Is the real world, as a part of complex reality?

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Abstract : By constructing the action of the Poincaré group on the dual of his Lie algebra, Souriau revealed, in 1970, the quantities of physics as objects of pure geometry. A first draft of a complex extension is presented

Introduction :

The approach of mathematician Jean-Marie Souriau [1] provided his own path through the difficult process of systematically geometrizing physics, by applying the tools of symplectic geometry. At the end of his work, he established a link with quantum mechanics through its geometric quantization. We propose to attempt to follow this path through a complex extension. Before that, we present a description of his construction of the classical elements of physics in a form more accessible to the widest possible audience. At the end of this article, the reader will find a first draft of the complex extension, whose sole merit is to show that, as soon as one undertakes this type of approach, new elements naturally appear in the formalism. These should not be interpreted as established results, but rather as indications, signals suggesting that this path deserves to be explored further, which will be the subject of the following article.

This article is aimed at engineers, students, and teachers in scientific disciplines. Its ambition is simple and explicit: to show that by using familiar tools such as matrix and differential calculus, it is possible to reach very high levels of understanding of modern physics.

The reader does not need prior mastery of differential geometry or notoriously difficult abstract formalisms. The starting point is deliberately elementary: matrices, linear transformations, invariants. Step by step, these tools lead to a deep geometric structure where the fundamental quantities of physics—energy, momentum, angular momentum, spin—naturally appear; that is to say, precisely the objects that experiment allows us to measure.

Some technical steps will require the reader to provisionally accept results without immediately having all the justifications. But the essential point lies elsewhere: at the end of the process, we discover that these physical quantities are not added from outside the model. They are already present, inscribed in the initial, very simple geometric structure.

In 1970, Jean-Marie Souriau built what can be considered a veritable cathedral of geometric physics. The aim of this work is not to present its complete architecture, but to offer a staircase of access. By following it, the reader can gradually ascend to a viewpoint where the overall

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structure becomes visible, and where one can perceive how geometry organizes and unifies seemingly disparate physical concepts.

This text is therefore intended as an invitation: to observe, through the experience of calculation, that simple tools, used rigorously, allow access to the upper platform, at the top of the cathedral, and to enjoy the view. The reader will be tempted to say to themselves, "*I didn't think it was possible to climb so high, to understand so much, with such simple tools.*" The reward for this effort is a renewed vision of physics, where geometry reveals the hidden framework within which the laws of motion operate.

1 – The construction of Euclid's group.

3D Euclidean space is defined by its metric:

(1) $$ds^2 = dx^2 + dy^2 + dz^2$$

We choose to introduce the column vector:

(2) $$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

And its transpose, the line vector:

(3) $$X^t = \begin{pmatrix} x & y & z \end{pmatrix}$$

This allows us to translate the expression of the metric (1) into the matrix form of a scalar product, using row-column multiplication:

(4) $$< X , X > = X^t X$$

We will now search for a group of matrices a that preserve this length. We have:

(5) $$X' = a X$$

Which means :

(6) $$< X' , X' > = < X , X >$$

First consider the translations :

(7) $$X' = X + \Delta X = X + C \text{ with } \Delta X = C = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Consider two points *A* and *B* in space. Translations preserve the length of the segment $X = X_B - X_A$ simply because:

(8) $$X' = {X_B}' - {X_A}' = X_B + \Delta X - X_A - \Delta X = X_B - X_A = X$$

Therefore, (6) is automatically satisfied. We will represent this "translation" action in the form of this matrix with format (4,4):

(9)

$$X' = TX = \begin{pmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + \Delta x \\ y + \Delta x \\ z + \Delta x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} X + C \\ 1 \end{pmatrix}$$

We have just constructed, in matrix form, the group of 3D translations. Let us take another matrix a, and express that it preserves the length:

$$X^{t'}X' = X^tX \tag{10}$$

$$(a\,X)^t\,(a\,X) = X^t\,X \tag{11}$$

But :

$$(AB)^t = B^tA^t \tag{12}$$

Therefore, relation (11) is satisfied if the matrices a obey the relation :

$$a^t\,a = I \;\; (\,identity\ matrix\,) \tag{13}$$

This implies that the inverses a^{-1} of matrices a have their transposes ${}^{t}a$. The determinant of these matrices is ±1. When it is +1, they are the matrices of 3D rotations. Since this set contains the identity matrix, it is a "special orthogonal" subgroup denoted by SO(3). When the set consists of elements whose matrices are -1, its elements associate a rotation with a P-symmetry, a "right-left" inversion. The union of the two forms the "three-dimensional orthogonal" group O(3). We will opt for another matrix representation of this group, using (4,4) matrices:

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

By combining these two elements, we obtain the three-dimensional Euclid group:

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & C \\ 0 & 1 \end{pmatrix} \tag{15}$$

The Euclid group is the isometry group of Euclidean space, defined by (1).

2 – **Construction of the Poincaré group**:

Let us consider the metric of space introduced in 1909 by Minkowski [2], whose points are located using the four coordinates$(\,x_0\,, x_1\,, x_2\,, x_3\,)$ where $(\,x_1\,, x_2\,, x_3)$ are the space coordinates and x_0 the time coordinate, the chronological coordinate.

$$ds^2 = {dx_0}^2 - {dx_1}^2 - {dx_2}^2 - {dx_3}^2 \tag{16}$$

We will opt for the representation of quadratic forms introduced by the Dane Jørge Gram in 1883 [3]. We will use a Gram matrix for this:

(17)
$$\begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & \pm 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \pm 1 \end{pmatrix}$$

In this case, that of special relativity, presented by Einstein in 1915 [4] is:

(18)
$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Which is equivalent to a signature:

(19)
$$(+ \; - \; - \; -)$$

The metric then takes the form:

(20)
$$ds^2 = dX^t \, G \, dX$$

We can immediately see that in Euclidean space the Gram matrix is simply the identity matrix:

(21)
$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We will similarly search for the matrices that will constitute the isometry group of the Minkowski space. We first find the subgroup of spacetime translations, which we will represent in the form of (5,5) matrices:

(22)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & \Delta x_0 \\ 0 & 1 & 0 & 0 & \Delta x_1 \\ 0 & 0 & 1 & 0 & \Delta x_2 \\ 0 & 0 & 0 & 1 & \Delta x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}$$

We will proceed in the same way by searching for matrices L which, acting on the points X of 4D space, preserve length, that is to say such that:

(23)
$$X'^t \, G \, X' = (LX)^t \, G \, L \, X = X^t \, L^t \, G \, L \, X = X^t \, G \, X$$

This will give us the axiomatic definition of the Lorentz group:

(24)
$$L^t \, G \, L = G$$

It is thus understood that the complete axiomatic definition of the orthogonal group O(3) is in fact:

(25)
$$a^t \, I \, a = I$$

In 1905, Henri Poincaré identified a group structure in the transformation introduced by Lorentz in 1904 [5]. He included spacetime translations in the process [6], which gave the isometry group of Minkowski space, which would henceforth be called the Poincaré group.

(26) $$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

We see that the Lorentz group represents 4D translations in a Minkowski space, which some interpreted as four-dimensional "Euclidean" rotations, where one of the dimensions, that of time, is then purely imaginary.

3 - **Minkowski space geodesics.**

Geodesics are the shortest paths, therefore falling under the variational approach:

(26)

$$\delta \int \sqrt{{dx_0}^2 - {dx_1}^2 - {dx_2}^2 - {dx_3}^2} = \delta \int \sqrt{\left(\frac{dx_0}{dp}\right)^2 - \left(\frac{dx_1}{dp}\right)^2 - \left(\frac{dx_2}{dp}\right)^2 - \left(\frac{dx_3}{dp}\right)^2}\ dp = 0$$

Or :

(27) $$\delta \int \sqrt{{\dot{x}_0}^2 - {\dot{x}_1}^2 - {\dot{x}_2}^2 - {\dot{x}_3}^2}\ dp = 0$$

We are therefore in a situation that leads to the Lagrange equations, with the Lagrangian:

(28) $$L = \sqrt{{\dot{x}_0}^2 - {\dot{x}_1}^2 - {\dot{x}_2}^2 - {\dot{x}_3}^2}$$

And Lagrange equations :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

But we saw [7,8] that if the derivatives were taken with respect to the length s, we also obtained the geodesics with the Lagrangian:

(29) $$L' = L^2 = {\dot{x}_0}^2 - {\dot{x}_1}^2 - {\dot{x}_2}^2 - {\dot{x}_3}^2$$

The Lagrange equations then give:

(30) $$\ddot{x}_0 = \ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = 0$$

Geodesics are lines in spacetime :

(31) $$x_0 = a_0 s + b_0$$
(32) $$x_1 = a_1 s + b_1$$
(33) $$x_2 = a_2 s + b_{21}$$
(34) $$x_3 = a_3 s + b_3$$

However, in shifting from the L -Lagrangian function to the L^2 -Lagrangian function , the Lagrange equations reveal real curves, see equations (31° to (34), but no longer satisfying the condition of non-negativity of the quantity under the radical. These real lines are then equipped with an imaginary length. They can be called virtual geodesics. They do not belong to the solution hypersurface. Light follows geodesics of zero length. From (16), by setting the condition of non-negativity of the length s or the proper time, and setting s = $c\tau$ and $x_0 = ct$, we obtain:

$$c^2 \geq \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2 = v^2 \tag{35}$$

The speed cannot exceed the value c.

4 – Connected components of the isometry group.

We have seen that the orthogonal group has two components, called connected components. These are components that preserve the orientation of objects and components that include a P-symmetry, which reverses this orientation. We can perform the same analysis with the Lorentz group. In Minkowski space, we will therefore seek to identify components involving time inversion (T-symmetry) and space inversion (P-symmetry). The Lorentz group contains the identity element, which reverses neither time nor space:

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

Among all the elements of this group of matrices, we will group together all those which, like the identity element, do not invert time or space. They will form the identity subgroup $\{ L_n \}$ Among these elements, we will find an infinite number of elements that invert space, such as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{37}$$

We will group them into a set $\{ L_s \}$. We will also have the elements that reverse time, but not space, such as:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{38}$$

We will group them into a set $\{ L_t \}$. Finally, there will be the elements that invert both space and time. Among these:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{39}$$

Elements that will constitute the set $\{ L_{st} \}$. Thus, the complete group will be the union of its four connected components:

$$\{ L \} = \{ L_n \} \cup \{ L_s \} \cup \{ L_t \} \cup \{ L_{st} \}. \tag{40}$$

The set:

$$\{ L_o \} = \{ L_n \} \cup \{ L_s \} \tag{41}$$

Forms the orthochronous subgroup, or restricted Lorentz group. The set of time-reversing elements will form the orthochronous subset:

$$\{ L_a \} = \{ L_t \} \cup \{ L_{st} \}. \tag{42}$$

From the orthochronous components, we can form the restricted Poincaré group:

$$\begin{pmatrix} L_o & C \\ 0 & 1 \end{pmatrix} \tag{43}$$

Until 1970, until the mathematician Jean-Marie Souriau [1] explained the function of these antichronous components of the group, the use of the Poincaré group was exclusively limited to the restricted group, equipped with its orthochronous components. Before addressing the central theme of this article, we will use the axiomatic definition of the group to construct its inverse, which we will need later. To do this, we start from (23) and multiply by G^{-1} on both sides. We obtain.

$$G^{-1} L^t \, G \, L = I \tag{44}$$

Taking into account that $G^{-1} = G$, we get:

$$L^{-1} = G \, L^t G \tag{45}$$

We will now move on to the main body of the article:

5 – The two ways a group acts. Adjoint action.

It is easier, mentally, to refer to the Euclid group, which encompasses the operations of translation, rotation, and symmetry. The initial idea is that a group "transports." We have a point X in space, and an element a of the group. In the case of the Euclid group, we consider a segment AB. By acting on this segment, we transport it to another point in space by performing a translation.

$$\Delta X = (\Delta x, \Delta y, \Delta z) \tag{46}$$

And a rotation :

$$\Omega = \{ \omega_1, \omega_2, \omega_{21} \} \ \text{(trois angles)} \tag{47}$$

At a point X, let a be the element of the group. This depends on the point, that is, the coordinates used to locate X, and on the six parameters defining the action of this group, the quantities ΔX and Ω. Let's summarize this action from the perspective of "transport":

The group, acting on the segment AB, transports it to $X' = X + \Delta X$ by rotating it by angles $\Omega = \{ \omega_1, \omega_2, \omega_{21} \}$. This gives the parameters of the segment A'B' (this operation preserving the length of this segment $\overline{AB}$, or “$\overrightarrow{AB}$”).

But we can consider it differently and say that these components of $\overrightarrow{A' \ B'}$ reflect how this vector is perceived from this new point of view $X' = X + \Delta X$ And this is the very essence of special relativity. Let's limit the Poincaré group to the action of its subgroup, the Lorentz group L. By applying an element of this group to a "motion," represented by a segment connecting two points *X* and *R* in four-dimensional Minkowski spacetime, we "see it from another angle" (the action of the Lorentz group being equivalent to this 4D rotation). We "change our frame of reference," and we observe that something is conserved (the quantity $\Delta s = s_B - s_A$)). In the particular case of photons, when they travel from a point *A* to a point *B* in Minkowski spacetime, the quantity Δs is zero. By observing this motion of the photon from another angle the quantity is conserved, therefore remains zero. And, referring to (16) this means that the measurement of the speed of these photons will also remain at c, whatever the "point of view" (the action of the Lorentz group meaning that at the same point we observe this motion using a measuring instrument moving at any speed (Michelson experiment).

Returning to Euclid's group, this means that if an observer observes a segment $\overrightarrow{AB}$ "by turning their head", or "by observing it from a different point", or "by observing it through its image in a mirror", or all three at once, the length of this segment will be the same

We can consider this vector $\overrightarrow{AB}$ as an object in Euclidean space. Now imagine that our operator changes its point of view, its frame of reference, but now observes not a vector but a matrix A_{ij}. This matrix, by maintaining more than an object, is an operator acting on objects in Euclidean space, a linear operator.

The group is also a linear operator, limited to rotations and translations. A linear operator is much more general. Let's imagine we decide to observe this matrix A from another viewpoint $\{ \Delta X , \Omega \}$. That is, "from afar," by "turning our heads," and then possibly "using a mirror." How will the components of this matrix appear to us, if we preserve its functionality, its identity?

The "identity" of a linear (matrix) operator are its determinant, its trace and ts eigenvalues. If we can conceive of a group action that preserves these three elements, then we will have what we are looking for, in particular, how a linear operator presents itself, seen from another point of view. If we consider the action:

$$a\,A\,a^{-1} \tag{48}$$

The first requirement is satisfied, since we know that the determinant of a^{-1} is equal to the inverse of the determinant of *a*.

The second follows from a theorem on matrices which tells us that the trace of a product of n matrices *Tr*(*ABC*) is equal to the trace of a product constructed using a circular permutation: *Tr*(*ABC*) = *Tr*(*CAB*), therefore:

$$Tr(a\,A\,a^{-1}) = Tr(a^{-1}a\,A) = Tr(A) \tag{49}$$

We recall the definition of the concept "eigenvectors v - eigenvalues λ ".
v is an eigenvector of the linear operator *A* if:

$$A\,v = \lambda\,v \tag{50}$$

Or :

$$(\lambda\,I - A) = 0 \tag{51}$$

This is a system of n linear equations. This system gives a non-trivial solution $v \neq 0$ if and only if:

(52) $$det(\lambda I - A) = 0$$

This gives us the polynomial equation that yields the eigenvalues. These will be preserved since the polynomial equations giving the eigenvalues are identical.

(53) $$det(\lambda I - A) = det(\lambda I - A') = det(\lambda I - a A a^{-1})$$

But we can write that:

(54) $$\lambda I = \lambda a a^{-1} = a \lambda I a^{-1}$$

Thus :

(55) $$det(\lambda I - a A a^{-1}) = det\,[a\,(\lambda I - A)a^{-1}] = det(\lambda I - A)$$

The action in question does indeed retain its inherent values. A group therefore gives us two possible actions, which we must interpret as changes in perspective:

- With respect to vectors A, that is to say the action *aA* called "regular left"

  (we call it "Right Regular Action")

- With respect to linear operators, the adjoint action (also called conjugation) is written $a\,A\,a^{-1}$

Tha action :

(56) $$a^{-1} A\, a$$

It is then only the adjoint action constructed with the element of the group a^{-1}.

6 – **Action of the group on its Lie algebra**.

We will continue to clarify elements of calculation using only matrix calculus, without delving into the details of symplectic geometry—that is, without explaining why the coadjoint action of the group on the dual of its Lie algebra reveals this second space, which Souriau calls the momentum space and which complements the Minkowski space. We have just constructed a tool: the adjoint action, acting on matrices. We will now apply this tool to a set of matrices constituting what is called the Lie algebra of the group.

The Poincaré group is a Lie group, a group of matrices. The differential δa of the group is a set of matrices (which do not form a group since it does not contain the identity element) called the Lie algebra of the group:

(57) $$\delta a = \delta \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} = \mathrm{Z} = \begin{pmatrix} \delta L & \delta C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Lambda & \gamma \\ 0 & 0 \end{pmatrix}$$

Let *G* be the Gram matrix of the Minkowski space, also called the "metric matrix". We will differentiate in the neighborhood of an element $L_{(0)}$, according to:

(58) $$L = L_{(0)} + \varepsilon\, \delta L = L_{(0)} + \Lambda$$

And we will pose:

(59) $$\Lambda = G\,\omega$$

By differentiating (19) and doing:

(60) $$L_{(0)} = I$$

That is, by differentiating in the neighborhood of the group's identity element, we show that the matrix Λ is antisymmetric:

(61) $$\Lambda^t = -\Lambda$$

We will construct the adjoint action of a^{-1} on this Lie algebra δa of the group a. That is to say, the expression:

(62) $$\mathrm{Z}' = \begin{pmatrix} G\omega' & \gamma' \\ 0 & 0 \end{pmatrix} = a^{-1}\,Z\,a$$

We need the inverse matrix a^{-1} of the one representing the Poincaré group. We find easily:

(63) $$a^{-1} = \begin{pmatrix} L^{-1} & -L^{-1}C \\ 0 & 1 \end{pmatrix}$$

By forming (62) we easily find, after having made explicit and taken into account that $GG = I$

(64) $$\omega' = L^t\,\omega\,L$$

(65) $$\gamma' = G\,L^t\omega\,C + G\,L^tG\,\gamma$$

This result represents the result of the adjoint action of the element a^{-1} of the group on the element of its Lie algebra Z' (57).

7 – Action du groupe sur le dual de son algèbre de Lie.

The aim of this article is not to explain the path (symplectic geometry) by which J-M Souriau establishes this link between geometry and physics, but rather to demonstrate to students, engineers, and teachers, who have access to matrix calculus, the simplicity, elegance, and clarity of the calculations. At this point, we must introduce duality. The element of the Lie algebra consists of an antisymmetric matrix ω, of size (4,4), therefore with six components. Let us write:

(66) $$\mathrm{Z} = \left\{ \omega_{\mathrm{sx}}, \omega_{\mathrm{sy}}, \omega_{\mathrm{sz}}, \omega_{\mathrm{fx}}, \omega_{\mathrm{fy}}, \omega_{\mathrm{fz}}, \gamma_{\mathrm{t}}, \gamma_{\mathrm{x}}, \gamma_{\mathrm{y}}, \gamma_{\mathrm{z}}, \right\}$$

It is therefore logical to imagine that the moment represents this set of ten components.:

(67) $$\mu = \left\{ \mathrm{s_x}, \mathrm{s_y}, \mathrm{s_z}, \mathrm{f_x}, \mathrm{f_y}, \mathrm{f_z}, \mathrm{E}, \mathrm{p_x}, \mathrm{p_y}, \mathrm{p_z} \right\}$$

The duality corresponding to the relationship:

(68) $$< \mathrm{Z}, \mu > = \mathrm{Cst}$$

The six components of an antisymmetric matrix *M* can be arranged in the form:

(69)
$$M = \begin{pmatrix} 0 & -s_z & s_y & f_x \\ s_z & 0 & -s_x & f_y \\ -s_y & s_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix}$$

Let's multiply this matrix by an antisymmetric matrix :

(70)
$$\begin{pmatrix} 0 & -s_z & s_y & f_x \\ s_z & 0 & -s_x & f_y \\ -s_y & s_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix}$$

I simply calculate the diagonal terms to get the trace and I obtain:

(71)
$$\begin{array}{c} s_z\,\omega_{12} - s_y\,\omega_{13} - f_x\,\omega_{14} \\ s_z\,\omega_{12} + s_x\,\omega_{23} - f_y\,\omega_{24} \\ -s_y\,\omega_{13} + s_x\,\omega_{23} - f_z\,\omega_{34} \\ -f_x\,\omega_{14} - f_y\,\omega_{24} - f_z\,\omega_{34} \end{array}$$

When I add them up, I see that I have twice what I'm looking for. So I just need to form:

(72)
$$\frac{1}{2}\mathrm{Tr}(M\omega)$$

With :

(73)
$$\mu(Z) = \frac{1}{2}\,Tr(M\omega) + P^t G\gamma$$

The combination $\{M, P\}$ of an antisymmetric matrix and a vector forms what is called a torsor. This term will take on its full meaning when we apply all of the above to the Euclid group. It is then appropriate to express the duality in terms of scalar conservation (a simple trick for handling matrix calculations):

(74)
$$\frac{1}{2}Tr(M.\omega) + P^t.G\,\gamma = \frac{1}{2}Tr(M'.\omega') + P'^t.G\,\gamma'$$

It will then suffice to substitute the quantities ω' and γ' into this equation, from (64) and (65), to construct the desired dual transformation:

(75)
$$\{M, P\} \rightarrow \{M', P'\}$$

Let's perform this replacement:

$$\frac{1}{2}Tr(M.\omega) + P^t.G\,\gamma = \frac{1}{2}Tr(M'.L^t\,\omega\,L) + P'^t.G(GL^t\omega C + GL^tG\gamma) \tag{76}$$

This equation must be satisfied for all ω and γ. Identification on γ gives, taking into account that $GG = I$:

$$P^t = P'^t\,L^t \tag{77}$$

Or :

$$P = L\,P' \tag{78}$$

The calculation giving the application $M \rightarrow M'$ has been returned in the appendix. The set represents the expression of the coadjoint action of the Poincaré group on its moment space, a result which was first given by the mathematician J-M Souriau in 1970 [1].

8 – **The Euclid group, a particular dynamic group.**

If the reader has followed all these calculations closely, constructing the action of the Euclid group on the dual of its Lie algebra—that is, on its moment—will be a useful exercise. The result is absolutely similar to what emerges from the Poincaré group:

$$P = a\,P' \tag{79}$$

$$M = a\,M'a^t + C\,P'^t a - a\,P'^t C \tag{80}$$

The formulas are the same, with the matrix of the orthogonal group O(3) replacing that of the Lorentz group. The objects P and M then acquire an elegant physical meaning. P is a force and M a torsional moment. The Euclid group is therefore a special dynamic group, that of statics. Any change of reference frame in this uniform medium preserves its "state of stress".

9 – **Significance for the physicist, the engineer, and the student.**

We started with a four-dimensional space, Minkowski spacetime, the spacetime of special relativity, defined by its metric (16).

In Section 2, we constructed its isometry group, the complete Poincaré group (26), formed from the spacetime translation subgroup (22) and the Lorentz subgroup (24).

In Section 3, we showed that the Lorentz group decomposes into four connected components: two orthochronous components (not reversing time), forming the subgroup known as the restricted Lorentz group, and two antichronous components (reversing time).

In Section 4, we considered two types of action: one acting on vectors and the other, adjoint action, acting on matrices. The fundamental idea of special relativity was to show that masses and photons correspond to different ways of moving through Minkowski space, the former along geodesics of zero length, the latter along geodesics of zero length. Here we find Souriau's favorite expression.

- *Tell me how you move, and I'll tell you what you are.*

We thus identify particles with specific motions. However, joining two distinct points A and B in spacetime with a geodesic is not enough to define the object in question. It lacks its attributes: energy, momentum, and spin. The idea is to group all these objects according to a matrix, constituting the object moment μ. We then consider a change of reference frame, which translates into a group action. The action on the left of the group preserves the length s. This action preserves the nature of the motion, the nature of the particle. We then seek to construct an action that preserves the identity of an object, which takes the form of a matrix, and we then define this operator, which is the coadjoint action. We then examine what this change of reference frame preserves: these other attributes of the motion that the action on the left does not allow us to "see." We obtain a wrench, an antisymmetric matrix, plus a four-vector P. Let's show that the coadjoint action preserves its length $P^t G\ P\ =\ (LP')^t G\ LP$ is satisfied due to the axiomatic definition of L. Therefore, the dot product $\langle\ P\ ,P\ \rangle$ is conserved. This is a property of motion. Let's introduce the scalar E and the 3-vector p by decomposing P according to:

$$P\ =\begin{pmatrix} E \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{81}$$

With c = 1 and :

$$p^2\ =\ {p_1}^2+\ {p_2}^2+\ {p_3}^2 \tag{82}$$

We get :

$$E^2-\ p^2\ =\ Cst \tag{83}$$

We can therefore change our frame of reference, "consider a motion from another angle," using an observational or measuring instrument determined by this change of frame in spacetime. This quantity will be conserved and will thus constitute a characteristic of this particular motion. Souriau thus reveals, starting solely from the space where these motions occur, defined by his metric alone, essential characteristics of the entire range of motions that can correspond to the geodesics of this spacetime. Quantities familiar to the relativistic physicist, such as energy E and momentum p, emerge from this purely geometric approach. But these two parameters are not sufficient to categorize a particular motion. Spin is missing.

The spin 3-vector is one of the two 3-vectors involved in the construction of the antisymmetric matrix M, the second being the "passage" 3-vector. Souriau shows that this third vector is merely an artifact that must be eliminated by opting for the specific coordinate system that accompanies the particle's motion. For the coadjoint action of the group on the spin, we refer the reader to the appendix; we wish to focus here on the action on the energy-momentum pair. That being said, it is quite remarkable that Souriau, in 1970, brought spin to the fore as an object of pure geometry, leading to the concept of a dynamical group. We can say that the entire approach illustrates Souriau's statement:

- *A little mathematics takes you away from reality; a lot brings you back to it.*

10 - **Physical meaning of T-symmetry.**

What does the existence of antichronous components in the complete Poincaré group mean?

Geometrically, this means that, in Minkowski space, if we start with a set of motions oriented in a past-future direction, by introducing antichronous elements, we create their T-symmetrics, the same motions, but with an inverse temporal orientation, oriented in a future-past direction. Can we then consider that they belong to the world of physics?

What impact will this have on the parameters of the movement?

Equation (78) tells us that time reversal, T-symmetry, reverses both energy and momentum (referring to the appendix we will see that it conserves spin).

Using the relation $E = mc^2$, we see that this reversal of energy is synonymous with a reversal of mass. Thus, the subsidiary question that arises is: can negative masses coexist with our positive masses in our spacetime?

In 1970, this emergence of objects with negative energy and mass was viewed with perplexity, as physicists preferred to continue limiting themselves to the restricted Poincaré group, restricted to its orthochronous components. The Janus cosmological model, on the other hand, exploited the complete group [10], involving the use of the geometric concept of a covering [11].

11-**Geometric translation of C-symmetry.**

In chapter 5 of a work published in 1964 [10], J-M Souriau adds a fifth dimension, x_5, of the spacelike type, to spacetime. He notes that the inversion of this additional dimension accounts for charge conjugation. In Chapter 5 of [1], entitled "A Method of Quantization," the link between quantization and the compactness of the additional dimension is illustrated. Article [9] is a simple outline, through the theory of dynamical groups, of the geometrization of quantum mechanics, by taking into account additional, closed dimensions. C-symmetry then follows from the inversion of the additional dimensions. By coupling this C-symmetry to P and T symmetries, through the Janus group, the Janus model then becomes the geometric translation of a CPT symmetry of the two sheets of the coating. But these additional dimensions remain real. It seems to us that a geometrization including gravitation and quantum mechanics could emerge from a fibration of spacetime using compact complex dimensions. Geometrically, if we add a single compact dimension (fibered in a circle), the symmetry can be illustrated in the figure below.

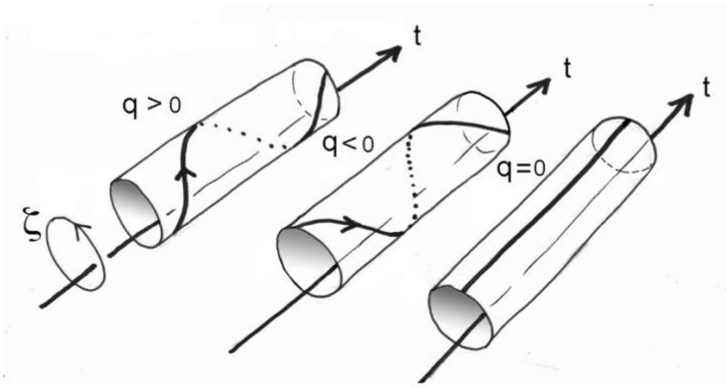

Fig.1 : Educational image illustrating C-symmetry. q is the charge

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12 - **When quantum field theory rejects a priori states of negative energy**.

In quantum field theory [12], we find P and T operations of space and time inversion. Two options then present themselves, depending on whether these operators are:

- Linear and unitary

- Antilinear and antiunitary

The imperative to avoid generating the existence of negative energy states, considered a priori impossible [12], leads this theory to the restrictive choice:

- Linear and unitary P
- Antilinear and antiunitary T

In 2014, in [13] it was suggested that the emergence of the Janus Cosmological Model calls into question this absolute principle of the a priori negation of negative energy states. The mathematician Nathalie Debergh takes up this idea, showing that with a linear and unitary operator T, such states naturally emerge from the Dirac and Schrödinger equations ([14],[15],[16]). In 2021 [17], the technique for constructing the elements of physics, based on calculating the action on the dual of its Lie algebra, was applied, extending it to the complex extension of the Poincaré group. This approach seems to us to constitute a tool for a geometrization of quantum mechanics, which remains to be done. However, this highly technical study has not garnered much attention from theoretical physicists. We will revisit this technique in the following section, attempting to make it more accessible.

13 – **Construction of the coadjoint action of the dual of the Lie algebra of the complexified Poincaré group**.

There are works in the literature that represent various attempts to translate quantum mechanics into a geometric approach. Our aim is not to be exhaustive in this regard, and interested readers can refer to the articles cited in reference [17]. What follows essentially reiterates part of that article, but with detailed calculations to make it more appealing to our target audience. It represents one way of presenting the extension of the group into complex numbers.

The real Poincaré group emerges as an isometry group of the Minkowski space, defined by its metric (16). Here we will focus on the complexified Minkowski space. In 1854, the French mathematician became interested in complex quadratic forms [18]:

$$\sum a_{ij}\ z^i \bar{z}^j \tag{84}$$

Where z^i is a complex vector, defined in the set $\mathbb{C}^4$and its conjugate (such that the imaginary part of each of these complex numbers is inverted):

$$z = a + ib \quad \longrightarrow \quad \bar{z} = a - ib \tag{85}$$

The following metric corresponds to the reduced canonical form:

$$ds^2 = z^i\ \bar{z}^i \tag{86}$$

In this case the dot product is always real and equal to the square of the norm of the vector:

$$< z\ , z > = \|z\|^2 = a^2 + b^2 \tag{87}$$

In the real case, considering matrices *M*, we introduced the transposition operation M^t, consisting of performing a symmetry with respect to the main diagonal:

$$\left(M_{ij}\right)^t = M_{ji} \tag{88}$$

The vector *X* could then be expressed in two forms. A "standard form": the column vector, and its transpose, the row vector. In reality, the Euclidean metric could then be written in the form:

$$ds^2 = dX^t\ I\ dX \tag{89}$$

Where *I* is the identity matrix (the Gram matrix of Euclidean space). We will use the adjoint of a matrix, where there is not only symmetry with respect to the main diagonal, but where these terms are also conjugate:

$$M = z_{ij} \qquad M_{ij} \ \rightarrow \ M^\dagger = \bar{z}_{ji} \tag{90}$$

This also applies to vectors. The adjoint of a column vector then becomes the row vector where the components are replaced by their conjugates. Thus, in a three-dimensional space:

$$dX = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \quad \rightarrow \quad dX^\dagger = (\ d\bar{x}_1 \quad d\bar{x}_2 \quad d) \tag{91}$$

So, the complexified Euclidean space has the (real) metric:

$$ds^2 = dX^\dagger\ I\ dX \tag{92}$$

Similarly, the complexified 4D Minkowski space $\mathbb{C}\ (1,3)$, will have the (real) metric:

$$ds^2 = dX^\dagger\, G\, dX \tag{93}$$

Where *G* is the classic Gram matrix (18). We will now search for the isometry group of this complexified Minkowski space. Let L be a complex matrix. Let's write that it preserves the length:

$$dX'^\dagger\, G\, dX' = d\,(LX^\dagger)\, G\, d(LX) = \;= dX^\dagger\, G\, dX \tag{94}$$

Earlier we saw how the transpose of a product is constructed. An analogy theorem works for the adjoint of a product:

$$(\,A\,B\,)^\dagger = B^\dagger A^\dagger \tag{95}$$

Thus, relation (94) becomes:

$$dX^\dagger L^\dagger G\, L\, dX = \;= dX^\dagger\, G\, dX \tag{96}$$

This gives us the axiomatic expression:

$$L^\dagger G\, L = G \tag{97}$$

This is nothing other than the complexified Lorentz group. By adding the complexified 4D translations, we find that the isometry group of the complexified Minkowski space is simply the complexified Poincaré group where L and C are complex. Formally, this is similar to (26).

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \tag{98}$$

We will then extend the calculations of the coadjoint action of the group on the dual of its Lie algebra. To perform this calculation, we will need to acquire new tools. In the real calculus, we saw the appearance of symmetric and antisymmetric matrices, defined by:

(99) Matrice symétrique : $M + M^t = 0$

(100) Matrice antisymétrique : $M - M^t = 0$

Here, we have :

(101) Matrice hermitienne : $M + M^\dagger = 0$

(102) Matrice anti hermitienne : $M - M^\dagger = 0$

In the real world, an antisymmetric matrix has its main diagonal composed of zeros. In the complex world, this diagonal will consist of purely imaginary numbers. Let's begin by constructing the inverse matrix of *L*:

$$L^\dagger G\, L = G \;\rightarrow\; (\,G^{-1} L^\dagger G\,)\, L = I \;\rightarrow\; L^{-1} = G^{-1} L^\dagger G \tag{103}$$

This is the same as (45), but with the transpose replaced by the adjoint. We will expand the matrix *L* around the identity element:

(104) $$L = I + \varepsilon\,\delta L$$

By introducing into (103) and neglecting the second order:

(105) $$(G\,\delta L)^t + G\,\delta L = 0$$

This means that $G\ \delta L$ is an anti-Hermitian matrix which we will denote by ω.

(106) $$G\,\delta L = \omega \quad and\, GG = I \quad \rightarrow \quad \delta L = G\,\omega$$

We will therefore write the element of the Lie algebra according to:

(107) $$Z = \begin{pmatrix} \delta L & \delta C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} G\,\omega & \gamma \\ 0 & 0 \end{pmatrix}$$

If we construct the group's action on its Lie algebra:

(108) $$Z' = a^{-1}\,Z\,a$$

Using calculations entirely similar to those performed in the real domain, we obtain the same equations, where the adjoint replaces the transpose.

(109) $$\omega' = L^{\dagger}\omega\,L$$

(110) $$\gamma' = G\,L^{\dagger}\omega\,C + G\,L^{\dagger}G\,\gamma$$

This time we will put the complex moment in the form of a pair consisting of an antihermitic matrix *M* and a complex four-vector *P*.

(111) $$\mu = \{\,M\,,P\,\}$$

An antisymmetric matrix of size (4,4) depends, in complex numbers, on 12 scalars.

The element Z of the Lie algebra of the Poincaré group belongs to a vector space of dimension 20 (the dimension of the group). We can therefore write:

(112) $$\mathrm{Z} = \left\{\, \omega_{\mathrm{sx}}\,, \omega_{\mathrm{sy}}\,, \omega_{\mathrm{sz}}\,, \omega_{\mathrm{fx}}\,, \omega_{\mathrm{fy}}\,, \omega_{\mathrm{fz}}\,, \gamma_{\mathrm{t}}\,, \gamma_{\mathrm{x}}\,, \gamma_{\mathrm{y}}\,, \gamma_{\mathrm{z}}\,, \right\}$$

It is therefore logical to imagine that the moment represents this set of ten complex components:

(113) $$\mu = \left\{\, \mathrm{s_x}\,, \mathrm{s_y}\,, \mathrm{s_z}\,, \mathrm{f_x}\,, \mathrm{f_y}\,, \mathrm{f_z}\,, \mathrm{E}\,, \mathrm{p_x}\,, \mathrm{p_y}\,, \mathrm{p_z} \,\right\}$$

The duality corresponding to the relationship:

(114) $$< \mathrm{Z}\,, \mu > = \mathrm{Cst}$$

Let us form the anti-Hermitian matrix:

(115)
$$\mathrm{M} = \begin{pmatrix} 0 & -\bar{\mathrm{s}}_z & \mathrm{s}_y & \mathrm{f}_x \\ \mathrm{s}_z & 0 & -\bar{\mathrm{s}}_x & \mathrm{f}_y \\ -\bar{\mathrm{s}}_y & \mathrm{s}_x & 0 & \mathrm{f}_z \\ -\bar{\mathrm{f}}_x & -\bar{\mathrm{f}}_y & -\bar{\mathrm{f}}_z & 0 \end{pmatrix}$$

Let's multiply this matrix by the anti-Hermitian matrix ω:

(116)
$$\begin{pmatrix} 0 & -\bar{\mathrm{s}}_z & \bar{\mathrm{s}}_y & \bar{\mathrm{f}}_x \\ \mathrm{s}_z & 0 & -\bar{\mathrm{s}}_x & \bar{\mathrm{f}}_y \\ -\mathrm{s}_y & \mathrm{s}_x & 0 & \bar{\mathrm{f}}_z \\ -\mathrm{f}_x & -\mathrm{f}_y & -\mathrm{f}_z & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \bar{\omega}_{12} & \bar{\omega}_{13} & \bar{\omega}_{14} \\ -\omega_{12} & 0 & \bar{\omega}_{23} & \bar{\omega}_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \bar{\omega}_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix}$$

Once again, keeping only the trace:

(117)
$$\begin{gathered} \bar{\mathrm{s}}_z\,\omega_{12} - \bar{\mathrm{s}}_y\,\omega_{13} - \bar{\mathrm{f}}_x\,\omega_{14} \\ \mathrm{s}_z\,\bar{\omega}_{12} + \bar{\mathrm{s}}_x\,\omega_{23} - \bar{\mathrm{f}}_y\,\omega_{24} \\ -\mathrm{s}_y\,\bar{\omega}_{13} + \mathrm{s}_x\,\bar{\omega}_{23} - \bar{\mathrm{f}}_z\,\omega_{34} \\ -\mathrm{f}_x\,\underline{\omega}_{14} - \mathrm{f}_y\,\bar{\omega}_{24} - \mathrm{f}_z\,\bar{\omega}_{34} \end{gathered}$$

These quantities are all of the form $a\bar{b} + \bar{a}b$ We obtain the desired scalar quantity by introducing: $\mu(Z) = \frac{1}{2}\,Tr(M\omega) + P^{\dagger}G\,\gamma$ où where M and γ are antihermitian. We can therefore write :

(118)
$$\mu \equiv \{M, P\} \text{ with } M^{\dagger} = -M \qquad P \in \mathbb{C}^4$$

(119)
$$\frac{1}{2}\,Tr(M\omega) + P^{\dagger}G\,\gamma = \frac{1}{2}\,Tr(M'\omega') + P'^{\dagger}G\,\gamma'$$

We get :

(120)
$$\frac{1}{2}\,Tr(M\omega) + P^{\dagger}G\,\gamma = \frac{1}{2}\,Tr(M'L^{\dagger}\omega\,L) + P'^{\dagger}G\,(\,GL^{\dagger}\omega\,C + G\,L^{\dagger}G\,\gamma\,)$$

(121)
$$\frac{1}{2}\,Tr(M\omega) + P^{\dagger}G\,\gamma = \frac{1}{2}\,Tr(M'L^{\dagger}\omega\,L) + P'^{\dagger}L^{\dagger}\omega\,C + P'^{\dagger}L^{\dagger}G\,\gamma\,)$$

Identifying the term γ gives:

(122)
$$P^{\dagger} = P'^{\dagger}L^{\dagger} \rightarrow P = L\,P'$$

In the trace, a circular permutation can be performed :

(123)
$$Tr(\,M'L^{\dagger}\omega\,L\,) = Tr\,(\,L\,M'\,L^{\dagger}\omega\,)$$

Separating the terms into ω :

(124) $$\frac{1}{2}Tr(\,M\,\omega) \;=\; \frac{1}{2}Tr(\,L\,M'\,L^{\dagger}\omega\,) \;+\; P'^{\dagger}L^{\dagger}\omega\,C$$

- The term $P'^{\dagger}L^{\dagger}\omega\,C$ is the dot product of two vectors.
- There is the row vector $P'^{\dagger}$
- And the column vector $L^{\dagger}\omega\,C$

If I have a scalar that is the product of two vectors (the components are complex this time):

(125) $$\begin{pmatrix} \mathrm{A} & \mathrm{B} \end{pmatrix}\begin{pmatrix} \mathrm{C} \\ \mathrm{D} \end{pmatrix} = \mathrm{AC} + \mathrm{BD}$$

I can form the matrix:

(126) $$\begin{pmatrix} \mathrm{C} \\ \mathrm{D} \end{pmatrix}\begin{pmatrix} \mathrm{A} & \mathrm{B} \end{pmatrix} = \begin{pmatrix} \mathrm{AC} & \mathrm{BC} \\ \mathrm{AD} & \mathrm{BD} \end{pmatrix}$$

We can see that:

(127) $$\begin{pmatrix} \mathrm{B} & \mathrm{B} \end{pmatrix}\begin{pmatrix} \mathrm{C} \\ \mathrm{D} \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} \mathrm{AC} & \mathrm{BC} \\ \mathrm{AD} & \mathrm{BD} \end{pmatrix}$$

The second term of the second member is equal to the product of a line-matrix by a column-matrix. This being equal to the reversed product. Hereafter, schematically, the product of a line-matrix by a column-matrix :

(128) $$P'^{\dagger}L^{\dagger}\omega\,C \;=\; \mathrm{Tr}(\,L^{\dagger}\omega\,C\,P'^{\dagger})$$

In the trace one can achieve a circular permutation :

(129) $$P'^{\dagger}L^{\dagger}\omega\,C \;=\; \mathrm{Tr}\,(\,CP'^{\dagger}L^{\dagger}\omega\,)$$

whence :

(130) $$\frac{1}{2}Tr(\,M\,\omega) \;=\; \frac{1}{2}Tr(\,L\,M'\,L^{\dagger}\omega\,) \;+\; Tr(\,C\,P'^{\dagger}L^{\dagger}\,)$$

(131) $$\boxed{\begin{aligned} M &= LM'L^{\dagger} + 2CP'^{\dagger}L^{\dagger} \\ P &= L\,P' \end{aligned}}$$

We then see that the group's action on the anti-Hermitian matrix is expressed in the form :

(132) $$\boxed{\mu' \;=\; a\,\mu\,a^{\dagger}}$$

This is the action of the complex Poincaré group, the isometry group of the complexified Minkowski space, on the dual of its Lie algebra. The formula is derived from that of the action, in real numbers (see the appendix) by replacing transposed with adjoint.

14- **Conclusion**.

In this work, we have shown that, starting from the sole geometric structure of Minkowski space and its isometry group, it is possible to derive, through essentially algebraic means, the fundamental quantities of relativistic physics. The approach introduced by Jean-Marie Souriau, based on the action of the Poincaré group on the dual of his Lie algebra, thus reveals energy, momentum, and spin as intrinsic geometric objects. The value of this presentation lies in the fact that it relies on accessible tools, while leading to a unified understanding of physical concepts generally introduced independently. It highlights the structuring role of symmetry groups and their actions in the organization of physical laws. The extension of this construction to the complex framework, proposed here as an exercise, constitutes only a first sketch. It nevertheless shows that, as soon as one undertakes this type of approach, new elements naturally appear in the formalism. These should not be interpreted as established results, but rather as indications, signals suggesting that this path deserves to be explored further, which will be the subject of the following article.

Appendix.

This appendix represents the completion of the calculation of the coadjoint action of the group on the dual of its Lie algebra, with respect to the matrix M, a result presented by Souriau in 1970 [1]. As will be seen, this formula shows that the antichronous components of the group do not modify the spin. This section makes use of theorems concerning matrices.

We therefore start from equation (76) of the text which gives us:

$$(1) \qquad \frac{1}{2}Tr(M.\omega) = \frac{1}{2}Tr(M'.L^t\,\omega\,L) + P'^t\,L^t\,\omega\,C$$

We will use the fact that the trace of a matrix product does not change if we perform a circular permutation within that product.

$$(2) \qquad \frac{1}{2}Tr(M.\omega) = \frac{1}{2}Tr(\,L\,M'.L^t\,\omega) + P'^t\,L^t\,\omega\,C$$

The second term of this equation is equal to the product of a row matrix P'^t by a column matrix $L^t\,\omega\,C$. Hereafter, schematically, this is equal to the trace of a row-matrix by a column-matrix:

$$(3) \qquad P'^t\,L^t\,\omega\,C = Tr\,(\,L^t\,\omega\,C\,P'^t)$$

And, within this trace, a new circular permutation is performed..

$$(4) \qquad Tr\,(\,L^t\,\omega\,C\,P'^t) = Tr\,(\,C\,P'^t L^t\,\omega\,)$$

Whence :

$$(5) \qquad \frac{1}{2}Tr(M.\omega) = \frac{1}{2}Tr(M'.L^t\,\omega\,L) + Tr\,(\,C\,P'^t L^t\,\omega\,)$$

Where ω is an antisymmetric matrix. We know that a matrix' trace is equal to the product of another matrix by a symmetrical matrix. Any matric can be symmetrized or antisymmetrized. In addition the trace of the product of a matrix by an antisymmetric matrix is zer. Whence :

(6) $$Tr(A\omega) = Tr[antisym(A) \times \omega]$$

We can apply that to the matrix $C\ P'^t L^t$why where we take the trace of its product by by an antisymmetric matrix ω.

(7) $$C\ P'^t L^t = sym\ (C\ P'^t L^t) + antisym\ (C\ P'^t L^t)$$

But :

(8) $$Tr\ [sym\ (C\ P'^t L^t) \times \omega] = 0$$

So that :

(9) $$Tr\ (\ C\ P'^t L^t\ \omega\ ) = Tr\ [antisym\ (C\ P'^t L^t) \times \omega]$$

(10) $$antisym\ (C\ P'^t L^t) = \frac{1}{2}\ [C\ P'^t L^t - (C\ P'^t L^t)^t]$$

(11) $$(C\ P'^t L^t)^t = L\ P\ C^t$$

(12) $$Tr\ (C\ P'^t L^t) = \frac{1}{2}\ Tr\ (\ C\ P'^t L^t - L\ P\ C^t\ )$$

Finally :

(13) $$M = L\ M'\ L^t + C\ P'^t L^t - L\ P\ C^t$$

The intrinsic invariance of the magnitude s of the spin vector then becomes apparent when the motion is inscribed along a world line (along a geodesic of Minkowski space where the length is real). Under these conditions, the vector C is collinear with P. In this case, the last two terms of the right-hand side are zero. But the transition vector of M is also zero, which gives the matrix M a row and a column composed of zeros. What remains is the omega submatrix, the spin submatrix. The action then represents an orthogonal rotation of the spin vector, which preserves its magnitude s. Incidentally, let us recall that the momentum matrix is

(14) $$\mu = \begin{pmatrix} M & -P \\ P^t & 0 \end{pmatrix}$$

So that the action of the group on its momentum is:

(15) $$\boxed{\mu = a\ \mu'\ a^t}$$

References :

[1] J.M.Souriau : Structure des systèmes dynamiques. Dunod Ed. France, 1970 and Structure of Dynamical Systems. Boston, Birkhaüser Ed. 1997
[2] Minkowski H**.** Raum und Zeit. *Physikalische Zeitschrift, 1909 vol. 10, p. 75–88.*

[3] Gram, J. P. (1883) *Über die Entwicklung reeller Funktionen in Reihen mittelst der Methode der kleinsten Quadrate* Journal für die reine und angewandte Mathematik**,** 94, p. 41–73.
[4] A.Einstein : Erlklärung der Perihelbewegung des Merkur aus allgemeinen Relarivitätstheorie. Königlish Preubische Akademie der Wissenschaften (Berlin) Sitzungberichet (25 nov. 1915) p. 831-838
[5] Lorentz, H. A. (1904) *Electromagnetic phenomena in a system moving with any velocity smaller than that of light* Proc. Royal Netherlands Academy**, 6**, 809–831.
[6] Poincaré H. *Sur la dynamique de l'électron. Rendiconti del Circolo Matematico di Palermo.* 23 juillet 1905 publicated : Dec 1905 / jan 1906 DOI:
[7] J-P Petit, G.D'Agostini. Mathematical and geometrical Incosistency of the Black Hole Model. 2026; Journal of Modern Physics. Part I Vol.17 N°2.
[8] J-P Petit, G.D'Agostini. Mathematical and geometrical Incosistency of the Black Hole Model. 2026; Journal of Modern Physics. Part II submitted
[9] J.P.Petit, H.Zejli : Study of symmetries through the action on torsors of the Janus symplectic group. Reviews in Mathematical Physics. Vol. 37, n001, 2024.
[10] J.P.Petit, F.Margnat, H.Zejli : A bimetric cosmological model on Andreï's twin universe approach. The European Physical Journal. Vol. 84 :N°1126 (2024)
[11] J.M.Souriau : Géométrie et Relativité. Ed. Hermann, 1964 (french edition only).
[12] Steven Weinberg : The quantum theory of fields, Cambridge University Press, 2005.
[13] J.P.Petit, G.D'Agostini : Negative Mass hypothesis in cosmology and the nature of dark energy. Astrophysics And Space Science,. *A* 29, 145-182 (2014)
[14] N.Debergh, J.P.Petit and G.D'Agostini : Evidence of negative energies and masses in the Dirac equation through a unitary time-reversal operator. , J. Phys. Comm. **2** (2018) 115012
[15] N.Debergh, J.P.Petit and G.D'Agostini : Evidence of negative energies and masses in the Dirac equation through a unitary time-reversal operator. , J. Phys. Comm. **2** (2018) 115012
[16] N.Debergh , J.P.Petit : On some reasons why the the reversal operator could be unitary. Phys. Lett A 442. 2022
[17] N.Debergh, G.D'Agostini, J-P Petit :On the Poincaré Algebra in a Complex Space-Time Manifold. Journal of Modern Physics 2021,12,pp. 218-228
[18] C.Hermite. *"Sur la théorie des formes quadratiques"*. Journal de Mathématiques Pures et Appliquées, 2^{e} série, tome 1, 1854, p. 215–234.