Bmatrix} + \begin{Bmatrix} 2 \\ 2\ 1 \end{Bmatrix}\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} + \begin{Bmatrix} 3 \\ 2\ 3 \end{Bmatrix}\begin{Bmatrix} 3 \\ 2\ 3 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}(\log \sqrt{-g})_{|1} = 0 \tag{6.43}$$

that is,

$$R_{22} = \frac{\partial^2}{\partial\theta^2}(\log|\sin\theta|) + (e^{-\lambda}r)' + 2(-e^{-\lambda}) + \cot^2\theta + e^{-\lambda}r\left(\frac{\lambda' + \nu'}{2} + \frac{2}{r}\right) = 0 \tag{6.44}$$

By virtue of (6.36) this simplifies to

$$(e^{-\lambda}r)' = 1 \tag{6.45}$$

This is precisely the same as Eq. (6.41) except that the unknown constant which appeared in (6.41) is now identified as 1. Integration immediately gives

$$e^{-\lambda}r = r - 2m \tag{6.46}$$

where $-2m$ is an arbitrary constant of integration. Thus, from the three equations $R_{00} = R_{11} = R_{22} = 0$, we have solved for the functions $\nu(r)$ and $\lambda(r)$ which appear in the line element (6.9)

$$\begin{aligned} e^{\nu} &= e^{-\lambda} = 1 - \frac{2m}{r} \\ e^{\lambda} &= \frac{1}{1 - 2m/r} \end{aligned} \tag{6.47}$$

Consider for a moment the result (6.47). As we noted, it was only necessary to use three of the ten equations in the system (6.25) to obtain what appears to be a complete solution (6.47). Apparently, then, we have a consistency problem remaining: Are the other seven equations in the system (6.25) *consistent* with the solution (6.47)? We shall show that the remaining diagonal element R_{33} of the Ricci tensor is indeed zero by virtue of the solution (6.47), so that (6.47) and the equation $R_{33} = 0$

are consistent. As before, we write

$$(6.48)\quad R_{33} = (\log\sqrt{-g})_{|3|3} - \left\{\begin{matrix}\alpha\\3\ 3\end{matrix}\right\}_{|\alpha} + \left\{\begin{matrix}\beta\\\tau\ 3\end{matrix}\right\}\left\{\begin{matrix}\tau\\\beta\ 3\end{matrix}\right\} - \left\{\begin{matrix}\tau\\3\ 3\end{matrix}\right\}(\log\sqrt{-g})_{|\tau} = 0$$

Discarding the identically zero terms, we get

$$(6.49)\quad R_{33} = -\left\{\begin{matrix}1\\3\ 3\end{matrix}\right\}_{|1} - \left\{\begin{matrix}2\\3\ 3\end{matrix}\right\}_{|2} + 2\left\{\begin{matrix}3\\1\ 3\end{matrix}\right\}\left\{\begin{matrix}1\\3\ 3\end{matrix}\right\} + 2\left\{\begin{matrix}3\\3\ 2\end{matrix}\right\}\left\{\begin{matrix}2\\3\ 3\end{matrix}\right\} - \left\{\begin{matrix}1\\3\ 3\end{matrix}\right\}(\log\sqrt{-g})_{|1} - \left\{\begin{matrix}2\\3\ 3\end{matrix}\right\}(\log\sqrt{-g})_{|2} = 0$$

Observe that, by virtue of (6.28) and (6.38), we have

$$(6.50)\quad \log\sqrt{-g} = 2\log r + \log|\sin\theta|$$

Hence

$$(6.51)\quad (re^{-\lambda}\sin^2\theta)' + \frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 2(-e^{-\lambda}\sin^2\theta - \cot\theta\sin\theta\cos\theta) + re^{-\lambda}\sin^2\theta\left(\frac{2}{r}\right) + \cos^2\theta = 0$$

This simplifies to

$$(6.52)\quad \sin^2\theta[(e^{-\lambda}r)' - 1] = 0$$

By virtue of (6.45) this is identically zero, so we see that the equation $R_{33} = 0$ is indeed consistent with (6.47). We leave it to the reader to verify that all the off-diagonal elements of the contracted Riemann tensor are identically zero when explicitly written in terms of the coordinates and that (6.47) is therefore a completely consistent solution of (6.25).

Let us now summarize the results of this section by exhibiting the *Schwarzschild line element*

$$(6.53)\quad ds^2 = \left(1 - \frac{2m}{r}\right)(dx^0)^2 - \frac{dr^2}{1 - 2m/r} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

This result must be considered to be the main achievement of general relativity theory in the field of celestial mechanics; it is an exact solution, which corresponds historically to Newton's treatment of the $1/r^2$ force law of classical gravitational theory. The rest of this chapter will be devoted to investigating the physical consequences of the line element (6.53).

It is evident that the Schwarzschild line element approaches the flat-space form (6.2) at large r. We may therefore identify t with the time measured by an observer at a large distance from the origin. Thus the coordinate time is, in this sense, a distinguished coordinate. It is important to keep in mind that the physical meaning of the coordinates is intimately related to the metric, as this example demonstrates.

The unknown constant of integration m which appears in the Schwarzschild line element can be determined by an appeal to correspondence with Newtonian theory. Recall that, in Sec. 4.3, we found that a geometric theory of gravitation will reduce in the classical limit of weak fields and slowly moving bodies to the Newtonian theory if $g_{00} \cong 1 + 2\varphi/c^2$; φ is the classical potential for the gravitational field. In the present case of a point mass, φ is simply $-\kappa M/r$, where M is the mass of the particle and $\kappa = 6.67 \times 10^{-8}$ dyne-cm^2/g^2 is the gravitational constant. Thus, in the classical limit, $g_{00} \cong 1 - 2\kappa M/c^2 r$. Comparing this with (6.53), we see that

$$(6.54)\quad m = \frac{\kappa M}{c^2}$$

The constant m has the units of distance and will be referred to as the *geometric mass* of the central body.

We have here obtained the Schwarzschild solution by imposing the conditions of spherical symmetry and time-independence. However, it can be proved (Birkhoff, 1923) that the requirement of time-independence is superfluous and that any spherically symmetric distribution of matter, even if in radial motion, leads to the *same* line element exterior to the matter distribution. This result is called *Birkhoff's theorem*. The derivation is straightforward but more cumbersome than that presented in the text, since λ and ν are treated as functions of r and t (see Prob. 6.1). A consequence of Birkhoff's theorem is that a radially pulsating distribution of matter can emit no gravitational waves since the metric exterior to the distribution is static. Such waves can therefore be emitted only by more complicated deformations of a massive body.

Before continuing to the next section, it should be noted that, on the spherical shell $r = 2m$, the coefficient of dr^2 in the Schwarzschild line element becomes infinite and the coefficient of $(dx^0)^2$ is zero; $r = 2m$ is called the *Schwarzschild radius*. For ordinary stars this is characteristically a very small number; for the sun the Schwarzschild radius is about

3 km, which is well *inside* the sun, where the free-space field equations (6.25) are *not* valid and the Schwarzschild line element is *not* an appropriate description of the space-time geometry. The existence of this singularity is therefore of no consequence for the description of planetary motion.

It is clear that a star of roughly solar mass would have to be compressed to exceedingly high density before the bulk of its mass could be inside the Schwarzschild radius. The study of just this situation has become of great interest in recent years since it now appears likely from theoretical studies that a significant fraction of the stars in the universe may actually reach and exceed such densities in the process of gravitational collapse, which occurs at the end of their existence as normal stars. We shall return to this subject later in this chapter and again in Chap. 14.

6.2 The Schwarzschild Solution in Isotropic Coordinates

In the preceding section we obtained the Schwarzschild solution (6.53) in terms of a set of spherical polar coordinates: r, θ, φ, and t. The choice of this particular set of coordinates was motivated by the radial symmetry, time-independence, and relative simplicity required of the basic line element (6.9). However, it is characteristic of general relativity that there are usually many convenient coordinate systems available in which to work, and the coordinates r, θ, φ, and t in which (6.53) is expressed are not the only coordinates which correspond to our intuitive notions of radial and angular markers. In this section we shall consider another convenient set of coordinates and investigate the Schwarzschild line element expressed in the new coordinates.

The main reason for seeking an alternative set of coordinates is that we would like to express ds^2 in a form which is independent of the particular *space* coordinates used. More specifically, we would like to put the line element in the form

$$ds^2 = A(r)(dx^0)^2 - B(r)\, d\sigma^2 \tag{6.55}$$

where $d\sigma^2$ is $dx^2 + dy^2 + dz^2$ in Cartesian coordinates or $dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$ in spherical coordinates, etc. This sort of line element agrees most closely with our intuitive notion of space, which is based mainly on Euclidean geometry. Indeed, to illustrate this, consider two vectors in three dimensions, ξ^i and η^i. In the metric (6.55) the cosine of the angle between these vectors, $\xi^i\eta_i/|\xi|\,|\eta|$, is the same as if we were in Euclidean space; this is due to the fact that the factor $B(r)$ cancels in the above ratio.

For this reason the line element (6.55) is called a *conformal line element.* The coordinates in which the line element takes the form (6.55) are called *isotropic coordinates.*

To obtain isotropic coordinates we shall attempt to use the following particularly simple coordinate transformation: The coordinates θ, φ, and t remain unchanged, while a radial coordinate $\rho(r)$ replaces r. In terms of these coordinates we ask that the Schwarzschild line element have the isotropic form

$$\begin{aligned} ds^2 &= \left(1 - \frac{2m}{r}\right) c^2\,dt^2 - \lambda^2(\rho)[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] \\ &= \left(1 - \frac{2m}{r}\right) c^2\,dt^2 - \lambda^2(\rho)\, d\sigma^2 \end{aligned} \tag{6.56}$$

These demands lead to the mathematical problem of finding two functions of ρ, $r(\rho)$ and $\lambda(\rho)$, for which the two forms of the Schwarzschild line element, (6.53) and (6.56), are consistent. Comparing the coefficients of the angular interval $(d\theta^2 + \sin^2\theta\, d\varphi^2)$ in (6.53) and (6.56), we see that we must have

$$r^2 = \lambda^2\rho^2 \tag{6.57}$$

A similar comparison of the radial intervals gives

$$\frac{dr^2}{1 - 2m/r} = \lambda^2\, d\rho^2 \tag{6.58}$$

Substituting for λ^2 from (6.57) and taking the square root, we obtain an ordinary differential equation for $r(\rho)$:

$$\frac{\pm\, dr}{\sqrt{r^2 - 2mr}} = \frac{d\rho}{\rho} \tag{6.59}$$

An easy integration then yields

$$\pm \log\,[(r^2 - 2mr)^{1/2} + (r - m)] = \log \rho + \text{const} \tag{6.60}$$

To evaluate the constant and determine the sign of the left side of (6.60), consider r much larger than $2m$; asymptotically we must have

$$\pm \log\,(2r) = \log \rho + \text{const} \tag{6.61}$$

For large radial distances we wish r and ρ to be asymptotically equal, so we must choose the plus sign and take the constant to be log 2. Equation (6.60) then gives

(6.62) $$\sqrt{r^2 - 2mr} + (r - m) = 2\rho$$

In order to solve this algebraic equation for r as a function of ρ, note that

(6.63) $$[(r - m) + \sqrt{r^2 - 2mr}][(r - m) - \sqrt{r^2 - 2mr}] = m^2$$

Dividing this by (6.62), we obtain

(6.64) $$(r - m) - \sqrt{r^2 - 2mr} = \frac{m^2}{2\rho}$$

Addition of (6.64) above to (6.62) yields

(6.65) $$r - m = \rho + \frac{m^2}{4\rho}$$

Thus, finally,

(6.66) $$r = \rho + \frac{m^2}{4\rho} + m = \rho\left(1 + \frac{m}{2\rho}\right)^2$$

From Eq. (6.57) and the above it follows that the function $\lambda(\rho)$ is

(6.67) $$\lambda = \frac{r}{\rho} = \left(1 + \frac{m}{2\rho}\right)^2$$

Let us now return to the isotropic form of the Schwarzschild line element (6.56) and express it in terms of the coordinate ρ. According to (6.66), the coefficient of dt is

(6.68) $$\left(1 - \frac{2m}{r}\right) = 1 - \frac{2m}{\rho(1 + m/2\rho)^2} = \frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2}$$

Thus, by (6.56), (6.67), and (6.68), the Schwarzschild line element in terms of *isotropic* coordinates is

(6.69) $$\begin{aligned} ds^2 &= \frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2}\, c^2\, dt^2 - \left(1 + \frac{m}{2\rho}\right)^4 (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\varphi^2) \\ &= \frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2}\, c^2\, dt^2 - \left(1 + \frac{m}{2\rho}\right)^4 d\sigma^2 \end{aligned}$$

We have now succeeded in putting the Schwarzschild line element in isotropic form; in such a form it should still be directly comparable with the approximate solution (4.142) obtained by a correspondence argument. In order to make a comparison, note that the constant of integration m which appears in the isotropic line element (6.69) evidently serves as a measure of how much the line element differs from the Lorentzian form $c^2\, dt^2 - d\sigma^2$; indeed, $m = 0$ gives precisely the Lorentzian form. Thus, for a weak gravitational field [in which case (4.142) is a valid approximation], we expect m/ρ to be a small quantity compared with 1 for physically significant values of ρ; that is, $m/\rho \ll 1$. We may then expand (6.69) to first order in m/ρ,

(6.70) $$\begin{aligned} ds^2 &\cong \left(1 - \frac{m}{\rho}\right)\left(1 - \frac{m}{\rho}\right) c^2\, dt^2 - \left(1 + \frac{2m}{\rho}\right) d\sigma^2 \\ &\cong \left(1 - \frac{2m}{\rho}\right) c^2\, dt^2 - \left(1 + \frac{2m}{\rho}\right) d\sigma^2 \end{aligned}$$

If we compare this with (4.142), we see that we must have within our approximation

(6.71) $$g_{00} \cong \left(1 - \frac{2m}{\rho}\right) \cong \left(1 + \frac{2\varphi}{c^2}\right) = \left(1 - \frac{2\kappa M}{c^2\rho}\right)$$

and we thereby obtain the same result as in Sec. 6.1,

(6.72) $$m = \frac{\kappa M}{c^2}$$

Thus both the original Schwarzschild solution and the above isotropic form lead by a correspondence argument to a consistent identification of the constant of integration m.

6.3 The General Relativistic Kepler Problem and the Perihelic Shift of Mercury

The principal results of the preceding sections are the Schwarzschild solution (6.53) and the identification of the constant of integration m in (6.54) and (6.72). In this section we shall use these results to study the motion of a test particle in a Schwarzschild field, which should directly correspond to planetary motion in the gravitational field of the sun.

This problem is the relativistic analogue of the classical Kepler problem of planetary motion in an inverse-square force field.

As a guide in investigating the relativistic problem, let us recall some of the main features of the classical problem. Kepler's first law states that a planet describes a closed elliptical orbit with the sun at a focal point. However (more realistically), the presence of such small influences as other planets moving in the sun's field causes a perturbation in the motion of a given planet, and the resulting orbit is not precisely elliptic. Indeed, one may think of the actual orbit as a slightly bumpy ellipse which may precess in the plane of motion; that is, the perihelion (point of closest approach to the sun) shifts about and does not always occur at the same angular position.

The fact that the idealized classical orbit is a closed ellipse is a result peculiar to the Newtonian inverse-square law; in fact, Newton himself found that, if the force of gravity were proportional to $1/r^{2+\delta}$ instead of $1/r^2$, then a planetary orbit would not be closed and a perihelic shift of order δ would occur. Indeed, this result was taken to indicate that, since planetary orbits are very nearly closed, the Newtonian inverse-square law must be quite accurate, as in fact it is.

Let us now ask what differences might be expected between the predictions of classical celestial mechanics and general relativistic celestial mechanics. Since Kepler's first law is experimentally verified to be correct to high accuracy, we might expect the relativistic theory merely to add a few bumps to the nearly elliptic orbits and contribute somewhat to perihelic motion. Since angles are much more conveniently measured in astronomy than are distances, it is natural to concentrate on perihelic motion. Conveniently enough, there is, in fact, a well-known discrepancy in classical mechanics concerning the perihelic motion of the planet Mercury. Because of Mercury's high velocity and eccentric orbit, the perihelion position can be accurately determined by observation; the difference between the classically predicted perihelic shift (due to perturbation by other planets) and the observed perihelic shift is 43 seconds of arc per century. Even though this is a very small difference, it is about a hundred times the probable observational error and represents a true discrepancy from the very precise predictions of celestial mechanics which has bothered astronomers since the middle of the last century (Leverrier, 1859).

The first attempt to explain this discrepancy consisted in hypothesizing the existence of a new planet, Vulcan, inside the orbit of Mercury, and much theoretical work was done to predict the position of Vulcan, using the known perturbation on Mercury's orbit. However, careful observation failed to discover the hypothetical planet, and the hypothesis was finally abandoned in 1915 when Einstein used general relativity theory

to explain the observed effect. Let us now proceed to investigate the general relativistic Kepler problem and, as an application, study the motion of planetary perihelia.

We must first decide which radial measure to use, either the r of (6.53) or the ρ of (6.69). A perihelic shift involves the angle between *successive minima* of the radial distance; since $\rho(r)$ is a monotonic function of r outside the Schwarzschild singularity [by virtue of (6.62)], the minima of both r and $\rho(r)$ for a planetary orbit occur at the same angular position, so we can use equally well either the original coordinate r or the isotropic coordinate ρ. It will prove more convenient to use r and the line element (6.53) since the resultant equations of motion will be more similar to the equations of motion for the classical Kepler problem and easier to interpret.

As we stated in Sec. 4.3, the motion of a body in a gravitational field follows a four-dimensional geodesic line. Hence, to find the orbit of a planet, we need the Euler-Lagrange equations for the following variational problem:

$$\delta \int ds = 0 \tag{6.73}$$

where ds is given by the Schwarzschild line element (6.53). As in Sec. 6.1, we may simplify calculations by considering the equivalent variational problem

$$\delta \int \left\{\left(1 - \frac{2m}{r}\right) c^2\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2[\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2]\right\} ds = 0 \tag{6.74}$$

(As before, a dot indicates differentiation with respect to s.) The three Euler-Lagrange equations for θ, φ, and t associated with this variational problem are the following:

$$\frac{d}{ds}(r^2\dot{\theta}) = r^2 \sin\theta \cos\theta\,\dot{\varphi}^2 \tag{6.75}$$

$$\frac{d}{ds}(r^2 \sin^2\theta\,\dot{\varphi}) = 0 \tag{6.76}$$

$$\frac{d}{ds}\left[\left(1 - \frac{2m}{r}\right)\dot{t}\right] = 0 \tag{6.77}$$

Note that we have not included the Euler-Lagrange equation for r; it is more convenient to divide the line element (6.53) by ds^2 to obtain a fourth differential equation,

$$(6.78)\qquad 1 = \left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2 \theta\, \dot{\varphi}^2)$$

Using the above four differential equations for t, r, θ, and φ as functions of s, it is possible to obtain and solve the equations of a planetary orbit. In classical mechanics the orbit of a body in a central force field lies in a plane. We can show that the same holds true in the present theory. By an appropriate orientation of the axes we can make $\theta = \pi/2$ and $\dot{\theta} = 0$ at some initial s. Then, from (6.75), it follows that, for all s,

$$(6.79)\qquad \theta = \frac{\pi}{2}$$

since the initial conditions determine a unique solution of (6.75), and (6.79) is surely such a solution. Substitution of $\theta = \pi/2$ in (6.76) allows us to integrate (6.76) immediately:

$$(6.80)\qquad r^2 \dot{\varphi} = h = \text{const}$$

Equation (6.77) integrates to

$$(6.81)\qquad \left(1 - \frac{2m}{r}\right) \dot{t} = l = \text{const}$$

Substituting the results (6.79), (6.80), and (6.81) into (6.78), we obtain the following differential equation for $r(s)$:

$$(6.82)\qquad 1 = \left(1 - \frac{2m}{r}\right)^{-1} c^2 l^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - \frac{h^2}{r^2}$$

As in the classical Kepler problem, one can simplify matters by considering r as a function of φ instead of s. Denoting differentiation with respect to φ by a prime, we then have

$$(6.83)\qquad r' = \frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}}$$

From (6.80) and (6.83) we obtain

$$(6.84)\qquad \dot{r} = \dot{\varphi} r' = \frac{h}{r^2} r'$$

The differential equation for $r(\varphi)$ is then obtainable from (6.82):

$$(6.85)\qquad \left(1 - \frac{2m}{r}\right) = c^2 l^2 - \frac{h^2}{r^4} r'^2 - \frac{h^2}{r^2}\left(1 - \frac{2m}{r}\right)$$

Following once more the example of the classical Kepler problem, we substitute for the dependent variable

$$(6.86)\qquad r = \frac{1}{u}$$

which implies

$$(6.87)\qquad r' = -\frac{u'}{u^2}$$

Using these relations, we can convert (6.85) to a differential equation for $u(\varphi)$:

$$(6.88)\qquad (1 - 2mu) = c^2 l^2 - h^2 u'^2 - h^2 u^2 (1 - 2mu)$$

This reduces to

$$(6.89)\qquad u'^2 = \left(\frac{c^2 l^2 - 1}{h^2}\right) + \frac{2m}{h^2} u - u^2 + 2mu^3$$

which is immediately integrable:

$$(6.90)\qquad \varphi = \varphi_0 + \int_{u_0}^{u} \frac{du}{\left(\frac{c^2 l^2 - 1}{h^2} + \frac{2m}{h^2} u - u^2 + 2mu^3\right)^{1/2}}$$

This is an *exact solution* to the problem; it expresses the angle φ as an integral of $u = 1/r$, and conversely it gives u as the inverse (implicit) function of φ.

Unfortunately, even though (6.90) is a complete solution to our problem, its form is not particularly enlightening; $u(\varphi)$ is given in implicit form, and the approximate classical form of the trajectory (an ellipse) is not at all evident in (6.90). To make the problem more transparent and to establish a closer connection with the classical Kepler problem (which involves a second-order differential equation), we shall convert the first-order equation (6.89) to a second-order equation by differentiation with respect to φ. We obtain

$$2u'u'' = \frac{2m}{h^2}u' - 2uu' + 6mu^2u' \tag{6.91}$$

One possible solution is then obtained by setting the common factor u' equal to zero:

$$u' = 0 \qquad u = \text{const} \qquad r = \text{const} \tag{6.92}$$

Thus circular motion occurs in relativity theory just as in classical theory. [This could also be inferred from the first-order equation (6.89).] The other possible solution, which is much more interesting, will result from canceling the common factor u' from (6.91):

$$u'' + u = \frac{m}{h^2} + 3mu^2 \tag{6.93}$$

This last equation is quite similar in structure to the orbit equation of the classical Kepler problem. Indeed, for the sake of completeness and comparison, let us recall the derivation of Binet's formula for the motion of a particle of mass m in a central field of force with potential function $mf(r)$. We assume, as before, that the motion takes place in a plane $\theta = 0$. Suppressing a common factor of m in the Lagrangian, we find

$$L = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] - f(r) \tag{6.94}$$

whence the differential equations of motion

$$\frac{d^2r}{dt^2} = r\left(\frac{d\varphi}{dt}\right)^2 - f'(r) \qquad r^2\frac{d\varphi}{dt} = H = \text{const} \tag{6.95}$$

where $f'(r) = df/dr$. Consider now the trajectory equation for $r = r(\varphi)$, and also introduce the function

$$u(\varphi) = \frac{1}{r(\varphi)} \tag{6.96}$$

We then have

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = r'(\varphi)\frac{H}{r^2} = -Hu'(\varphi) \tag{6.97}$$

where $u'(\varphi) = du/d\varphi$. Hence the first differential equation in (6.95) becomes

$$\frac{d^2r}{dt^2} = -H^2u''(\varphi)u^2 = H^2u^3(\varphi) - f'(r) \tag{6.98}$$

Rearranging this, we obtain

$$u'' + u = \frac{1}{H^2}\frac{f'(r)}{u^2} \tag{6.99}$$

which is Binet's general formula describing the $(1/r, \varphi)$ relation for a central force.

For the special case of a Newtonian potential, $f(r) = -\kappa M/r$, Binet's equation (6.99) becomes

$$u'' + u = \frac{\kappa M}{H^2} \tag{6.100}$$

where H is twice the constant areal velocity:

$$H = r^2\frac{d\varphi}{dt} = \text{const} \tag{6.101}$$

The analogous term in the relativistic equation (6.93) is m/h^2, which, by virtue of (6.72) and (6.80), is explicitly given by

$$\frac{m}{h^2} = \frac{\kappa M}{c^2r^4(d\varphi/ds)^2} = \frac{\kappa M}{c^2r^4(d\varphi/dt)^2(dt/ds)^2} \tag{6.102}$$

Furthermore, we know from Sec. 4.3 that, for slowly moving bodies in weak gravitational fields, $(dt/ds)^2$ is approximately $1/c^2$; substituting this in (6.102), we obtain an approximate form for m/h^2:

$$\frac{m}{h^2} \cong \frac{\kappa M}{r^4(d\varphi/dt)^2} = \frac{\kappa M}{H^2} \tag{6.103}$$

Thus we see that the relativistic equation (6.93) differs from the classical equation (6.100) through the addition of the quadratic term $3mu^2$ and has a slightly different constant term m/h^2. One might furthermore expect the term $3mu^2$ to be small relative to the leading constant term; we may easily verify that this is indeed the case by forming the ratio of it and the constant term m/h^2. This ratio is $3u^2h^2$, which, by virtue of (6.80), is $3r^2\dot{\varphi}^2 \cong 3[r(d\varphi/dt)]^2 \cdot 1/c^2$. The quantity $r(d\varphi/dt)$ is the lateral velocity of the planet (the velocity perpendicular to r), so the above ratio may be written as $3v^2_{\text{lateral}}/c^2$, which is always very small and equal to 7.7×10^{-8} in the case of Mercury. The close similarity between the relativistic equation (6.93) and the classical theory (6.100) is now quite clear.

Equation (6.93) may be interpreted, by the above comments, as the Binet equation of motion in classical mechanics for a field of force with the potential

$$f(r) = \frac{-\kappa M}{r} - \frac{\gamma}{r^3} \qquad \gamma = mH^2 = \kappa M h^2 \tag{6.104}$$

Observe, however, that for different values of H, the indicated modification of Newton's law would be different. Thus this analogy may be helpful in a geometric discussion of the trajectory, but has no real physical significance.

For later argument we bring (6.104) into the form

$$f(r) = -\frac{\kappa M}{r}\left(1 + \frac{h^2}{r^2}\right) \tag{6.104a}$$

If v_l denotes the lateral velocity of the planet, we may use the above approximate equation $u^2h^2 = (1/r^2)h^2 \cong (v_l/c)^2$ and write

$$f(r) \cong -\frac{\kappa M}{r}\left[1 + \left(\frac{v_l}{c}\right)^2\right] \tag{6.104b}$$

Since the planetary orbits are very nearly circular, we may also assume $v_l = 2\pi r/T$, where T is the period of revolution of the planet. Finally, by Kepler's third law, we know that r^3/T^2 is the same for all planets, and hence we have $r(v_l/c)^2 = C$ as common value for all planets. Thus

$$f(r) \cong -\frac{\kappa M}{r}\left[1 + \frac{C}{r}\right] \tag{6.104c}$$

a formula in which the angular velocity of the planet has been eliminated. This formulation of (6.104) will be of value later, when we shall discuss contributions to the perihelic shift of nonrelativistic origin.

Let us now investigate the relativistic equation (6.93) with a view to calculating the perihelion shift. We saw above that the term $3mu^2$ represents a small addition to the classical equations, so let us try a perturbation approach. Define

$$A = \frac{m}{h^2} \cong \frac{\kappa M}{H^2} \tag{6.105}$$

and the small dimensionless quantity

$$\epsilon = 3mA \cong \frac{3\kappa^2 M^2}{c^2 H^2} \tag{6.106}$$

The relativistic orbit equation then takes the form

$$u'' + u = A + \frac{\epsilon u^2}{A} \tag{6.107}$$

To solve this we assume a solution of the form

$$u(\varphi) = u_0(\varphi) + \epsilon v(\varphi) + O(\epsilon^2) \tag{6.108}$$

and attempt to find $u_0(\varphi)$ and $v(\varphi)$.

Substituting this form for u in the differential equation (6.107), we obtain

$$u_0'' + \epsilon v'' + u_0 + \epsilon v = A + \epsilon u_0^2/A + O(\epsilon^2) \tag{6.109}$$

Equating the zeroth-order terms in ϵ, we have

$$u_0'' + u_0 = A \tag{6.110}$$

which is essentially the classical equation (6.100). The solution is easily checked to be

$$u_0 = A + B\cos(\varphi + \delta) \tag{6.111}$$

where B and δ are arbitrary constants. By an appropriate orientation of the axes we may make δ equal to zero, in which case we obtain the familiar equation of an ellipse,

$$u_0 = A + B\cos\varphi \tag{6.112}$$

Similarly, equating the first-order ϵ terms in (6.109), we obtain

$$\begin{aligned} v'' + v = \frac{u_0^2}{A} &= A + 2B\cos\varphi + \frac{B^2}{A}\cos^2\varphi \\ &= \left(A + \frac{B^2}{2A}\right) + 2B\cos\varphi + \frac{B^2}{2A}\cos 2\varphi \end{aligned} \tag{6.113}$$

Note that we need only a nonhomogeneous solution to this equation since the zeroth-order solution already contains a term $B \cos \varphi$, which is the general solution to the homogeneous equation. Despite the cumbersome appearance of (6.113) it is readily solved; since it is *linear* in v, we may write v as the sum $v = v_a + v_b + v_c$, where v_a, v_b, and v_c are solutions of the equations

$$(6.114) \quad v_a'' + v_a = A + \frac{B^2}{2A} \quad v_b'' + v_b = 2B \cos \varphi \quad v_c'' + v_c = \frac{B^2}{2A} \cos 2\varphi$$

that is, we superpose the three solutions (6.114) to get (6.113). The nonhomogeneous solutions to (6.114) are easily checked to be

$$(6.115) \quad v_a = A + \frac{B^2}{2A} \quad v_b = B\varphi \sin \varphi \quad v_c = -\frac{B^2}{6A} \cos 2\varphi$$

so a nonhomogeneous solution to (6.113) is

$$(6.116) \quad v = v_a + v_b + v_c = \left(A + \frac{B^2}{2A}\right) + B\varphi \sin \varphi - \frac{B^2}{6A} \cos 2\varphi$$

Combining this with the zeroth-order solution (6.112), we have the entire solution for the orbit to first order in ϵ:

$$(6.117) \quad u = u_0 + \epsilon v$$
$$= \left(A + \epsilon A + \frac{\epsilon B^2}{2A}\right) + \left(B \cos \varphi - \frac{\epsilon B^2}{6A} \cos 2\varphi\right) + \epsilon B\varphi \sin \varphi$$

Using this solution, we can readily calculate the perihelion shift. Since only the last term is nonperiodic, it is clear that whatever irregularities occur in the perihelion position must be due to this term. To clarify further the effect of the nonperiodic term, note that, to first order in ϵ,

$$(6.118) \quad \cos (\varphi - \epsilon\varphi) = \cos \varphi \cos \epsilon\varphi + \sin \varphi \sin \epsilon\varphi = \cos \varphi + \epsilon\varphi \sin \varphi$$

so the solution may be written as

$$(6.119) \quad u = A + B \cos (\varphi - \epsilon\varphi) + \epsilon \left(A + \frac{B^2}{2A} - \frac{B^2}{6A} \cos 2\varphi\right)$$

In this form the effect of the various terms on the orbit is apparent. The basic elliptical orbit is represented by $A + B \cos \varphi$. The effect of the last term is to introduce small *periodic* variations in the radial distance

of the planet. Such effects are difficult to detect, and since they are periodic, they cannot influence the perihelic motion. However, the $\epsilon\varphi$ which appears in the cosine argument does indeed introduce a nonperiodicity, and since φ can become large, the effect is not negligible. Accordingly, let us write (6.119) in the form

$$(6.120) \quad u = A + B \cos (\varphi - \epsilon\varphi) + (\text{periodic terms of order } \epsilon)$$

The perihelion of a planet occurs when r is a minimum or when $u = 1/r$ is a maximum. From (6.120) we see that u is maximum when

$$(6.121) \quad \varphi(1 - \epsilon) = 2\pi n$$

or approximately

$$(6.122) \quad \varphi = 2\pi n(1 + \epsilon)$$

Therefore successive perihelia will occur at intervals of

$$(6.123) \quad \Delta\varphi = 2\pi(1 + \epsilon)$$

instead of 2π as in periodic motion. Thus the perihelion *shift* per revolution is given by

$$(6.124) \quad \delta\varphi = 2\pi\epsilon = 2\pi \left(\frac{3\kappa^2 M^2}{c^2 H^2}\right)$$

For the case of Mercury, Eq. (6.124) gives a total shift of 43.03″ per century. This is in excellent agreement with the observational result of 43.11 ± 0.45″ which is unaccounted for classically. (For a more extensive discussion of the observational problem, see the review article of Finlay-Freundlich, 1955, and the article of Shapiro, 1972.) This fact is of crucial importance; historically it was the first major observational test of general relativity theory. Recently, however, Dicke (Dicke, 1964) has questioned the excellence of this test by calling into question the exact shape of the sun and hence its classical gravitational field. We discuss this further in the following section.

6.4 The Sun's Quadrupole Moment and Perihelic Motion

In Sec. 6.3 we derived a formula for the perihelic motion of a planet due to the relativistic correction to Newton's law of gravitation and showed that the result is in good agreement with observation in the case of Mercury. We used this fact to strengthen our confidence in Einstein's field equations. Thus it is very important to discuss this effect critically and evaluate all classical contributions.

We mentioned in Sec. 6.3 that attempts were made in the nineteenth century to explain the advance of the perihelion of Mercury by hypothesizing a planet Vulcan between Mercury and the sun, whose influence would cause the discrepancy between the predictions of classical celestial mechanics and the actual motion. This explanation was given up since the planet Vulcan was never observed. However, a very similar explanation might be given for at least part of the effect, namely, that a very small flattening of the solar sphere into an ellipsoid would also lead to perihelic shifts. The perturbation on planetary orbits by a surplus mass located in a ring around the equator of the sun would cause perturbations similar to those of a small planet just grazing the surface of the sun. Such a solar flattening has been measured by Dicke and Goldenberg (Dicke and Goldenberg, 1967), who find a flattening of 5.0 ± 0.4 parts in 10^5. It is somewhat difficult to reconcile this value with the rotation rate of the surface of the sun; it is necessary to make the awkward assumption that the interior of the sun rotates faster than the surface. This leads to some doubt whether the mass distribution of the sun is flattened by the same amount as the visual sphere. If one assumes that it is, however, this measurement indicates that the resultant quadrupole moment contributes about 3.4″ per century to the perihelic shift of Mercury, so that relativity theory and observation would differ by about 8 per cent for this crucial test. However, the measurements of H. Hill (1974) yield a flattening of only 1.0 ± 0.7 parts in 10^5, a result fully consistent with a uniformly rotating sun. It is clearly of interest to study this question further and to evaluate the quadrupole-induced perihelic shift and the relativistic perihelic shift for all the planets.

We shall first calculate the potential created by a sphere which is widened by a bulge around its equator. This potential will depend only on the distance r from the center of the sphere and the azimuthal angle θ from the polar axis. It has to be unchanged if we go from any point in space to its mirror image in the equatorial plane. Hence, if we develop the potential $f(r,\theta)$ in spherical harmonics, the leading two terms will be of the form

$$(6.125) \qquad f(r,\theta) = -\frac{\kappa M}{r}\left[1 + D\,\frac{3\cos^2\theta - 1}{r^2}\right] + O\left(\frac{1}{r^4}\right)$$

That is, the usual potential of a sphere has been corrected by the addition of a quadrupole term. The factor D can easily be calculated on the basis of the details of the deformation, but its precise value is not important for our discussion. Since we assume that the deformation from the spherical form is small and will use the potential relatively far from the center of the sun, we shall neglect the terms $O(1/r^4)$. Since all planets near the sun lie essentially in the plane of the ecliptic where $\theta = \pi/2$, the potential $f(r,\theta)$ acts on them as if it has the purely radial dependence

$$(6.126) \qquad f(r) = -\frac{\kappa M}{r} - \frac{B}{r^3}$$

The motion of the planets lies in the equatorial plane and may be described by the relation $r = r(\varphi)$. We have, by Binet's formula (6.99), the following differential equation for the function $u(\varphi) = r(\varphi)^{-1}$ in the field of force (6.126):

$$(6.127) \qquad u'' + u = \frac{1}{u^2H^2} f'(r) = \frac{1}{H^2}(\kappa M + 3Bu^2) = A + \frac{\epsilon}{A}u^2$$

where $H = r^2(d\varphi/dt)$, and we have defined parameters

$$(6.128) \qquad A = \frac{\kappa M}{H^2} \qquad \epsilon = \frac{3\kappa M}{H^4}B$$

The dimensionless quantity ϵ may be considered as small since the factor B depends on the deformation of the sphere and is very small. We can therefore use the result of the perturbation theory applied to the formally identical problem (6.107). We found in (6.124) the corresponding perihelic shift per revolution $\delta\varphi = 2\pi\epsilon$, which becomes in the present problem, by virtue of (6.128),

$$(6.129) \qquad \delta\varphi = \frac{6\pi\kappa M}{H^4}B$$

Astronomers observe the shift in the perihelic motion after many revolutions of the planet. They express, therefore, the perihelic shift per century by the formula

$$(6.130) \qquad S = \frac{\delta\varphi}{T} = \frac{6\pi\kappa M}{H^4 T}B$$

where T is the period of revolution expressed in units of centuries. We may also relate the constant H of areal velocity to the period T. If r is the mean distance of the planet from the sun, we may integrate the relation

$$r^2 \frac{d\varphi}{dt} = H \tag{6.131}$$

over one revolution and obtain the approximate relation

$$2\pi r^2 = HT \tag{6.132}$$

(Note that this is not valid for some of the minor planets with a large eccentricity, e.g., Icarus.) Thus (6.130) becomes

$$S = 6\pi\kappa MB \frac{T^3}{(2\pi)^4 r^8} \tag{6.133}$$

Finally, we make use of Kepler's third law of planetary motion, which asserts that

$$\frac{T^2}{r^3} = C \tag{6.134}$$

has the same value for all planets. Hence (6.133) reduces to

$$S = \left(\frac{3\kappa MBC^{3/2}}{(2\pi)^3}\right) r^{-7/2} \tag{6.135}$$

Since only r varies from planet to planet, we recognize that the quadrupole-induced perihelic shift for different planets would vary as the $-\frac{7}{2}$ power of their distance from the sun.

Let us next compare the result with the prediction of the general relativity theory. We have the following formula from Sec. 6.3:

$$\delta\varphi = 2\pi \left(\frac{3\kappa^2 M^2}{c^2 H^2}\right) \tag{6.136}$$

from which follows the perihelic shift per century,

$$S = \frac{6\pi\kappa^2 M^2}{c^2} \frac{1}{H^2 T} \tag{6.137}$$

in which the period T of the planet is expressed in centuries. We derive, by use of (6.132) and (6.134),

$$S = \frac{3\kappa^2 M^2 C^{1/2}}{2\pi c^2} r^{-5/2} \tag{6.138}$$

The perihelic shift as predicted by Einstein's formula thus varies as the $-\frac{5}{2}$ power of the distance of the planet from the sun.

It is very important to note that the r dependence of the relativistic effect is different from that of the quadrupole effect. Thus, in principle, it is possible to evaluate both contributions. For example, since both effects are small perturbations to the usual classical orbit, the total effect may be written as a linear sum

$$S = \lambda \frac{3\kappa^2 M^2 C^{1/2}}{2\pi c^2} r^{-5/2} + \frac{3\kappa MBC^{3/2}}{(2\pi)^3} r^{-7/2} \tag{6.139}$$

where according to relativity theory $\lambda = 1$. Thus it is necessary to measure the shift of two or more planets to obtain an observational value for λ and B; one would thus test relativity theory as well as measure the solar quadrupole moment.

The values of the perihelic shifts as presently known and as calculated from relativity theory are given in Table 6.1. The quadrupole-moment correction suggested by Dicke is not included. The separation of relativistic and quadrupole effects is not feasible at present, but much more precise measurements should be possible with the use of planetary radar reflection (Shapiro et al., 1971). Such measurements should soon provide much more accurate values of the shift for the inner planets.

TABLE 6.1

	Distance r from sun, $\times 10^9$ m	Shift S, seconds of arc/century	
		Calculated	Observed
Mercury†	58	43.03	43.11 ± 0.45
Venus†	108	8.6	8.4 ± 4.8
Earth†	149	3.8	5.0 ± 1.2
Icarus‡	161	10.3	9.8 ± 0.8

† Data from Duncombe (1956).
‡ Data from Shapiro et al. (1971).

6.5 The Trajectory of a Light Ray in a Schwarzschild Field

In this section we shall treat a second interesting case of motion in the sun's gravitational field, the trajectory of a light ray. This problem is particularly interesting because, as with Mercury's perihelion shift, the predictions can be subjected to observational test within the solar system.

In order to treat this problem we need to make two assumptions about the propagation of light rays in a Riemann space: (1) As with the case of a massive test particle, we assume that the trajectory is a geodesic line in a four-dimensional space. (2) In special relativity the path of a light ray (which lieson the light cone) is characterized in space-time by its null line element, $ds^2 = 0$. We assume that the same is true in general relativity. Thus, in short, the light-ray trajectories are null-geodesic lines.

When discussing null geodesics we must observe that the curve parameter s which we have been using until now is no longer admissible since $s=0$ holds on null geodesics. We have to return to the original concept of parallel displacement, i.e., to ask that a null vector dx^α/dq be parallel-displaced in terms of the arbitrary parameter q according to the general law

$$(6.140) \qquad \frac{d}{dq}\left(\frac{dx^\alpha}{dq}\right) + \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \frac{dx^\beta}{dq}\frac{dx^\gamma}{dq} = 0$$

By the general theory this vector will preserve its length; that is, it will remain a null vector. It is easy to see that the above differential equations for the null geodesic are equivalent to the variational problem

$$(6.141) \qquad \delta \int g_{\alpha\beta}\frac{dx^\alpha}{dq}\frac{dx^\beta}{dq}\,dq = 0$$

The parameter q belongs to the family of distinguished parameters discussed in Sec. 2.3. Recall that all parameters of this family are linearly related. In the case of the Schwarzschild metric, we find the equations of motion for φ and t as before [(6.80) and (6.81)]:

$$(6.142) \qquad r^2\dot{\varphi} = \tilde{h} = \text{const} \qquad \left(1 - \frac{2m}{r}\right)\dot{t} = \tilde{l} = \text{const}$$

The dots now denote differentiation with respect to q, and we have assumed as before that $\theta = \pi/2$. Instead of Eq. (6.82), however, we now obtain, since $ds^2 = 0$,

$$(6.143) \qquad 0 = \left(1 - \frac{2m}{r}\right)^{-1} c^2\tilde{l}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - \frac{\tilde{h}^2}{r^2}$$

Thus, proceeding as before with the substitutions $u(\varphi) = 1/r(\varphi)$, we obtain

$$(6.144) \qquad 0 = c^2\tilde{l}^2 - \tilde{h}^2u'^2 - \tilde{h}^2u^2(1 - 2mu)$$

and by differentiation with respect to φ,

$$(6.145) \qquad u'(u'' + u - 3mu^2) = 0$$

Temporarily discarding the special solution u = const, we finally arrive at the equation for a light-ray trajectory

$$(6.146) \qquad u'' + u = 3mu^2$$

Before continuing with (6.146), let us return for a moment to the formal solution u = const, which entered the theory through differentiation of (6.144). This solution would describe light rays circling the attracting center at a fixed distance $r = r_0$. Such singular solutions occurred also in the theory of planetary motion, and in that theory they have physical reality. The situation in the present case is different. Observe that the general equation (6.93) admits $u = u_0$ as a solution for an appropriate choice of the initial angular momentum. However, $u = u_0$ is a solution of the light-ray equation (6.146) only if $u_0^{-1} = r_0 = 3m$. Hence the singular solutions of $u' = 0$ cannot be changed continuously into solutions of the more general equation (6.146) except at $r_0 = 3m$. Thus these solutions are in general unstable. Indeed, the general equation for light rays should be of second order so that rays through every point and in every direction are possible. This condition is fulfilled by (6.146), but not by $u' = 0$.

It is interesting to note that (6.146) can also be deduced from (6.93) by intuitive reasoning. Equation (6.93) describes the orbit or trajectory of a particle in the Schwarzschild field:

$$(6.147) \qquad u'' + u = \frac{m}{h^2} + 3mu^2$$

Using the expression for m given by (6.72) and the (exact) expression for m/h^2 given by (6.102), we can write this as

$$(6.148) \qquad u'' + u = \frac{\kappa M}{c^2r^4}\left(\frac{ds}{d\varphi}\right)^2 + 3\,\frac{\kappa M u^2}{c^2}$$

This equation for the geodesics follows directly from the variational problem (6.74) and involves no approximation. In order to specialize this to the case of a light ray, we must additionally set $ds^2 = 0$. Since the angular interval $d\varphi$ will in general be nonzero as the light ray sweeps by the sun, we conclude that, for the limiting case of a null geodesic,

$$\frac{m}{h^2} = \frac{\kappa M}{c^2 r^4}\left(\frac{ds}{d\varphi}\right)^2 = 0 \tag{6.149}$$

It follows that the equation of the trajectory is, in agreement with our preceding derivation,

$$u'' + u = 3mu^2 \qquad \text{(null geodesic)} \tag{6.150}$$

As with the orbit equation of Sec. 6.3, we can show that the term $3mu^2$ is small relative to the other terms of the equation. To do this, form the ratio of $3mu^2$ to the term u; that is, consider $3mu$. Using the definition of the Schwarzschild radius $r_s = 2m$ (Sec. 6.1), we may also write this ratio as $\frac{3}{2}(r_s/r)$. As we mentioned in Sec. 6.1, the Schwarzschild radius of the sun is of the order of a kilometer; thus, for a trajectory outside the sun's surface, the above ratio is evidently very small. This allows us to regard $3mu^2$ as a small perturbation term in Eq. (6.150). Accordingly, let us call

$$3m = \epsilon \tag{6.151}$$

and write the equation of the light-ray trajectory as

$$u'' + u = \epsilon u^2 \tag{6.152}$$

As in Sec. 6.3, we shall use a standard perturbation approach to treat the above equation; we suppose a solution to (6.152) of the form

$$u = u_0 + \epsilon v + O(\epsilon^2) \qquad \epsilon = 3m \tag{6.153}$$

Substituting this in (6.152), we obtain

$$u_0'' + u_0 + \epsilon v'' + \epsilon v = \epsilon u_0^2 + O(\epsilon^2) \tag{6.154}$$

Equating the zeroth-order terms in ϵ, we have

$$u_0'' + u_0 = 0 \tag{6.155}$$

This has the solution (see Fig. 6.1)

$$u_0 = A \sin(\varphi - \varphi_0) \tag{6.156}$$

which, by an appropriate orientation of axes, may be written without the arbitrary constant φ_0:

$$u_0 = A \sin \varphi \tag{6.157}$$

In terms of the first-order radius $r = 1/u_0$, this becomes

$$r \sin \varphi = \frac{1}{A} \tag{6.158}$$

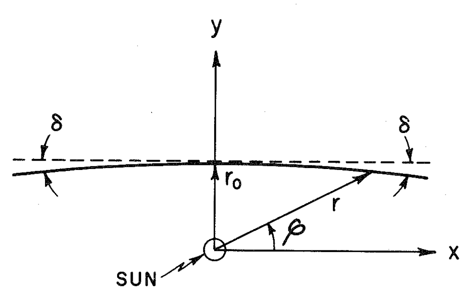
Since $r \sin \varphi$ is simply the Cartesian coordinate y, this evidently represents a straight line parallel to the x axis. This is indeed precisely what we should expect: in first approximation the light ray is not deflected at all by the sun's gravitational field. From Eq. (6.158) it is clear that the distance of closest approach to the origin (the sun) is $1/A$, so we shall call this constant r_0 and write the zeroth-order solution as

$$u_0 = \frac{1}{r_0} \sin \varphi \tag{6.159}$$

Next, equating the first-order ϵ terms of (6.154), we obtain

$$v'' + v = u_0^2 = \frac{1}{r_0^2}\sin^2\varphi = \frac{1}{2r_0^2}(1 - \cos 2\varphi) \tag{6.160}$$

Fig. 6.1
Deflection of light by the sun. The dotted line is the undeflected path $r \sin \varphi = r_0$, and the solid line is the deflected path. δ is the angle between the undeflected path and the asymptote to the deflected path.



To solve this we use a trial solution with unknown coefficients:

(6.161)
$$v = \alpha + \beta \cos 2\varphi$$

Differentiation gives

(6.162)
$$v'' = -4\beta \cos 2\varphi$$

so that

(6.163)
$$v'' + v = \alpha - 3\beta \cos 2\varphi$$

Comparing this term by term with (6.160), we see that (6.161) will be a solution if

(6.164)
$$\alpha = \frac{1}{2r_0^2} \qquad \beta = \frac{1}{6r_0^2}$$

Thus a solution of the differential equation (6.160) is

(6.165)
$$v = \frac{1}{2r_0^2} + \frac{1}{6r_0^2} \cos 2\varphi$$

Using this and the zeroth-order solution (6.159), we have the full first-order solution to the trajectory equation (6.152):

(6.166)
$$u = \frac{1}{r_0} \sin \varphi + \frac{\epsilon}{2r_0^2} (1 + \tfrac{1}{3} \cos 2\varphi)$$

As we have seen above, the trajectory of a light ray as given by (6.166) is essentially a straight line $[u = (1/r_0) \sin \varphi]$ with a perturbation of order ϵ. The effect of this perturbation will alter the trajectory to produce a small overall deflection; that is, light approaches the sun along an asymptotic straight line, is deflected by the gravitational field, and recedes again on another asymptotic straight line. The total deflection can be measured observationally for the case of starlight grazing the sun and arriving finally on the earth. Let us therefore see what total deflection is predicted by (6.166) for such a situation.

The asymptotes of the trajectory will clearly correspond to those values of the angle φ for which r becomes infinite or (equivalently) u becomes zero in (6.166). These asymptotes are nearly parallel to the x axis and correspond to φ being close to zero or π. Thus considering the asymptote near $\varphi = 0$ first and calling δ the small angle between it and the x axis, we approximate $\sin \varphi$ by δ and $\cos 2\varphi$ by 1. Then, setting $u = 0$ in (6.166), we obtain

(6.167)
$$0 = \frac{1}{r_0} \delta + \frac{4}{3} \frac{\epsilon}{2r_0^2}$$

or

(6.168)
$$\delta = -\frac{2\epsilon}{3r_0} = -\frac{2m}{r_0}$$

The minus sign indicates the light ray is bent inward by the sun. A similar procedure for the other asymptote, for which φ is taken to be $\pi - \delta$, yields the same value, $\delta = -2m/r_0$. Thus the total deflection of the light ray, the angle between the asymptotes, is

(6.169)
$$\Delta = \frac{4m}{r_0} = \frac{4\kappa M}{c^2 r_0}$$

For a light ray which just grazes the sun, Eq. (6.169) predicts a deflection of 1.75″. The early attempts to compare this prediction with observational data utilized photographs taken during solar eclipses. The positions of stellar images near the sun during an eclipse were compared with the positions 6 months later, with the sun no longer in the field of view. This procedure is inherently difficult since very small displacements of the images have to be measured. As a result the observational results obtained have ranged from 1.5″ to nearly 3″ (von Klüber, 1960).

With the advent of large radio telescopes and the discovery of the pointlike sources of intense radio emission called *quasars* the deflection can now be measured using long-base-line interferometric techniques when such a source passes near the sun. Measurements range from 1.57 to 1.82″, each with an accuracy of about 0.2″. It should be possible in time to reduce this error to about 0.01″ and obtain an extremely accurate test of the theory (Sramek, 1971).

6.6 Travel Time of Light in a Schwarzschild Field

Another interesting problem concerning the behavior of light in a Schwarzschild field is the question of travel time between two given points. Because space-time is curved in the presence of a gravitational field, this travel time is greater than it would be in flat space, and the difference can be tested experimentally.

We can easily calculate the time delay. It is simple to show that the curvature of the path, as discussed in the preceding section, makes a negligible contribution to the time delay when one considers a ray of light traveling between the earth and another planet. Thus we can approximate the path by a straight line, which we choose to be parallel to the x axis; $r \sin \varphi = r_0$, $\theta = \pi/2$. Clearly r_0 is the distance of closest approach to the sun (see Fig. 6.1). The relationship between the time and space coordinates along the world-line of a light ray is given by setting the Schwarzschild line element to zero:

$$0 = \left(1 - \frac{2m}{r}\right) c^2\, dt^2 - \frac{dr^2}{(1 - 2m/r)} - r^2\, d\varphi^2 \tag{6.170}$$

From the equation of the path of the ray we can reexpress $r^2\, d\varphi^2$ in terms of r and dr, so that we obtain

$$\begin{aligned} c^2\, dt^2 &= \frac{dr^2}{(1 - 2m/r)^2} + \frac{r_0^2\, dr^2}{(1 - 2m/r)(r^2 - r_0^2)} \\ &= \frac{dr^2\,(1 - 2mr_0^2/r^3)}{(1 - r_0^2/r^2)(1 - 2m/r)^2} \end{aligned} \tag{6.171}$$

We now take the square root of this, expand to obtain $c\,dt$ to first order in m,

$$c\, dt = \frac{dr}{\sqrt{1 - r_0^2/r^2}}\left(1 + \frac{2m}{r} - \frac{mr_0^2}{r^3}\right) \tag{6.172a}$$

then integrate

$$\begin{aligned} ct &= \left(\sqrt{r_p^2 - r_0^2} + \sqrt{r_e^2 - r_0^2}\right) \\ &\quad + 2m \log \frac{\left(\sqrt{r_p^2 - r_0^2} + r_p\right)\left(\sqrt{r_e^2 - r_0^2} + r_e\right)}{r_0^2} \\ &\quad - m\left(\frac{\sqrt{r_p^2 - r_0^2}}{r_p} + \frac{\sqrt{r_e^2 - r_0^2}}{r_e}\right) \end{aligned} \tag{6.172b}$$

The integration is taken from $r = r_0$ to r_p, the planet radius, and from $r = r_0$ to r_e, the earth radius. It is evident that the first term above is the flat-space result for the earth-planet distance, while the other two terms represent an effective increase in the distance. For the solar system we may regard r as a very reasonable radial coordinate and t as an approximate physical time. Note that, as may be expected, the main contribution to the increase in travel time comes from the part of the trajectory closest to the sun, i.e., for small values of r as is evidenced in the terms proportional to m in the integrand (6.172a).

The experimental verification of the delay has been carried out by sending pulsed radar signals from the earth to Venus and Mercury and timing the echoes as the positions of earth and the planet change relative to the sun. For Venus near superior conjunction the measured delay amounts to about 200 μs. The measurements are within 5 per cent of the calculated delays. These measurements constitute the first entirely new test of general relativity in over 50 years (Shapiro, 1972).

6.7 Null Geodesics and Fermat's Principle

We may obtain an interesting interpretation of our principle that light rays travel along null geodesics in space-time and connect it with a well-known theorem of classical optics. We assume a line element that is time-independent or, in invariant language, stationary. The spatial coordinates will be denoted by x^i and the time coordinate by t. The path of a light ray is then characterized by the following two conditions for a null geodesic:

$$\delta \int \mathfrak{L}(x^i, \dot{x}^i, \dot{t})\, dq = 0 \tag{6.173}$$

and

$$ds^2 = A^2 c^2\, dt^2 + g_{ik}\, dx^i\, dx^k = 0 \tag{6.174a}$$

$$\mathfrak{L}(x^i, \dot{x}^i, \dot{t}) = A^2 c^2 \left(\frac{dt}{dq}\right)^2 + g_{ik} \frac{dx^i}{dq} \frac{dx^k}{dq} = 0 \tag{6.174b}$$

where a dot denotes differentiation with respect to q.

We wish now to compare the integral

$$J = \int \mathfrak{L}(x^i, \dot{x}^i, \dot{t})\, dq \tag{6.175}$$

along the actual light trajectory with the same integral taken over an arbitrary trajectory in space-time which is near the null geodesic, has the same endpoints P_1 and P_2 in the three-space, and satisfies the condition of light velocity $\mathfrak{L}(x^i, \dot{x}^i, \dot{t}) = 0$. There are many nonstationary curves between P_1 and P_2 for which the condition (6.174) is fulfilled. They may start at different moments t_1 at P_1 and end at different moments t_2 at P_2.

On the one hand, it is evident that the integral (6.175) is zero for all competing trajectories since its integrand is identically zero. On the