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GOLDEN OLDIE

Republication of: Contributions to Einstein's theory of gravitation

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Contributions to Einstein's theory of gravitation.
by Ludwig Flamm.

A. Einstein has significantly improved the understanding of his new theory of gravitation by the summary which he gave recently.[1])*[1] The popular accounts given by M. Born[2]) and E. Freundlich[3]) have also been clarifying. However, the exact solutions for gravitational fields with spherical symmetry, which were found and discussed by K. Schwarzschild[4]) are particularly instructive. In the present lines, the author wishes to add further conclusions to these. In particular, the remarkable properties of the gravitational field will be made quite clear. In this way, the physical assumptions of the general theory of relativity might perhaps appear even more transparent. Furthermore, the motion of light in a gravitational field can be treated more accurately than before. Likewise, a strict numerical computation of the constants relevant for the gravitational field of the Sun might be desirable.

§1. It is advisable to first consider the case of the gravitational field inside a ball of incompressible fluid, which was discussed by Schwarzschild later. The presence of the gravitational field manifests itself by the fact that Minkowski's line element can no longer be reduced to the special form

$$d\,s^2 = d\,t^2 - d\,x^2 - d\,y^2 - d\,z^2$$

by an appropriate choice of coordinates. Rather, in the simplest case it now reads

$$d\,s^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 d\,t^2 - \frac{3}{\kappa\rho_0}\left(d\,\chi^2 + \sin^2\chi\,d\,\vartheta^2 + \sin^2\chi\,\sin^2\vartheta\,d\,\varphi^2\right),$$

where the coordinates have already been chosen in a particularly suitable way. Here, χ is a coordinate which increases radially from the centre of the fluid ball, reaching the value χ_a at the boundary surface, and ϑ and φ are the usual spherical coordinates. The constant ρ_0 stands for the density of the fluid ball and κ stands for the gravitational constant of Einstein's theory, which has the value

$$\kappa = \frac{8\pi k^2}{c^2},$$

where k^2 is the usual gravitational constant

$$k^2 = 6.68 \cdot 10^{-8}\mathrm{cm}^3\mathrm{g}^{-1}\mathrm{sec}^{-2}$$

1) Die Grundlagen der Allgemeinen Relativitätstheorie, Ann. d. Phys. **49**, 769, 1916. Published separately by Johann Ambrosius Barth, Leipzig 1916.

*1 Translator's note: The footnotes in the original text were numbered starting from 1 in each column. Here they will be numbered sequentially throughout the text.

2) Physikalische Zeitschrift **17**, 51, 1916.

3) Naturwissenschaften **4**, 363 and 386, 1916. Published separately by Julius Springer, Berlin 1916.

4) Berliner Sitzungsberichte 1916, p. 189 and 424.

and c is the speed of light.

It is practical to write for the line element

$$d\,s^2 = d\,\tau^2 - d\,\sigma^2, \tag{1}$$

that is, to split it into a temporal part

$$d\,\tau = \frac{3\cos\chi_a - \cos\chi}{2} d\,t$$

and the spatial part

$$d\,\sigma^2 = R_0{}^2 \left(d\,\chi^2 + \sin^2\chi\,d\,\vartheta^2 + \sin^2\chi\,\sin^2\vartheta\,d\,\varphi^2\right).$$

Here the new constant

$$R_0 = \sqrt{\frac{3}{\kappa\rho_0}}$$

was introduced. As Schwarzschild remarks, the above expression for $d\,\sigma^2$ represents nothing other than the line element of the spherical space with radius of curvature R_0; hence, this must be the geometry inside the fluid ball.

In order to gain an overview of the relations it is enough, because of the spherical symmetry, to consider the geometry of any planar section through the origin, thus for $\vartheta = \frac{\pi}{2}$. In this case the line element reads

$$d\,\sigma_e{}^2 = R_0{}^2 \left(d\,\chi^2 + \sin^2\chi\,d\,\varphi^2\right). \tag{2}$$

One can see that these are the same metric properties as on a sphere of radius R_0. According to Fig. 1, one can think of this as being generated by rotating the circle around the vertical diameter AB. Then the meridional element of arc-length is

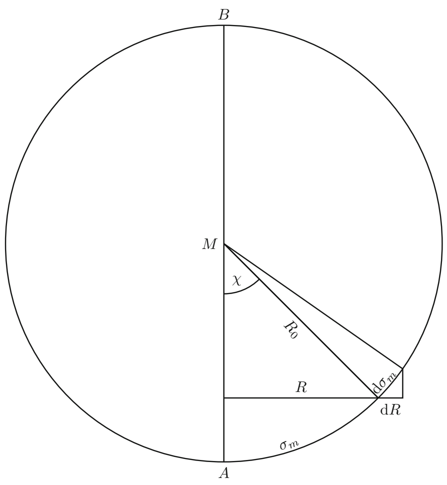

Fig. 1

$$d\,\sigma_m = R_0\,d\,\chi,$$

if R_0 is the radius of the meridian circle, the element of arc-length on the parallel circles is

$$d\,\sigma_n = R \cdot d\,\varphi,$$

if φ is the angle of rotation and

$$R = R_0 \sin\chi$$

denotes the distance from the axis of rotation. Hence, the line element on this sphere is given by the formula (2) as well.

A striking difference of the geometry on the sphere compared to Euclidean geometry consists, for example, in the fact that the circumferences of concentric circles

$$U = 2\pi R$$

do not increase proportionally to the radius σ_m, but instead they are obtained from it by the complicated formula

$$U = 2\pi R_0 \sin\frac{\sigma_m}{R_0}.$$

Thus, the circumference reaches a maximum for

$$\sigma_m = \frac{\pi}{2} R_0,$$

to decrease again and to shrink back to zero completely for

$$\sigma_m = \pi R_0.$$

One has

$$d\,U = 2\pi d\,\sigma_m \cdot \cos\chi, \tag{3}$$

and, hence,

$$d\,U \leq 2\pi d\,\sigma_m,$$

while in Euclidean geometry only the equality sign holds. Exactly the same relations must hold for radius and circumference of the circles within every central planar section of the fluid ball.

For a better understanding of the following, a small rearrangement will be made. Since

$$\frac{d\,U}{2\pi} = d\,R$$

equation (3) can also be written as

$$d\,R = d\,\sigma_m \cdot \cos\chi$$

and, in this form, can be read off directly from Fig. 1, because χ also represents the angle between the element of arc-length and the horizontal. Therefore, the formula (2) can also be given in the form

$$d\sigma_e{}^2 = \frac{dR^2}{\cos^2\chi} + R^2 d\varphi^2. \tag{2'}$$

But then this is the general expression for the line element on a surface of rotation, where χ denotes the angle that the tangent to the meridional curve, which can be quite arbitrary, makes with the equatorial plane.

§2. Now we can, in an analogous way, examine the first case which was treated by S c h w a r z s c h i l d, the gravitational field of a point mass. The line-element in its simplest form has the same structure as above and reads

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \dfrac{\alpha}{R}} - R^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right).$$

One quantity is completely new in this expression, the constant α, and has the value

$$\alpha = \frac{2k^2 M_0}{c^2},$$

where M_0 stands for the central mass, as would be obtained from astronomical measurements. The line-element will be decomposed again according to formula (1).

The spatial part of the line-element is again of a non-Euclidean nature. Once more, due to the spherical symmetry it seems to be enough to restrict the consideration to the metric properties in an arbitrary planar section through the origin, *i.e.*, to $\vartheta = \frac{\pi}{2}$. Now one obtains

$$d\sigma_e{}^2 = \frac{dR^2}{1 - \dfrac{\alpha}{R}} + R^2\, d\varphi^2. \tag{4}$$

This line-element is also of the form (2'), because one can put

$$\cos^2\chi = 1 - \frac{\alpha}{R},$$

and it is also identical with a line-element on a surface of rotation. Denoting by z the coordinate in the direction of the rotational axis, the equation for the meridional curve follows from

$$\frac{dz}{dR} = \operatorname{tg}\chi = \sqrt{\frac{\alpha}{R - \alpha}}$$

as

$$z^2 = 4\alpha(R - \alpha).$$

This describes a parabola with parameter

$$p = 2\alpha,$$

whose equation is referred to the axis and directrix as coordinate lines, as seen in Fig. 2. Thus, the expression (4) is identical with the line-element of that surface which is obtained by rotating this parabola around the directrix ZZ'.

The meridional radius of principal curvature of this surface of rotation is

$$\rho_m = \frac{d\,\sigma_m}{d\,\chi} = -\frac{2R}{\sin\chi}.$$

The orthogonal principal curvature radius is given by the line $\overline{MP}$ in Fig. 2 and it has the value

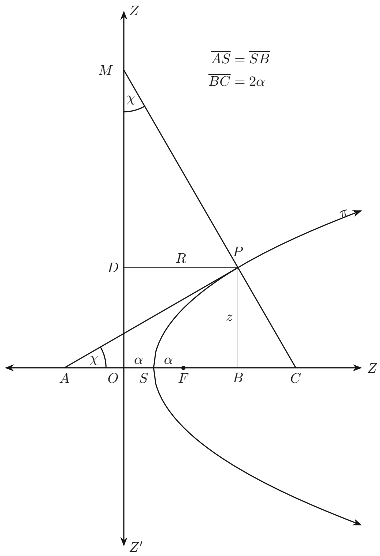

Fig. 2

$$\rho_n = \frac{R}{\sin\chi}.$$

So one finds for the G a u ß curvature of this rotational paraboloid

$$K = \frac{1}{\rho_m \rho_n} = -\frac{\alpha}{2R^3}.$$

However, for the sphere discussed in the previous section the G a u ß curvature is

$$K = \frac{1}{{R_0}^2},$$

i.e., positive and constant. For the present rotational paraboloid the curvature is negative and variable; it decreases in absolute value with increasing R. And for this very reason the geometry on that surface is even more involved. Though on the sphere the dependence of the circumference on the radius was not as simple as in Euclidean geometry, it was uniform over the entire sphere. For the rotational paraboloid at hand it is additionally dependent on the location of the circles on the surface. The metric properties in any planar section through the point mass are of exactly the same nature.

§3. The mass point, which generates the gravitational field, is found at the vertex S of the meridional parabola. The surface of rotation of the branch $S\pi$ of the parabola, as seen in Fig. 3, already maps to the full sectional plane through the centre, preserving the metric properties. The peculiarity that the point mass has a finite circumference, an equator of length $2\pi\alpha$, as Schwarzschild has already emphasized, is clearly noticeable in the figure.

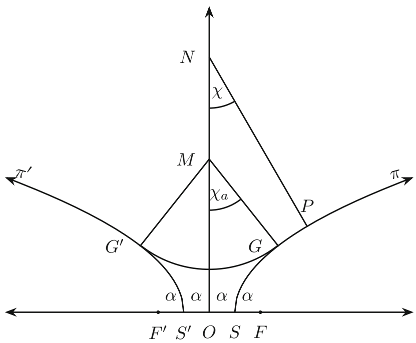

Fig. 3

In the exterior of the fluid ball, as discussed in the first paragraph, the line-element is the same as for a central point mass. The relationship between the interior of the fluid ball and its exterior is given by boundary conditions, which simply consist of the fact that on the boundary the coefficients of the two line elements together with their first derivatives have the same value.

For our representation of the metric properties in a plane through the centre, this implies that the sphere, which represents the geometry in the interior of the fluid, must touch the rotational paraboloid which is valid in the exterior. In Fig. 3 this is indicated by the circular arc GAG', which touches the two parabolas. Now, the surface of rotation for the combined curve $AG\pi$ uniquely represents the entire sectional plane through the centre preserving the metric properties. The angle χ, which grows on the spherical segment up to a maximal value χ_a at the boundary, slowly decreases along the subsequent part of the rotational paraboloid down to zero. The whole could be regarded as some kind of a funnel surface.

§4. Thus, the dimensions of the elementary rulers, which are represented by $d\,\sigma$, are subject to such influences in the gravitational field that using them for measurements—this is called "natural measurements of space"—does not, in general, lead to a Euclidean geometry. A quite analogous fact holds for the measurement of time by elementary clocks, the so called "natural time measurement", which is represented by $d\tau$. In general, it is no longer possible to simply identify $d\tau$ with dt as in the Minkowski world, but the connection is more involved. However, the speed of light is given by the equation[*2]

$$ds^2 = 0$$

as in the Minkowski world. From this one finds

$$\frac{d\sigma}{d\tau} = 1;$$

thus, the naturally measured speed of light is constant. In particular, the units are usually chosen from the outset in such a way that this constant assumes the value 1. In this sense, also in the general theory of relativity the postulate of the constancy of the speed of light holds.

It is of importance to have a closer look at the significance of the proposition just derived. Expressed in coordinates, which are mere parameters in the formulation of the gravitational field, the speed of light is by no means constant; in fact, it has different values in different directions even at the same location. But, when measured with material rods and clocks, the propagation of light also appears homogeneous and isotropic in a gravitational field. Choosing as an elementary clock the molecule which emits the red cadmium line, and fixing its period as the unit of time, one immediately realises that — due to that constancy of the speed of light— the metric unit of length must be covered at all times and in all locations by the same number of wavelengths of the red cadmium line. Fixing, in addition, the elementary rod to be the lattice distance of the sodium chloride crystal, then one arrives at a like conclusion in view of the constancy of the "naturally measured" speed of light. Thus, the theory of general relativity is based on the fundamental assumption that, for instance, the ratio between the wavelength of the red cadmium line and the lattice constant of the sodium chloride crystal is an absolute constant. Even in a general gravitational field, it must be completely independent of the location, the orientation and the instant of time.

Therefore, if the elementary rulers and clocks are influenced by the gravitational field, then the propagation of light must be influenced in exactly the same way, so that when compared to each other they do not show any differences. The same must hold for all the other physical phenomena. This might be why one can describe the motion of a point particle by the simple variational equation

$$\delta \int_{P_1}^{P_2} ds = 0, \tag{5}$$

even though the expression contains only quantities related to the metric properties. The influence of the gravitational field on the motion of a point particle is entirely in line with the change exhibited by elementary rods and clocks; relative to one another, the

[*2] Translator's note: The original has $d^2 s = 0$ here.

behaviour of the phenomena has remained the same. In this way, the general principle of relativity can be given a more concrete formulation.

§5. Since light propagation satisfies the condition*3

$$d\,s^2 = 0,$$

it must be contained as a special case within the equations (5) for the general motion of a point particle. Indeed, one only needs to put to zero the one constant h in the intermediate integrals for the particle motion that have been derived by Schwarzschild[5]) in order to satisfy the above condition. Hence, in the gravitational field of a central mass the equations for the propagation of light read:

$$\left.\begin{aligned} \left(1-\frac{\alpha}{R}\right)\left(\frac{d\,t}{d\,s}\right)^2 - \frac{1}{1-\frac{\alpha}{R}}\left(\frac{d\,R}{d\,s}\right)^2 - R^2\left(\frac{d\,\varphi}{d\,s}\right)^2 &= 0 \\ R^2\frac{d\,\varphi}{d\,s} &= \Delta, \\ \left(1-\frac{\alpha}{R}\right)\frac{d\,t}{d\,s} &= 1 \end{aligned}\right\} \tag{6}$$

The second equation represents Kepler's second law*4 and, since the speed of light at infinity equals 1, the constant Δ has a very clear meaning. It is nothing else than the length of the perpendicular from the centre to the light ray, if it were not changing its direction coming from infinity; it is, so to speak, the distance Δ, by which the light ray at infinity misses the centre.

After introducing

$$\frac{1}{R} = x,$$

the equation for the trajectory derived from the system (6) reads quite analogously to Schwarzschild's

$$\left(\frac{d\,x}{d\,\varphi}\right)^2 = \frac{1}{\Delta^2} - x^2 + ax^3$$

and, hence, represents the equation for a light ray in the present gravitational field. It follows that

$$d\,\varphi = \frac{\Delta \cdot d\,x}{\sqrt{1-\Delta^2(x^2-\alpha x^3)}}$$

and, hence, also leads to an elliptic integral as with the planetary motion. Upon introducing the new variable

$$\mu = \Delta \cdot x\sqrt{1-\alpha x}$$

one has, up to quantities of second order in α,

$$\Delta \cdot x = \mu\left(1 + \frac{\alpha}{2\Delta}\mu\right)$$

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*3 Translator's note: The equation below was again changed from the original $d^2s = 0$.

5) ibid., p. 195, equations (15), (16) and (17).

*4 Translator's note: the original says here "Flächensatz" = area theorem.

and one obtains

$$d\varphi = \frac{d\mu}{\sqrt{1-\mu^2}} + \frac{\alpha}{\Delta}\frac{\mu\, d\mu}{\sqrt{1-\mu^2}}.$$

Denoting by φ_a the angle that is swept out by the radius vector, when an element of the light ray from infinity reaches its closest approach to the central mass, which is the case for

$$\frac{dx}{d\varphi} = 0,$$

then one has

$$\varphi_a = \left(\arcsin\mu - \frac{\alpha}{\Delta}\sqrt{1-\mu^2}\right)_0^1 = \frac{\pi}{2} + \frac{\alpha}{\Delta},$$

since μ meanwhile increases from 0 to exactly the value 1. With the light ray element hurrying on from the perihelion back to infinity, the radius vector sweeps out the angle φ_b, for which one finds

$$\varphi_b = \frac{\pi}{2} + \frac{\alpha}{\Delta},$$

as well. The radius vector would have swept out the angle π in total if the light ray was propagating in a straight line. Hence, the light ray incurs the deflection

$$\varepsilon = \varphi_a + \varphi_b - \pi = \frac{2\alpha}{\Delta} \tag{7}$$

in the gravitational field, which coincides with the formula[6]) calculated by E i n s t e i n for this case. However, the derivation given here is completely identical to the calculation of the perihelion movement of the planets[7]). Hence, the influence of the gravitational field on the light rays has the same cause. Therefore, one can also regard the deflection of light rays as a perihelion shift.

The value of the coordinate R reached by the light ray element at the perihelion can also be calculated. It follows from the system of equations (6) for

$$\frac{dR}{ds} = 0$$

in the form of the relation

$$R^2 - \left(1 - \frac{\alpha}{R}\right)\Delta^2 = 0.$$

Finally, up to quantities of second order in α, this gives

$$R = \Delta - \frac{\alpha}{2}. \tag{8}$$

Thus, one could use the quantity R in the denominator in equation (7) instead of Δ without affecting the accuracy of the final formula.

§6. The numerical calculations in relation to the gravitational field of the Sun have nearly all been carried out in a more or less approximate way by E i n s t e i n and

6) Ann. d. Phys. **49**, 822, 1916.

7) A . E i n s t e i n, Berliner Sitzungsberichte 1915, p. 831.

S c h w a r z s c h i l d. However, in the following the author wishes to present a unified and exact recalculation of the relevant constants. The astronomical quantities employed were taken from the table of constants at the end of the "Theoretische Astronomie" by W. K l i n k e r f u e s[8]). Thus, in particular, the calculations used the equatorial horizontal parallax of the Sun

$$\pi = 8.80''$$

and the speed of light

$$c = 2.9986 \cdot 10^{10} \text{cm} \, \text{sec}^{-1},$$

as fixed at the astronomy conference in Paris. The equatorial radius of the Earth was taken from B e s s e l's value.

For the product of gravitational constant and solar mass one obtains

$$k^2 M_0 = 1.324 \cdot 10^{26} \text{cm}^3 \, \text{sec}^{-2}.$$

From this one determines the value of the important quantity α as

$$\alpha = 2.945 \cdot 10^5 \text{cm},$$

which was given by S c h w a r z s c h i l d only approximately as 3 km. It is easy to see that due to the bending of the light rays the true solar radius, which is determined from the apparent solar radius in the usual way, is in reality the quantity denoted by $\varDelta$ in the previous section. We denote this quantity by $\varDelta_a$ if it is referred to the special case of a light ray grazing the surface of the Sun. From the apparent solar radius according to A u w e r s one obtains

$$\varDelta_a = 6.9545 \cdot 10^{10} \text{cm}$$

and one should determine the corresponding value for R_a for the solar radius from equation (8). However, with the current status of the accuracy of astronomical measurements one can still identify the quantities R_a and $\varDelta_a$. With the numbers given above one obtains for the total deflection of a light ray grazing the surface of the Sun

$$\varepsilon_a = 1.75''$$

in good agreement with the calculation by E i n s t e i n.

Finally, here are the constants which one obtains when the Sun is regarded as a ball of incompressible fluid. The relation

$$\sin^2 \chi_a = \frac{\alpha}{R_a}$$

reveals the angle χ_a at the solar surface as a small quantity for whose calculation the formula

$$\chi_a = \sqrt{\frac{\alpha}{R_a}}$$

8) 2nd edition by H. B u c h h o l z, Braunschweig 1899, p. 927.

suffices. This yields

$$\chi_a = 7'4.5'';$$

therefore this funnel surface mapping the geometry within a central cut of the solar system is quite flat. Again, for the radius of curvature

$$R_0 = \frac{R_a}{\sin \chi_a}$$

of the spherical space inside the Sun the simplified calculation

$$R_0 = R_a \sqrt{\frac{R_a}{\alpha}}$$

suffices. The numerical calculation shows that the radius of curvature R_0 in the interior is 486 times larger than the radius R_a of the Sun. Or, computed in centimetres,

$$R_0 = 3.38 \cdot 10^{18}\text{cm},$$

which would represent a distance reaching from the Sun out to the planetoid belt.

Summary.

Every planar cut inside a ball of incompressible fluid carries the same geometry as a sphere (§1). Around a central mass every central cut has the same geometry as a surface which is generated by rotating a parabola around its directrix (§2). The spherical shell which represents every central cut of the fluid ball preserving its metric properties must touch the segment of the rotational paraboloid which serves the same purpose in the exterior (§3). The principle of a constant, and notably "naturally measured", speed of light is a fundamental assumption also for the general theory of relativity (§4). The deflection of light rays passing by a central mass can also be interpreted as the motion of the perihelion similarly to the planets (§5). The deviations from the Euclidean geometry in the solar system are extremely small (§6).

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(Received 3. September 1916.)

[*5] Translator's note: In the original text: "der k. k. Technischen Hochschule". k. k. = kaiserliche und königliche = imperial and royal was the usual pair of adjectives added to the names of state-owned institutions in the Austro-Hungarian empire.