Doc. 25

[p. 844] Session of the physical-mathematical class on November 25, 1915

The Field Equations of Gravitation

by A. Einstein

In two recently published papers[1] I have shown how to obtain field equations of gravitation that comply with the postulate of general relativity, i.e., which in their general formulation are covariant under arbitrary substitutions of space-time variables.

Historically they evolved in the following sequence. First, I found equations that contain the NEWTONIAN theory as an approximation and are also covariant under arbitrary substitutions of determinant 1. Then I found that these equations are equivalent to generally-covariant ones if the scalar of the energy tensor of "matter" vanishes. The coordinate system could then be specialized by the simple rule that $\sqrt{-g}$ must equal 1, which leads to an immense simplification of the equations of the theory. It has to be mentioned, however, that this requires the introduction of the hypothesis that the scalar of the energy tensor of matter vanishes.

I now quite recently found that one can get away without this hypothesis about the energy tensor of matter merely by inserting it into the field equations in a slightly different way. The field equations for vacuum, onto which I based the explanation of the Mercury perihelion, remain unaffected by this modification. In order not to [2]
force the reader constantly to consult the previous publications, I repeat here the considerations in their entirety.

One derives from the well-known RIEMANN-covariant of rank four the following covariant of rank two:

$$G_{im} = R_{im} + S_{im} \tag{1}$$

$$R_{im} = -\sum_{l} \frac{\partial \begin{Bmatrix} im \\ l \end{Bmatrix}}{\partial x_l} + \sum_{l\rho} \begin{Bmatrix} il \\ \rho \end{Bmatrix} \begin{Bmatrix} m\rho \\ l \end{Bmatrix} \tag{1a}$$

$$S_{im} = \sum_{l} \frac{\partial \begin{Bmatrix} il \\ l \end{Bmatrix}}{\partial x_m} - \sum_{l\rho} \begin{Bmatrix} im \\ \rho \end{Bmatrix} \begin{Bmatrix} \rho l \\ l \end{Bmatrix}. \tag{1b}$$

[1] *Sitzungsber.* 44, p. 778, and 46, p. 799 (1915). [1]

[p. 845] The ten generally-covariant equations of the gravitational field in spaces where "matter" is absent are obtained by setting

$$G_{im} = 0. \tag{2}$$

These equations can be simplified by choosing the system of reference such that $\sqrt{-g} = 1$. S_{im} then vanishes because of (16), and one gets instead of (2)

$$R_{im} = \sum_l \frac{\partial \Gamma^l_{im}}{\partial x_l} + \sum_{\rho l} \Gamma^l_{i\rho}\Gamma^\rho_{ml} = 0 \tag{3}$$

$$\sqrt{-g} = 1. \tag{3a}$$

We have set here

$$\Gamma^l_{im} = -\begin{Bmatrix} im \\ l \end{Bmatrix}, \tag{4}$$

which quantities we call the "components" of the gravitational field.

When there is "matter" in the space under consideration, its energy tensor occurs on the right-hand sides of (2) and (3), respectively. We set

$$G_{im} = -\kappa\left(T_{im} - \frac{1}{2}g_{im}T\right), \tag{2a}$$

where

$$\sum_{\rho\sigma} g^{\rho\sigma}T_{\rho\sigma} = \sum_\sigma T^\sigma_\sigma = T. \tag{5}$$

T is the scalar of the energy tensor of "matter," and the right-hand side of (2a) is a tensor. If we specialize the coordinate system again in the familiar manner, we get in place of (2a) the equivalent equations

$$R_{im} = \sum_l \frac{\partial \Gamma^l_{im}}{\partial x_l} + \sum_{\rho l} \Gamma^l_{i\rho}\Gamma^\rho_{ml} = -\kappa\left(T_{im} - \frac{1}{2}g_{im}T\right) \tag{6}$$

$$\sqrt{-g} = 1. \tag{3a}$$

We assume, as usual, that the divergence of the energy tensor of matter vanishes when taken in the sense of the general differential calculus (energy-momentum theorem). Specializing the choice of coordinates according to (3a), this means basically that the T_{im} should satisfy the conditions

$$\sum_\lambda \frac{\partial T^\lambda_\sigma}{\partial x_\lambda} = -\frac{1}{2}\sum_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_\sigma}T_{\mu\nu} \tag{7}$$

or

$$\sum_{\lambda} \frac{\partial T_{\sigma}^{\lambda}}{\partial x_{\lambda}} = -\sum_{\mu\nu} \Gamma_{\sigma\nu}^{\mu} T_{\mu}^{\nu}. \tag{7a}$$

When one multiplies (6) by $\partial g^{im}/\partial x_{\sigma}$ and sums over i and m, one gets[2] because [p. 846]
of (7) and because of the relation

$$\frac{1}{2}\sum_{im} g_{im} \frac{\partial g^{im}}{\partial x_{\sigma}} = - \frac{\partial lg\sqrt{-g}}{\partial x_{\sigma}} = 0$$

that follows from (3a), the conservation theorem of matter and gravitational field combined in the form

$$\sum_{\lambda} \frac{\partial}{\partial x_{\lambda}} (T_{\sigma}^{\lambda} + t_{\sigma}^{\lambda}) = 0, \tag{8}$$

where t_{σ}^{λ} (the "energy tensor" of the gravitational field) is given by

$$\kappa t_{\sigma}^{\lambda} = \frac{1}{2}\delta_{\sigma}^{\lambda} \sum_{\mu\nu\alpha\beta} g^{\mu\nu} \Gamma_{\mu\beta}^{\alpha} \Gamma_{\nu\alpha}^{\beta} - \sum_{\mu\nu\alpha} g^{\mu\nu} \Gamma_{\mu\sigma}^{\alpha} \Gamma_{\nu\alpha}^{\lambda}. \tag{8a}$$

The reasons that motivated me to introduce the second term on the right-hand sides of (2a) and (6) will only become transparent in what follows, but they are completely analogous to those just quoted (p. 785).

When we multiply (6) by g^{im} and sum over i and m, we obtain after a simple calculation

$$\sum_{\alpha\beta} \frac{\partial^2 g^{\alpha\beta}}{\partial x_{\alpha} \partial x_{\beta}} - \kappa(T + t) = 0, \tag{9}$$

where, corresponding to (5), we used the abbreviation

$$\sum_{\rho\sigma} g^{\rho\sigma} t_{\rho\sigma} = \sum_{\sigma} t_{\sigma}^{\sigma} = t. \tag{8b}$$

It should be noted that our additional term is such that the energy tensor of the gravitational field occurs in (9) on footing equal with the one of matter, which was not the case in equation (21) l.c.

Furthermore, one derives in place of equation (22) l.c. and in the same manner as there, with the help of the energy equation, the relations

---

[2]On the derivation see *Sitzungsber.* 44 (1915), pp. 784–785. For the following I ask [3]
the reader also to consult, for a comparison, the deliberations given there on p. 785.

$$\frac{\partial}{\partial x_\mu}\left[\sum_{\alpha\beta}\frac{\partial^2 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta} - k(T + t)\right] = 0. \tag{10}$$

Our additional term insures that these equations carry no additional conditions when compared to (9); we thus need not make other hypotheses about the energy tensor of
[p. 847] matter other than that it complies with the energy momentum theorem.

With this, we have finally completed the general theory of relativity as a logical structure. The postulate of relativity in its most general formulation (which makes space-time coordinates into physically meaningless parameters) leads with compelling necessity to a very specific theory of gravitation that also explains the movement of the perihelion of Mercury. However, the postulate of general relativity cannot reveal to us anything new and different about the essence of the various processes in nature than what the special theory of relativity taught us already. The opinions I recently
[4] voiced here in this regard have been in error. Every physical theory that complies with the special theory of relativity can, by means of the absolute differential calculus, be integrated into the system of general relativity theory—without the latter providing any criteria about the admissibility of such physical theory.