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**Hermann Weyl**

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# *3. On the Theory of Gravitation; by Hermann Weyl.*

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## A. Additions to the General Theory

### § 1. A Hamiltonian principle

The gravitational equations have been reduced to a H a m i l t o nian principle by H i l b e r t[1], based on M i e's theory[2], more generally by H . A . L o r e n t z[3] and by the founder of the theory of gravitation himself[4]. Its definitive formulation fails though, since we do not know the H a m i l t o nian function (world density of the action) for the matter, indeed do not even know which independent state variables should be used to describe the matter. Under these circumstances, it seems of importance to me to formulate *a H a m i l t o nian principle that is sufficiently broad so as to encompass our current sound knowledge of matter* (in E i n s t e i n's broad sense, i.e. knowledge of the energy-momentum tensor). From this principle, which will differ somewhat in form from those that have been presented hitherto, the following laws should thus emerge, as though from a common source:

1. The *inhomogeneous gravitational equations* according to which the energy-momentum tensor determines the curvature of the world. In this context, the energy-momentum tensor will be comprised only of that valid for the electromagnetic field in the æther and of the "kinetic" energy-momentum tensor of the matter in the more restricted sense $\varrho u_i u_k$, in which the invariant mass-density $\varrho$ and the components $u_i$ $(i = 1, 2, 3, 4)$ of the four-velocity appear. Here no account is thus taken of the still unclear constitution of important attributes of matter and of its cohesional forces;

2. the *M a x w e l l - L o r e n t z equations*, that, as in electron theory, gain a concrete meaning from the fact that a convectional current appears as the only electron current;

3. the law for *ponderomotive forces* in the electromagnetic field and the *mechanical equations*, which determine the motion of masses under the influence of these forces and of the gravitational field.

Let $x_i$ be the four coördinates used to fix world points[5],

$$g_{ik} dx_i \, dx_k \tag{1}$$

[1] Gött. Nachr. 1915. Meeting from 20 November.

[2] Ann. d. Phys. **37** p. 511, **39** p. 1, 1912; **40** p. 1, 1913.

[3] Four papers from the years 1915 and 1916 from the Versl. K. Akad. van Wetensch.

[4] A . E i n s t e i n, Sitzungsber. d. Preuß. Akad. d. Wiss. **42** p. 1111, 1916.

[5] In my notation, I follow A . E i n s t e i n's article "Die Grundlage der allgemeinen Relativitätstheorie", Ann. d. Phys. **49** p. 769, 1916, in particular the convenient rule of omitting the summation signs.

Let $x_i$ be the four coördinates used to fix world points[5],

$$g_{ik} dx_i \, dx_k \tag{1}$$

the invariant quadratic differential form (with the index of inertia 3)[6], whose coefficients comprise the gravitational potential, and $\varphi_i \, dx_i$ the invariant linear differential form, whose coefficients $\varphi_i$ are the components of the electromagnetic four-potential. I refer to the integral over some world domain $\mathfrak{G}$

$$-\frac{1}{2}\int H \, d\omega \quad \text{with} \quad H = g^{ik}\left(\begin{Bmatrix} ik \\ r \end{Bmatrix}\begin{Bmatrix} rs \\ s \end{Bmatrix} - \begin{Bmatrix} ir \\ s \end{Bmatrix}\begin{Bmatrix} ks \\ r \end{Bmatrix}\right)$$

as the *field action of gravitation*[7] (contained in this world domain), and to the integral

$$\frac{1}{2}\int L \, d\omega \quad \text{with} \quad L = \frac{1}{2}F_{ik}F^{ik} = \frac{1}{2}g^{ij}g^{kh}F_{ik}F_{jh}$$

as the *field action of electricity.* Here

$$F_{ik} = \frac{\partial \varphi_k}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_k}$$

are the components of the electromagnetic field, and $d\omega$ is the four-dimensional volume element

$$\sqrt{g} dx_1 \, dx_2 \, dx_3 \, dx_4, \quad -g = \det|g_{ik}|.$$

In this phenomenological theory, in addition to the "field", the "substance" arises, a three-dimensional, moving continuum that we can consider (mathematically) to be divided up into infinitesimal elements. Each element is endowed with a certain, unchanging mass or "mass charge" $dm$ and an unchanging electric charge $de$; as a representation of its history, a certain world

[5]In my notation, I follow A. Einstein's article "Die Grundlage der allgemeinen Relativitätstheorie", Ann. d. Phys. **49** p. 769, 1916, in particular the convenient rule of omitting the summation signs.

[6]Every quadratic form can be transformed linearly to a sum and difference of squares; the number of negative terms that arise is called the index of inertia. The fact that this is uniquely determined by the form is precisely the statement of the "law of inertia for quadratic forms".

[7]Editor's remark: Weyl used the now-outdated notation for coordinates and the Christoffel symbols. His $\begin{Bmatrix} ik \\ r \end{Bmatrix}$ is today's $\begin{Bmatrix} r \\ ik \end{Bmatrix}$.

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line, whose direction is characterized by the ratio of the four differentials $dx_1 : dx_2 : dx_3 : dx_4$, is associated with it. I call the quantity

$$\int \left\{ dm \int \sqrt{g_{ik}\, dx_i\, dx_k} \right\} \tag{2}$$

the *substance action of gravitation*, where the outer integral extends over the entire substance whereas the inner one extends over that part of the world line of the substance element $dm$ that lies within the domain $\mathfrak{G}$. We assume that the motion of the substance is related to the gravitational field such that the square root, i.e. the proper time $ds$, appearing in the inner integral, is always positive. We convert (2), which is possible, into an integral $\int \varrho d\omega$, which extends over the world domain $\mathfrak{G}$. The invariant space-time function $\varrho$ is called the absolute mass density. The integral describing the *substance action of electricity*

$$\int \left\{ de \int \varphi_i\, dx_i \right\}$$

is constructed in complete analogy to (2). The absolute electric charge density $\varepsilon$ is defined by

$$\int_{\mathfrak{G}} \varepsilon\, d\omega = \int \left\{ de \int ds \right\}.$$

The H a m i l t o nian principle reads:

*The sum of the field and substance actions of gravitation and electricity is an extremum in every world domain with respect to arbitrary variations in the electromagnetic and gravitational fields that vanish at the boundaries and with respect to analogous space-time variations of the moving substance elements.*[8]

Variation of $g^{ik}$ (with a fixed electromagnetic field and fixed world lines of the substance) yields the E i n s t e i n gravitational equations (I), variation of the electromagnetic potential $\varphi_i$ the M a x w e l l - L o r e n t z equations

$$\frac{1}{\sqrt{g}} \frac{\partial \left(\sqrt{g} F^{ik}\right)}{\partial x_k} = J^i = \varepsilon \frac{dx_i}{ds}, \tag{II}$$

[8]We use rational units, i.e. chosen so that the speed of light in vacuum = 1 and likewise E i n s t e i n's constant $8\pi\kappa$ ($\kappa = k/c^2$, $k$ the gravitational constant); electrostatic units following H e a v i s i d e.

and, finally, the variation of the substance elements' world lines yields the mechanical equations

$$\text{(III)} \qquad \varrho\left(\frac{d^2x_i}{ds^2}+\begin{Bmatrix} hk \\ i \end{Bmatrix}\frac{dx_h}{ds}\frac{dx_k}{ds}\right)=p^i,$$

where $p^i$ are the contravariant components of the force, the covariant components of which are given by

$$p_i = F_{ik}J^k.$$

These laws are, of course, not independent of each other. In fact, the mechanical equations (III) together with the equation of continuity for the matter are a mathematical consequence of laws (I) and (II), as can be verified by a simple calculation.

## § 2. Energy-momentum balance

According to the authors cited above — we return from the aforementioned phenomenological theory to a strict one, which can today, granted, only be formulated in general terms — the world is ruled by an action principle of the following form

$$\int_{\mathfrak{G}} (H - M)\, d\omega = \text{extremum}.$$

Here, the world density $M$ of the action associated with the material processes is a universal function of the independent state variables characterizing these processes, their derivatives (first, perhaps also higher ones) with respect to the coördinates $x_i$, and of $g^{ik}$. To refer to a concrete example, $M$ depends, in M i e's theory, not only on $g^{ik}$, but also on the four components $\varphi_i$ of the electromagnetic potential and on the field components $F_{ik}$ that result from $\varphi_i$ via differentiation. The derivation of the mechanical equations in the phenomenological theory mentioned above suggested to me that perhaps in general *the principle of conservation of energy and momentum expresses the fact that the H a m i l t o nian principle is obeyed by precisely those infinitely small variations that are brought about by an infinitesimal deformation of the world such that the state variables are "carried along" by this deformation.* This is indeed the case and it seems to lead to the simplest and most natural derivation for the principle of energy.

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If we set $M\sqrt{g} = \mathfrak{M}$, then the energy-momentum tensor $T_{ik}$ is defined by the equation for the total differential of $\mathfrak{M}$:

$$\frac{1}{\sqrt{g}}\delta\mathfrak{M} = -T_{ik}\delta g^{ik} + \frac{1}{\sqrt{g}}(\delta\mathfrak{M})_0,$$

where $(\delta\mathfrak{M})_0$ collects the terms containing the differentials of the material state variables linearly (e.g. of $\varphi_i$ and $F_{ik}$). The contravariant term $g^{ik}$ transforms under an arbitrary coördinate transformation

$$\bar{x}_i = \bar{x}_i(x_1, x_2, x_3, x_4)$$

according to the equations

$$\bar{g}^{ik} = g^{\alpha\beta}\frac{\partial\bar{x}_i}{\partial x_\alpha}\frac{\partial\bar{x}_k}{\partial x_\beta}.$$

If this transformation is infinitesimal:

$$\bar{x}_i = \bar{x}_i + \varepsilon\cdot\xi_i(x_1, x_2, x_3, x_4)$$

($\varepsilon$ denotes the infinitesimal constant, i.e. tending to zero), then for the difference of $g_{ik}$ and $\bar{g}_{ik}$

$$\bar{g}^{ik}(\bar{x}) - g^{ik}(x) = \delta g^{ik},$$

with the arguments $(x)$ and $(\bar{x})$ referring to the same world point in the old and new coördinate systems, one finds the equation

$$\delta g^{ik} = \varepsilon\left(g^{\alpha k}\frac{\partial\xi_i}{\partial x_\alpha} + g^{i\beta}\frac{\partial\xi_k}{\partial x_\beta}\right).$$

Applying the same procedure to the state variables of the material processes and making use of the fact that the invariant $M$ remains unchanged under such an infinitesimal transformation, we obtain the law governing how the energy-momentum tensor depends on $g^{ik}$ and the material state variables.

Consider a world domain $\mathfrak{G}$ that, when represented by the coördinates $x_i$, corresponds to a mathematical domain $\mathfrak{X}$ in the domain of definition of these coördinates $x_i$. If the above infinitesimal transformation has the property that the variations $\xi_i$ together with with their derivatives vanish at the boundary of the domain $\mathfrak{G}$, then this world domain in the new variables $\bar{x}_i$ corresponds exactly to the same mathematical domain $\mathfrak{X}$. I set

$$\Delta g^{ik} = \bar{g}^{ik}(x) - g^{ik}(x) = \delta g^{ik} + \left\{\bar{g}^{ik}(x) - \bar{g}^{ik}(\bar{x})\right\}$$

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$$= \delta g^{ik} - \varepsilon \cdot \frac{\partial g^{ik}}{\partial x_\alpha}\xi_\alpha,$$

i.e. I take the difference of $g^{ik}$ and $\bar{g}^{ik}$ at two space-time points, the second of which has the same coördinate values in the new coördinate system as the first in the old one; in other words, I perform a *virtual displacement*. For all other quantities, $\Delta$ has the same meaning. If I use the abbreviation $dx$ for the integration element $dx_1\,dx_2\,dx_3\,dx_4$, then $\int \mathfrak{M}\,dx$ is an invariant and hence

$$\int_{\mathfrak{x}} \mathfrak{M}\,dx = \int_{\mathfrak{x}} \bar{\mathfrak{M}}(\bar{x})\,d\bar{x} = \int_{\mathfrak{x}} \bar{\mathfrak{M}}(x)\,dx; \quad \text{therefore} \quad \int_{\mathfrak{x}} \Delta\mathfrak{M}\cdot dx = 0.$$

However

$$\Delta\mathfrak{M} = -\mathfrak{T}_{ik}\Delta g^{ik} + (\delta\mathfrak{M})_0 \qquad (\mathfrak{T}_{ik} = \sqrt{g}\cdot T_{ik}).$$

Here, the following has to be taken into account: In the transformed coördinate system, as in the original one, – I choose M i e's theory as an example – the equations

$$\frac{\partial\bar{\varphi}_k(\bar{x})}{\partial\bar{x}_i} - \frac{\partial\bar{\varphi}_i(\bar{x})}{\partial\bar{x}_k} = \bar{F}_{ik}(\bar{x})$$

hold. Since the labelling of the variables is immaterial, one has

$$\frac{\partial\bar{\varphi}_k(x)}{\partial x_i} - \frac{\partial\bar{\varphi}_i(x)}{\partial x_k} = \bar{F}_{ik}(x).$$

The relations

$$\frac{\partial\varphi_k}{\partial x_i} - \frac{\partial\varphi_i}{\partial x_k} = F_{ik}$$

thus remain true when we switch over from the functions $\varphi_i$, $F_{ik}$ to the functions $\bar{\varphi}_i$, $\bar{F}_{ik}$ of the same variables $x_i$; i.e. they remain unchanged for the variation $\Delta$ (in contrast to the variation $\delta$). According to the general action principle in which we leave $g^{ik}$ unchanged, i.e. *due to the laws of the material processes*, one has

$$\int_{\mathfrak{x}} (\Delta\mathfrak{M})_0\,dx = 0 \quad \text{and thus also} \quad \int \mathfrak{T}_{ik}\Delta g^{ik}\cdot dx = 0.$$

If we insert the expression for $\Delta g^{ik}$ here and get rid of the derivatives of the displacement components via integration by parts, then we have

$$\int \left\{ \frac{\partial\mathfrak{T}_i^k}{\partial x_k} + \frac{1}{2}\frac{\partial g^{rs}}{\partial x_i}\mathfrak{T}_{rs} \right\} \xi_i\,dx = 0,$$

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and thus the energy-momentum equations

$$\frac{\partial \mathfrak{T}_i^k}{\partial x_k} + \frac{1}{2}\frac{\partial g^{rs}}{\partial x_i}\mathfrak{T}_{rs} = 0 \tag{3}$$

have been proved.

For a variation of the gravitational field vanishing at the boundary of the world domain $\mathfrak{G}$,

$$\delta \int H\, d\omega = \int (R_{ik} - \tfrac{1}{2} g_{ik} R)\delta g^{ik} \cdot d\omega$$

holds, where $R_{ik}$ is the symmetric R i e m a n n curvature tensor and $R$ is the invariant

$$R = g^{ik} R_{ik}.$$

If we apply the above considerations to $H$ instead of $M$ (the fact that $H$ also contains the differential quotients of $g^{ik}$ is immaterial here), then without further calculation, we find that equation (3) is fulfilled identically upon replacing $T_{ik}$ by the tensor

$$R_{ik} - \tfrac{1}{2} g_{ik} R.$$

The energy momentum balance is thus not only a mathematical consequence of the material processes, as we have just shown, but also of the gravitational equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -T_{ik}.$$

What takes place in E i n s t e i n's theory of the old classification into geometry, mechanics and physics, is the juxtaposition of material processes and gravitation. Mechanics is the eliminant of the two, if you will; for the existence of the energy-momentum balance is on the one hand a consequence of the laws for material processes, on the other hand , it is the necessary condition for the matter to be able to imprint a metric on the world in accordance with the law of gravity. Hence there are four extra equations contained in the system of material and gravitational laws; indeed there must be four arbitrary functions in the general solution, since the equations, because of their invariant nature, leave the coördinate system of the $x_i$ completely undetermined.[9]

[9]Cf. the derivation of the energy-momentum balance by A . E i n s t e i n, Sitzungsber. d. Preuß. Akad. d. Wiss. **42**, p. 1111, 1916 and the comments by D . H i l b e r t, Gött. Nachr. 1917 (meeting of 23 Dec. 1916) on causality.

## § 3. Connexion with observation. Light rays and trajectories in the static gravitational field

The "objective" world that physics strives to distil out of our immediately experienced reality – its testable content – can only be grasped using mathematical concepts. However, to characterize the meaning this system of mathematical concepts has as applied to reality, we somehow have to attempt to describe its connexion to the immediately tangible, a task for epistemology, which, of its very nature, cannot be accomplished with concepts from physics alone, but by a constant reference to that which is vividly experienced in the mind. The connexion between the frequency of an electromagnetic field and the sensory perception "colour" is, for example, of this sort. Quite generally, it seems that it is the intensity of the energy-momentum current striking the sensory epithelium that is primarily responsible for the corresponding intensity of perception and the nature of the space-time variability for its quality. Here I want to describe in more detail this connexion for a rather simplified relationship between subject and object.

Imagine that we are in the four-dimensional physical world of single, moving point masses, which emit light – the stars. For simplicity, we use geometrical optics according to which the world lines of the light signals emitted from the stars are singular geodesic lines. In general, the equations of a geodesic world line read

$$\frac{d^2x_i}{ds^2} + \left\{ \begin{matrix} kh \\ i \end{matrix} \right\} \frac{dx_k}{ds}\frac{dx_h}{ds} = 0, \tag{4}$$

where $s$ is an appropriate parameter. It follows that

$$F \equiv g_{ik}\frac{dx_i}{ds}\frac{dx_k}{ds} = \text{constant}.$$

The singular geodesic world lines are defined by the fact that for them in particular this constant is zero (while it is positive for the world line of point masses). We simplify the perceiving consciousness, the "monad", to a "point eye". At every moment of its life, it occupies a certain point of space-time, it traces out a world line; it perceives the points of this world line as following one another in "temporal succession". We consider a certain moment; let the gravitational potentials have the values $g_{ik}$ a the point $P$ occupied by the monad at this given moment; let $dx_i$ be the components of the element $\mathfrak{e}$ of its world line at that point and the ratios of the $dx_i$ describe the world line's direction (velocity). We must assume that the direction is time-like, i.e. that

$$ds^2 = g_{ik}\, dx_i\, dx_k > 0$$

holds. Instead of the differentials $dx_i$, I write $x_i$ from now on, since all of our considerations refer to the point $P$.

Two line elements $x_i$, $x_i{}'$ are called orthogonal when

$$g_{ik} x_i x_k{}' = 0$$

holds. I now claim that all line elements (originating in $P$) that are orthogonal to the time-like $\mathfrak{e}$ are themselves space-like. That is, they span an infinitesimal, three-dimensional domain $\mathfrak{R}$ upon which the form $-ds^2$ imprints a positive definite metric. The monad experiences this domain $\mathfrak{R}$ as its immediate "spatial vicinity". To prove our assertion, we take $\mathfrak{e}$ to be the fourth coördinate axis; then the first three components of $\mathfrak{e}$ are equal to zero and $g_{44} > 0$. We can set

$$ds^2 = \sum_{i,k=1}^{4} g_{ik} x_i x_k = g_{44} \left( x_4 + \frac{g_{14}}{g_{44}} x_1 + \frac{g_{24}}{g_{44}} x_2 + \frac{g_{34}}{g_{44}} x_3 \right)^2 - \text{quad. F.}(x_1 x_2 x_3).$$

If we introduce

$$x_4 + \frac{g_{14}}{g_{44}} x_1 + \frac{g_{24}}{g_{44}} x_2 + \frac{g_{34}}{g_{44}} x_3$$

instead of the above $x_4$ for the fourth coördinate, then one finds

$$ds^2 = g_{44} x_4{}^2 - Q(x_1 x_2 x_3).$$

Since $ds^2$ has the index of inertia 3, the quadratic form $Q$ must be positive definite. All those elements and only those for which $x_4 = 0$ holds are orthogonal to $\mathfrak{e}$. This proves our assertion. We see, furthermore, that every line element can be split up uniquely into two summands, one of which is parallel to $\mathfrak{e}$ (has components proportional to those of $\mathfrak{e}$) and the other orthogonal to $\mathfrak{e}$. We call the direction of the second summand the "spatial direction" of the line element. Such spatial directions orthogonal to $\mathfrak{e}$ form angles with one another that can be calculated in the usual manner with the help of the quadratic form $-ds^2$, which is positive for such directions. Using this prescription, we identify (in the natural way) the angle between the spatial directions of light signals emitted from two stars and meeting at $P$, with the difference in the angles of the two directions in which the point eye sights the two stars at the given moment. We consider this difference in direction to be, at least approximately, something directly ascertainable; indeed, in seeing, the qualitative can also have the feature of being "spatially extended" (an aspect that cannot be traced back to the qualitative content of perception). By making use of appropriate instruments for observation, the observations of angles can be made more exact; however the task remains for the consciousness of determining if two directions are distinguishable or indistinguishable (coïncidence of cross-hairs and position of the star, reading off from a graduated circle). – At any rate, this simple scheme suffices in principle for the description of how the observation of stars can be used to test E i n s t e i n's theory.

In connexion with what preceded, I want to show in the simplest way how one can derive *F e r m a t's principle of fastest arrivals* for a static gravitational field from the general principle "the world line of a light signal is a singular geodesic line". If we choose to describe a geodesic line by the parameter $s$ in accordance with equation (4), then it is characterized by the variational principle

$$\delta \int F \, ds = 0, \tag{5}$$

valid for a virtual displacement for which the ends of the segment of the world line being considered remain fixed. (Alternatively, one can start from the equation

$$\delta \int \sqrt{F} \, ds = 0,$$

except for the case of singular geodesic lines). In the static case, we set $x_4 = t$; the quadratic fundamental form (1) has the structure

$$f \, dt^2 - d\sigma^2,$$

where $d\sigma^2$ is a positive quadratic form of the spatial differentials $dx_1$, $dx_2$, $dx_3$ whose coefficients as well as $f$, the square of the speed of light,[10] do not

[10] Translators' note: For light, $ds^2 = 0$ holds, which implies $f = (d\sigma/dt)^2$ and is here denoted as the "square of the speed of light", which should not be confused with the constant $c$.

depend on the time $t$. In this case, if we only vary $t$, then

$$\delta \int F\,ds = 2\int f\frac{dt}{ds}d\,\delta t = \left[2f\frac{dt}{ds}\delta t\right] - 2\int \frac{d}{ds}\left(f\frac{dt}{ds}\right)\delta t\,ds. \tag{6}$$

Therefore,

$$f\frac{dt}{ds} = \text{constant} = E$$

must hold. If we abandon the requirement that $\delta x_1$, $\delta x_2$, $\delta x_3$ as well as $\delta t$ disappear at the ends of the interval of integration, then it follows from (6) that we have to replace (5) by

$$\delta \int F\,ds = [2E\,\delta t] = 2\delta \int E\,dt. \tag{7}$$

If we vary the spatial trajectory of the light signal arbitrarily, holding the ends fixed, and imagine that the varied curve be traversed at the speed of light, then

$$F = 0, \qquad d\sigma = \sqrt{f}\cdot dt$$

holds for the original as well as the varied curve and (7) becomes

$$\delta \int dt = 0 \quad \text{or} \quad \delta \int \frac{d\sigma}{\sqrt{f}} = 0,$$

i.e. F e r m a t's principle. Time has been eliminated entirely; the last formulation refers only to the spatial path of the light ray and holds for every segment of it, if this segment is varied arbitrarily while keeping its initial and final points fixed.

We can apply the same method to ascertain a minimal principle for the trajectory of a point mass in a static gravitational field. Let us assume that the point mass $m$ moreover has an electric charge $e$ and is subjected to an electrostatic field with the potential $\Phi$. According to § 1, the variational principle then reads

$$\delta\left\{m\int ds + e\int \Phi\,dt\right\} = 0, \tag{8}$$

where $ds$ is the differential of proper time. If we vary $t$ and not the spatial coördinates, then the left hand side is

$$= \int\left\{mf\frac{dt}{ds} + e\Phi\right\}d\delta t.$$

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Hence

$$mf\frac{dt}{ds} + e\Phi = \text{constant} = E \tag{9}$$

holds and if we abandon the requirement that $\delta x_1$, $\delta x_2$, $\delta x_3$ as well as $\delta t$ vanish at the ends of the interval of integration, then the variational principle (8) must be replaced by

$$\delta\left\{m\int ds + e\int \Phi\, dt\right\} = [E\,\delta t] = \delta\int E\,dt. \tag{10}$$

If we introduce the expression

$$ds = \sqrt{f\,dt^2 - d\sigma^2}$$

in (9) and set

$$U = \frac{E - e\Phi}{\sqrt{f}}$$

for brevity, then we find the law

$$\boxed{\frac{U\,d\sigma}{\sqrt{f(U^2 - m^2)}} = dt.} \tag{11}$$

If we suppose the varied spatial trajectory with fixed ends to be traversed according to this same law for the speed, then we find that (9) is also valid for the varied path. Therefore, we find from (10)

$$\delta\int\left\{\frac{m^2 f}{E - e\Phi} - (E - e\Phi)\right\}dt = \delta\int\frac{\sqrt{f}(m^2 - U^2)}{U}dt = 0.$$

We can insert expression (11) for $dt$ here, since this expression remains valid for the variation, by assumption; in this way, time is completely eliminated and we find *that the spatial trajectory is characterized by the minimal principle*[11]

$$\boxed{\delta\int\sqrt{U^2 - m^2}\,d\sigma = 0.}$$

[11]Cf. also T. Levi-Civita, "Statica Einsteinia", Rend. d. R. Accad. dei Lincei **26** p. 464, 1917.



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## B. Theory of the Static, Rotationally Symmetric Gravitational Field

### § 4. Point mass without and with electric charge

For what follows, it will be necessary to make a few comments about Schwarzschild's determination of the gravitational field of a point mass at rest[12]. Upon introduction of appropriate coördinates, a three-dimensional, spherically symmetric line element necessarily has the form

$$d\sigma^2 = \mu({dx_1}^2 + {dx_2}^2 + {dx_3}^2) + l(x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3)^2,$$

where $\mu$ and $l$ depend only on the distance

$$r = \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2}.$$

The length scale for measuring this distance can be chosen so as to render $\mu = 1$; we make this choice. For the four-dimensional line element, we have to choose the form

$$ds^2 = f\,{dx_4}^2 - d\sigma^2,$$

where $f$ is also only a function of $r$. If we now set

$$1 + lr^2 = h$$

and the square root of the determinant $hf$ equal to $w$, then a short calculation, which for convenience we perform for the point $x_1 = r$, $x_2 = 0$, $x_3 = 0$, yields the value

$$-\frac{2lr}{h} \cdot \frac{w'}{w} \quad \text{for} \quad H = g^{ik}\left(\begin{Bmatrix} ik \\ r \end{Bmatrix}\begin{Bmatrix} rs \\ s \end{Bmatrix} - \begin{Bmatrix} ir \\ s \end{Bmatrix}\begin{Bmatrix} ks \\ r \end{Bmatrix}\right).$$

The prime denotes a derivative with respect to $r$. Moreover, let

$$-\frac{lr^3}{h} = \left(\frac{1}{h} - 1\right) r = v;$$

then one has to solve the variational problem

$$\delta \int vw'\,dr = 0 \quad \text{or} \quad \delta \int wv'\,dr = 0;$$

[12]Sitzungsber. d. Kgl. Preuß. Akad. d. Wiss. **7.** p. 189. 1916. [Translator's remark: See GRG **35.** p. 951. 2003 for an English translation.]

here $v$ and $w$ are to be taken to be the independent functions to be varied. The variation of $v$ gives

$$w' = 0, \quad w = \text{constant}$$

and by making appropriate use of the not yet fixed unit of time: $w = 1$. The variation of $w$ gives

$$v' = 0, \quad v = \text{constant} = -2a;$$
$$f = \frac{1}{h} = 1 - \frac{2a}{r}.$$

$a$ is related to the mass $m$ via the equation $a = \kappa m$; we call *$a$ gravitational radius of the mass $m$.*

In order to illustrate the geometry characterized by the line element $d\sigma^2$, we confine ourselves to the plane $x_3 = 0$ passing through the origin. If we introduce polar coördinates in it

$$x_1 = r\cos\vartheta, \qquad x_2 = r\sin\vartheta,$$

then we get

$$d\sigma^2 = h\,dr^2 + r^2\,d\vartheta^2.$$

This line element characterizes the geometry that is valid on the paraboloid of rotation

$$z = \sqrt{8a(r-2a)}$$

in a Euclidean space with the orthogonal coördinates $x_1, x_2, z$ if the paraboloid is projected orthogonally onto the $z = 0$ plane with the polar coördinates $r, \vartheta$. The projection covers the exterior of the circle $r \geq 2a$ twice, but does not cover the interior at all. Via natural analytic continuation, the true space will cover the domain $r \geq 2a$ doubly in the coördinate space of the $x_i$ used to represent it. The two coverings are separated by the sphere $r = 2a$ on which the mass lies and at which the metric becomes singular and one has to refer to the two halves as the "outside" and "inside" of the point mass.

This may become clearer upon introducing another coördinate system into which I need to transform Schwarzschild's equations at any rate in order to proceed further. The transformation equations read

$$x_1{}' = \frac{r'}{r}x_1, \quad x_2{}' = \frac{r'}{r}x_2 \quad x_3{}' = \frac{r'}{r}x_3; \qquad r = \left(r' + \frac{a}{2}\right)^2 \cdot \frac{1}{r'}.$$

If I remove the primes after carrying out the transformation, then

$$(12) \qquad d\sigma^2 = \left(1 + \frac{a}{2r}\right)^4 ({dx_1}^2 + {dx_2}^2 + {dx_3}^2), \qquad f = \left(\frac{r - a/2}{r + a/2}\right)^2$$

results. In the new coördinates, the line element of the gravitational space is thus *conformal* to Euclidean space; the linear enlargement factor is

$$\left(1 + \frac{a}{2r}\right)^2.$$

$d\sigma^2$ is regular for all values $r > 0$, $f$ is always positive and becomes zero only for

$$r = \frac{a}{2}.$$

The circumference of the circle ${x_1}^2 + {x_2}^2 = r^2$ is

$$2\pi r \left(1 + \frac{a}{2r}\right)^2;$$

if we allow $r$ to run over its range of values beginning with $+\infty$, then this function decreases monotonically until it reaches the value $4\pi a$ for

$$r = \frac{a}{2},$$

after which it begins to increase again as $r$ is decreased further toward zero, and grows finally without bound. According to the above interpretation, the domain

$$r > \frac{a}{2}$$

would correspond to the outside and

$$r < \frac{a}{2}$$

to the inside of the point mass. When continued analytically,

$$\sqrt{f} = \frac{r - a/2}{r + a/2}$$

becomes negative in the inside region, meaning that for a point at rest, the cosmic time ($x_4$) and proper time run in opposite directions. (In Nature, it is of course only possible to realize a portion of the solution, which does not extend up to the singular sphere.)

If the point mass carries an electric charge and if $\Phi$ is the electrostatic potential, then, if we choose C.G.S. units, the action principle is

$$\delta \int \left( vw' + \frac{\kappa}{c^2} \frac{\Phi'^2 r^2}{w} \right) dr = 0.$$

As above, the variation of $v$ gives

$$w' = 0, \qquad w = \text{constant} = 1,$$

and variation of $\Phi$

$$\frac{d}{dr}\left(\frac{r^2\Phi'}{w}\right) = 0, \quad \text{leading to} \quad \Phi = \frac{e}{r}.$$

Thus the same formula results for the electrostatic potential as without taking gravity into account. The constant $e$ is the electric charge (in the usual electrostatic units). If one varies $w$ however, then one finds

$$v' + \frac{\kappa}{c^2}\frac{\Phi'^2 r^2}{w^2} = 0,$$

which leads to

$$v = -2a + \frac{\kappa}{c^2}\frac{e^2}{r}, \quad \frac{1}{h} = f = 1 - \frac{2a}{r} + \frac{\kappa}{c^2}\frac{e^2}{r^2}.$$

As one can see, $f$ contains not only the mass dependent term $-2a/r$, but also an additional electric term. $a = \kappa m$ is again the gravitational radius of the mass $m$. In complete analogy, the length

$$a' = \frac{e\sqrt{\kappa}}{c}$$

must be termed the *gravitational radius of the electric charge*. For distances $r$ comparable with $a$, the mass term is $\sim 1$, whereas for $r \sim a'$, this is true for the electric term. $f$ remains positive for all values of $r$ if $|a'| > a$ holds; for an electron, the fraction $a'/a$ is of the order of magnitude $10^{20}$. For distances comparable to

$$a'' = \frac{e^2}{mc^2},$$

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the mass term and electric term in the gravitational potential $f$ are of the same order of magnitude; not until $r$ is much larger than $a''$ does the principle of superposition hold, meaning that the electric potential is determined by the electric charge and the gravitational potential by the mass according to the usual formulæ. It then follows that the quantity $a''$, which arises in other contexts as the "electron radius", can at least be treated as the radius of its sphere of impact. The relation $a' = \sqrt{a \cdot a''}$ holds.

Having derived the field of an electrically charged point mass, it is easy to use the last paragraph of § 3 to calculate the motion of a test body subject to this field, where its mass and charge are weak relative to those generating the field; as in the charge-free case (planetary motion[13]), this problem is solved exactly using elliptic functions .

## § 5. The field of a rotationally symmetric distribution of masses

The acquisition of exact solutions to the equations of gravity seems important to me with regard to the question of processes at work within the atom. After all, it is possible that at such scales it is essential to take the non-linearity of the exact laws of nature into account. It has long been well-known to mathematicians that in comparison to linear equations, the properties of non-linear differential equations, in particular regarding their singularities, are extremely complicated, unexpected and, at present, completely uncontrollable. It is well-known to physicists that peculiar processes must be at work inside the atom that have no analogue in the visible world, where the sum of forces is determined by the principle of superposition. I believe that these two things could be closely related and, indeed, that we could expect the definitive interpretation of quantum theory to stem from this relationship. To this end, granted, one that still lies in the distant future, it seems to me of interest to determine exactly the gravitational field of an axially symmetric distribution of masses and charges according to Einstein's theory. This will be done here for the static case; the study leads to surprisingly simple results.

The coördinates that play a rôle are: 1. the time $x_4 = t$; 2. a spatial coördinate singled out from the others, the angle $x_3 = \vartheta$ about the axis of rotation with the period $2\pi$; $\vartheta =$ constant is a meridional half-plane ending at the axis of rotation. To label the points in this half-plane, we have 3. two coördinates $x_1, x_2$ that we are now going to normalize. The line element

[13] K. Schwarzschild, ibid.

must have the form

$$ds^2 = f\,dx_4{}^2 - d\sigma^2,$$

where

$$d\sigma^2 = (h_{11}dx_1{}^2 + 2h_{12}dx_1\,dx_2 + h_{22}dx_2{}^2) + l dx_3{}^2;$$

the coefficients $f$, $l$; $h_{11}$, $h_{12}$, $h_{22}$ are functions of $x_1$ and $x_2$ alone. According to a general theorem on positive, quadratic differential forms of two variables, it is possible to choose the coördinates $x_1$, $x_2$ such that the term in brackets with the coefficients $h$ takes on the "isothermic" form

$$h(dx_1{}^2 + dx_2{}^2);$$

the pair of variables $x_1$, $x_2$ is then determined up to a conformal mapping. After making this choice of variables, we define for two arbitrary functions $\alpha$, $\beta$ of $x_1$, $x_2$

$$[\alpha, \beta] = \frac{\partial\alpha}{\partial x_1}\frac{\partial\beta}{\partial x_1} + \frac{\partial\alpha}{\partial x_2}\frac{\partial\beta}{\partial x_2}.$$

If I introduce $r = \sqrt{lf}$, then the square root of the determinant is

$$\sqrt{g} = w = hr.$$

For the action density $H$, the formula

$$2\mathfrak{H} = 2H\sqrt{g} = \left\{ {ik \atop r} \right\} \frac{\partial(g^{ik}\sqrt{g})}{\partial x_r} - \left\{ {ir \atop r} \right\} \frac{\partial(g^{ik}\sqrt{g})}{\partial x_k}$$

holds in general. In our case, the first term is

$$\sum_{i=1}^{4}\sum_{r=1}^{2} \left\{ {ii \atop r} \right\} \frac{\partial(g^{ii}\sqrt{g})}{\partial x_r} = \sum_{i=3}^{4}\sum_{r=1}^{2},$$

for which one immediately finds

$$\mathfrak{H}' = \frac{1}{2h}\left(\left[\frac{w}{l}, l\right] + \left[\frac{w}{f}, f\right]\right).$$

However, the second term is

$$\mathfrak{H}'' = -\frac{1}{\sqrt{g}}\sum_{i=1}^{2} \frac{\partial\sqrt{g}}{\partial x_i}\cdot\frac{\partial(g^{ii}\sqrt{g})}{\partial x_i} = \frac{[w, r]}{w}.$$

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Now

$$\left[\frac{w}{f}, f\right] = -\, w\frac{[f,f]}{f^2} + \frac{[w,f]}{f}$$
$$= -\, w[\lg f, \lg f] + r[h, \lg f] + h[r, \lg f]$$

holds, so for $\lg h = \mu$ we get

$$\frac{1}{h}\left[\frac{w}{f}, f\right] = -r[\lg f, \lg f] + r[\mu, \lg f] + [r, \lg f].$$

If one does the same for the other term in $\mathfrak{H}'$ and observes

$$2 \lg r = \lg l + \lg f,$$

then

$$\mathfrak{H}' = r[\mu, \lg r] + [r, \lg r] - \frac{1}{2}r([\lg f, \lg f] + [\lg l, \lg l])$$

results. If we introduce

$$\lambda = \lg \sqrt{l/f},$$

then we get

$$\frac{1}{2}([\lg f, \lg f] + [\lg l, \lg l]) = [\lg r, \lg r] + [\lambda, \lambda].$$

Thus we get

$$\mathfrak{H}' = [\mu, r] - r[\lambda, \lambda].$$

Finally we have

$$\mathfrak{H}'' = \frac{[w,r]}{w} = \frac{[r,r]}{r} + \frac{[h,r]}{r} = 4[\sqrt{r}, \sqrt{r}] + [\mu, r].$$

Altogether, we have

$$\mathfrak{H} = \frac{1}{2}(\mathfrak{H}' + \mathfrak{H}'') = [\mu, r] - \frac{1}{2}r[\lambda, \lambda] + 2[\sqrt{r}, \sqrt{r}].$$

To formulate the action principle, we have to calculate

$$\delta \int \mathfrak{H}\, dx_1\, dx_2$$

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for variations $\delta\mu$, $\delta\lambda$, $\delta r$ that vanish on the boundary of the (arbitrary) domain of integration. If we define in general

$$\Delta^2 \alpha = \frac{\partial^2\alpha}{\partial x_1{}^2} + \frac{\partial^2\alpha}{\partial x_2{}^2} \quad \text{and} \quad \Delta\,\alpha = \frac{1}{r}\left\{\frac{\partial}{\partial x_1}\left(r\frac{\partial\alpha}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(r\frac{\partial\alpha}{\partial x_2}\right)\right\}$$

then through integration by parts

$$\delta\int \mathfrak{H}\,dx_1\,dx_2 \quad \text{becomes} \quad \delta\int \mathfrak{H}^*\,dx_1\,dx_2,$$

where

$$\begin{aligned}\delta\mathfrak{H}^* = &-\delta\mu\cdot\Delta^2\,r + \delta\lambda\cdot r\Delta\,\lambda \\ &-\delta r\left(\Delta^2\,\mu + \frac{1}{2}[\lambda,\lambda] + \frac{2}{\sqrt{r}}\Delta^2\sqrt{r}\right).\end{aligned}$$

For (uncharged) matter at rest with negligible stresses, the energy-momentum tensor is comprised of the single component

$$T_{44} = \frac{\varrho}{g^{44}} \quad (\varrho = \text{absolute mass density}),$$

and

$$\delta\mathfrak{M} \equiv -\sqrt{g}T_{ik}\delta g^{ik} = -\varrho\sqrt{g}\cdot\tfrac{\delta g^{44}}{g^{44}} = \varrho h r\delta\lg f = \varrho^*(\delta r - r\delta\lambda)$$
$$(\varrho^* = h\varrho).$$

holds. According to the action principle, each coefficient of the differential $\delta\mathfrak{M}$ must agree with that of $\delta\mathfrak{H}^*$. That yields (coefficient of $\delta\mu$):

$$\boxed{\Delta^2\,r = 0.}$$

Thus $r$ is a potential function in the $x_1$–$x_2$ plane. If we denote the conjugate potential function by $z$ so that $z + ir$ is an analytic function of $x_1 + ix_2$, then the transition from $x_1$, $x_2$ to $z$, $r$ is a conformal mapping. We can thus assume right from the outset that

$$z = x_1, \quad r = x_2$$

holds. In the definition of the operator symbols [ ], $\Delta$, $\Delta^2$ one therefore has to replace $x_1$, $x_2$ by $z$, $r$. The coördinate system is now completely determined except for an arbitrary additive constant in $z$. In order for the line element to remain regular along the axis of rotation, $r$ must vanish there. I call $z$, $r$, *$\theta$ canonical cylindrical coördinates; the corresponding canonical form of the line element reads*

$$f\,dt^2 - \left\{ h(dz^2 + dr^2) + \frac{r^2\,d\vartheta^2}{f} \right\}.$$

Here the "Euclidean" case is contained for $f = 1$, $h = 1$. In order to be able to express ourselves in geometric terms, we portray the gravitational space by using a Euclidean image space with the cylindrical coördinates $z$, $r$, $\vartheta$. The mapping of the two spaces onto one another via the canonical coördinates is uniquely determined up to an arbitrary translation of the Euclidean image space in the direction of the $z$-axis. In the image space

$$\Delta = \frac{1}{r}\left\{ \frac{\partial}{\partial z}\left( r\frac{\partial}{\partial z} \right) + \frac{\partial}{\partial r}\left( r\frac{\partial}{\partial r} \right) \right\}$$

is the standard potential operator for axially symmetric functions.

Equating the coefficients of $\delta\lambda$ in

$$\delta\mathfrak{H}^* = \delta\mathfrak{M}$$

yields

(13) $$\Delta\,\lambda = -\varrho^*,$$

equating the coefficients of $\delta r$:

(14) $$\Delta^2\,\mu + \frac{1}{2}[\lambda, \lambda] - \frac{1}{2r^2} = -\varrho^*.$$

We first consider (13) and introduce $\psi = \lg\sqrt{f}$; then

$$\lambda = \lg r - 2\psi$$

holds and thus

(15) $$\Delta\,\psi = \frac{1}{2}\varrho^*,$$

or, when we add the factor $8\pi\kappa$, measured again in the C.G.S. system, to the right hand side

$$\boxed{\Delta\,\psi = 4\pi\kappa\varrho^*;}$$

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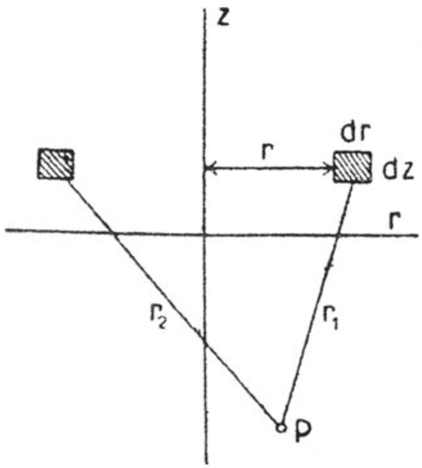


Fig. 1.

those solutions of $\psi$ are namely permissible that are regular along the axis of rotation. Thus using the canonical coördinate system, we have arrived at the ordinary P o i s s o n equation; since it is linear, the principle of superposition is valid for $\psi = \lg \sqrt{f}$.

For the infinitely thin ring described by rotating the surface element $dr\,dz$ in the canonical $r$-$z$ plane about the $z$-axis, the solution that one finds to the P o i s s o n equation upon setting

$$2\pi \varrho^* r\,dr\,dz = m$$

is

$$\psi = -\frac{\kappa m}{R}$$

as is well known. $R$, the "distance" from the ring to the point $P$ under consideration [Aufpunkt], is the arithmetic-geometric mean of the distances $r_1$ and $r_2$ of the point $P$ to the two intersection points of the ring with the meridional plane containing $P$:

$$\frac{1}{R} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{d\omega}{\sqrt{r_1 \cos^2 \omega + r_2 \sin^2 \omega}};$$

all of the equations to be understood as in the Euclidean case, but referring to the canonical coördinates! If only this ring is endowed with mass, then one gets

$$\sqrt{f} = e^{-\frac{\kappa m}{R}},$$

which for large $R$ is equal to

$$1 - \frac{\kappa m}{R}.$$

$m$ proves to be the gravitational mass contained in the ring and $\varrho^*$ is thus the *mass density in the canonical coördinate system.* — We have arrived at the following simple result:

*If the* (axially symmetric) *distribution of mass is known in the canonical coördinate system and if* $c^2\psi$ *is its N e w t o nian potential, then according to E i n s t e i n's theory we have*

$$\sqrt{f} = e^{\psi}.$$

We introduce $\psi$ in place of $\lambda$ in equation (14) as well; we have

$$[\lambda, \lambda] = \frac{1}{r^2} - \frac{4}{r}\frac{\partial \psi}{\partial r} + 4[\psi, \psi].$$

If we then multiply (14) by $\frac{1}{2}$, add (15) or equivalently

$$\Delta^2 \psi + \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{1}{2}\varrho^*$$

and take the unknown to be

$$\gamma = \lg \sqrt{hf} = \frac{\mu}{2} + \psi,$$

then we find

(16) $$\boxed{\Delta^2 \gamma = -[\psi, \psi],}$$

i.e. a P o i s s o n equation in the $r$-$z$ plane. In order for the line element to remain regular on the axis of rotation, $\gamma$ must vanish there; thus the unique solution of the P o i s s o n equation for $\gamma$ in the meridional half-plane is to be taken that vanishes at infinity and along the axis of rotation. Incidentally, if we content ourselves with the approximation that results by neglecting quadratic terms, then we must set $\gamma = 0$, $h = 1/f$.—

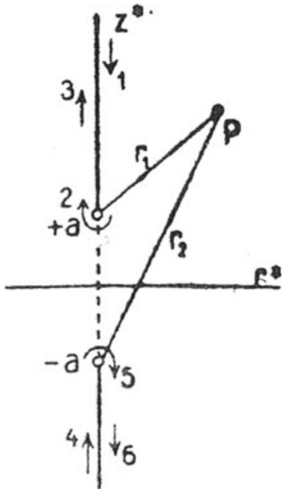


Fig. 2.

It is very instructive to trace how the *point mass* fits in to the general theory of rotationally symmetric mass distributions that has just been derived. We begin by taking the form (12) and introduce cylindrical coördinates instead of the "orthogonal" coördinates $x_i$:

$$x_1 = r\cos\vartheta, \quad x_2 = r\sin\vartheta, \quad x_3 = z;$$

the $r$ appearing in (12) must of course be replaced by $\sqrt{r^2+z^2}$. We then carry out the following conformal transformation in the meridional half-plane

$$(r+iz) - \frac{(a/2)^2}{r+iz} = r^* + iz^* \qquad (\vartheta = \vartheta^*);$$

then our line element indeed assumes the canonical form and the calculation yields:

$$f = \frac{\frac{r_1+r_2}{2} - a}{\frac{r_1+r_2}{2} + a}, \qquad hf = \frac{\left(\frac{r_1+r_2}{2} - a\right)\left(\frac{r_1+r_2}{2} + a\right)}{r_1 r_2},$$

where the meaning of $r_1$, $r_2$ can be taken from Fig. 2. In the canonical space with the cylindrical coördinates $z^*$, $r^*$, $\vartheta^*$, $\psi = \lg\sqrt{f}$ is the N e w t o nian potential of uniformly distributed mass along the line segment

$$r^* = 0, \qquad -a \leq z^* \leq +a :$$

in canonical coördinates, the "point mass" thus appears not as a sphere, but as a line segment, the meridional *half*-plane as the *entire* plane with a cut along the two solid half-lines, and the axis of rotation as the cut (connected at infinity), which must be traversed as indicated in the figure by the arrows and numbers. The right half of the entire plane corresponds to the outside, the left half to the inside of the point mass. Our interpretation as asserted in § 4. is confirmed anew: if we were not to take into account the "inside", then we would not arrive at the correct solution. One can convince oneself that $\lg\sqrt{hf}$ is indeed that solution to equation (16) in the entire plane $r^*$, $z^*$ with a cut that vanishes along the edges of the cut. —One would have been led naturally to this exact solution of the equations of gravity by studying the field of a line segment of length $2\kappa m$ covered by the mass $m$. Having found the field, then measuring out the "line segment" with the invariant spatial line element $d\sigma^2$ would then have shown however, that in reality it is not a line segment at all, but rather the surface of a sphere: in the exact theory of gravity, one can only determine a posteriori to which mass distribution a solution that one arrived at by some ansatz corresponds.

## § 6. The field of a rotationally symmetric distribution of charges

If the masses at rest carry static electric charges, then, in addition to the gravitational field, an electrostatic field arises that can be derived from the potential $\Phi = \Phi(x_1, x_2)$. As in the beginning of § 5., $x_1$ and $x_2$ are isothermic coördinates in the meridional half-plane. The action density for electricity is determined from

$$L = -\frac{[\Phi\Phi]}{hf}, \qquad L\sqrt{g} = -[\Phi\Phi]e^{\lambda} = -[\Phi\Phi]\frac{r}{f}.$$

The integral of $\delta(L\sqrt{g})$ over some region in the $x_1$-$x_2$-plane, provided the variations of $\delta\Phi$, $\delta\lambda$ vanish on the boundary of that region, is equal to the integral of

$$\delta\mathfrak{L}^* = -\,[\Phi\Phi]\frac{r}{f}\delta\lambda + 2r\Delta_f\,\Phi\cdot\delta\Phi,$$

$$\Delta_f = \frac{1}{r}\left\{\frac{\partial}{\partial x_1}\left(\frac{r}{f}\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{r}{f}\frac{\partial}{\partial x_2}\right)\right\}.$$

If we take into account not only the field-action, but also the substance-action as in § 1., then, to begin with, the general action principle yields through variation of $\Phi$:

$$\Delta_f \, \Phi = -\varepsilon^* = -h\varepsilon \quad (\varepsilon = \text{absolute charge density}). \tag{17}$$

In order to determine the gravitational field, by setting $\delta\Phi = 0$ as of now, we obtain the equation

$$\delta\mathfrak{H}^* = \delta\mathfrak{M} + \delta\mathfrak{L}^*. \tag{18}$$

It tells us again that $\Delta^2 r = 0$; thus the introduction of the *canonical coördinates* is possible and we again set $x_1 = z$, $x_2 = r$. Eq. (17) now reads:

$$\boxed{\Delta_f \, \Phi = \frac{1}{r}\left\{\frac{\partial}{\partial z}\left(\frac{r}{f}\frac{\partial\Phi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{r}{f}\frac{\partial\Phi}{\partial r}\right)\right\} = -\varepsilon^*.} \tag{19}$$

The arbitrary additive constant that appears in $\Phi$ will be chosen, as usual, so that $\Phi$ vanishes at infinity.

Equating the coefficients of $\delta\lambda$ in (18) results in Eq. (13) of the last section, with the modification that on the right hand side, in addition to the mass term $\varrho^*$, the additional term $1/f[\Phi\Phi]$ arising from the likewise gravitating electric energy appears; thus:

$$\Delta_f \, f = \frac{1}{r}\left\{\frac{\partial}{\partial z}\left(\frac{r}{f}\frac{\partial f}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{r}{f}\frac{\partial f}{\partial r}\right)\right\} = \varrho^* + \frac{1}{f}[\Phi\Phi]. \tag{20}$$

Eq. (14), § 5. can be adopted as is. We consider the expression for $\frac{1}{2}\Delta_f \, (\Phi^2)$; as a result of the equations

$$\frac{1}{2}\frac{\partial\Phi^2}{\partial z} = \Phi\frac{\partial\Phi}{\partial z}, \qquad \frac{1}{2}\frac{\partial\Phi^2}{\partial r} = \Phi\frac{\partial\Phi}{\partial r}$$

and of the fundamental electrostatic law (19), it is equal to

$$-\varepsilon^*\Phi + \frac{1}{f}[\Phi, \Phi].$$

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If we then introduce

$$f - \frac{1}{2}\Phi^2 = F, \qquad \varrho^* + \varepsilon^*\Phi = \sigma^*,$$

then we can replace equation (20) by

$$\boxed{\frac{1}{r}\left\{\frac{\partial}{\partial z}\left(\frac{r}{f}\frac{\partial F}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{r}{f}\frac{\partial F}{\partial r}\right)\right\} = \sigma^*.} \tag{21}$$

If the mass and charge are located only on an elementary ring with radius $r$ and cross-section $dr\,dz$ in canonical coördinates, and if we set

$$2\pi\sigma^* r\,dr\,dz = m, \qquad 2\pi\varepsilon^* r\,dr\,dz = e,$$

then it follows from equations (19) and (21) that

$$F = \text{constant} - \frac{m}{e}\Phi$$

necessarily holds. Choosing the units of time appropriately leads to constant = 1 and we have

$$f = 1 - \frac{m}{e}\Phi + \frac{1}{2}\Phi^2,$$

or, by introducing the C.G.S. system

$$f = 1 - \frac{2m\kappa}{e}\Phi + \frac{\kappa}{c^2}\Phi^2.$$

If one substitutes this value into (19), then one finds the linear potential law of standard electrostatics for the quantity

$$\int \frac{d\Phi}{1 - \frac{2m\kappa}{e}\Phi + \frac{\kappa}{c^2}\Phi^2}. \tag{22}$$

If one makes use of this to determine the above integral as a function of location in the meridional half-plane and from that $\Phi$ and $f$ — we shall carry out the calculation in a moment — then one can see that $m$ is the gravitational mass contained in the ring and $e$ its charge. Consequently, $\sigma^*$ and $\varepsilon^*$ are the mass and charge density in the canonical coördinate system;

*it is* particularly *noteworthy, that not* $\varrho^*$, *but* $\sigma^* = \varrho^* + \varepsilon^*\Phi$ *turns out to be the mass density.*[14]

The problem can be solved more explicitly if we assume that mass and charge are distributed arbitrarily, but in the same fashion, i.e. that the ratio $\sigma^* : \varepsilon^*$ is a constant independent of position. We denote the Euclidean volume integral over $\sigma^*$ in the space of canonical coördinates, i.e. the total mass, by $m$ and the analogous integral of $\varepsilon^*$, the total charge, by $e$. The aforementioned constant ratio will then be $m : e$. (22) is still the standard electric potential of the charge distribution $\varepsilon^*$ in the canonical space found without taking gravity into account. We introduce (cf. § 4.) the gravitational radii $a$, $a'$ of the mass $m$ and charge $e$ (concentrated to a point) and, favouring the case $a' > a$, set

$$\frac{a}{a'} = \sin\varphi_0.$$

Calculating integral (22), we arrive at the following result:

*If the charge distribution (which is proportional to the mass distribution by assumption) is known in the canonical coördinate system and its "elementary" potential, i.e. the potential found according to the elementary theory without taking gravity into account, when multiplied by the constant factor*

$$\frac{\sqrt{\kappa}}{c}\cos\varphi_0,$$

*is* $\varphi$, *then*

$$\Phi = \frac{e}{a'}\frac{\sin\varphi}{\cos(\varphi - \varphi_0)}, \qquad \sqrt{f} = \frac{\cos\varphi_0}{\cos(\varphi - \varphi_0)} \tag{23}$$

*holds exactly.* In particular, the case of the ring yields

$$\varphi = \frac{a'\cos\varphi_0}{R},$$

[14] If one takes $\varrho^* = 0$, then the standard radius $a''$ results for the region over which the charge of the electron is distributed. However, it cannot be ruled out that the term $\varepsilon^*\Phi$ can be almost completely compensated by a negative $\varrho^*$; I therefore refer the reader to M i e's theory. The whole point is to explain *why* the electron has such a small mass, i.e. where the *pure number* $a/a'$ of the order of magnitude $10^{-20}$ comes from! Hence, the true charge of the electron may be concentrated in a much smaller region, and $a''$ merely represents the "radius of impact".

where R is the "distance" between the point under consideration $P$ and the ring. For large $R$ this means that $\sqrt{f}$ is given by the asymptotic formula

$$1 - \frac{a}{R},$$

which implies that $m$ is indeed the gravitational mass, as claimed above.

With the current assumptions, we still need to calculate the second coefficient $h$ from the canonical line element. To that end, we have Eq. (14), § 5. at our disposal. If we treat it in the same way we did there, then for $\gamma = \lg \sqrt{hf}$ we first arrive at

$$\Delta^2 \gamma + \frac{[\sqrt{f}, \sqrt{f}]}{f} = \frac{1}{2} \frac{[\Phi, \Phi]}{f}.$$

If we then switch to the C.G.S. system — the factor $\frac{1}{2}$ on the right hand side is then to be replaced by $\kappa/c^2$ — and use expressions (23), then the equation for $\gamma$ takes on the simple form

$$\boxed{\Delta^2 \gamma = [\varphi, \varphi].} \tag{24}$$

If we do not assume the proportionality of charge and mass, then the solution cannot be obtained by such simple means. Now the numbers for the electron and the atomic nucleus are such that $a/a'$ is very small, of the order $10^{-20}$ and $10^{-17}$ respectively. Given this situation, the effect of gravity can be neglected entirely with respect to that of charge. If we specialize our equations accordingly, i.e. by choosing $a = 0$, $\varphi_0 = 0$ then we arrive at the statement:

*If the (axially symmetric) distribution of charge at rest, the effect of which is so strong as to render the effect of gravity negligible in comparison, is known in the canonical coördinate system and its elementary potential multiplied by $\sqrt{\kappa}/c$ is $\varphi$ then, taking gravity into consideration, we have*

$$\Phi = \frac{c}{\kappa} \mathrm{tg} \varphi \qquad \sqrt{f} = \frac{1}{\cos \varphi}.$$

The appearance of trigonometric functions, which are so closely related to integers through their periodicity, is rather surprising; the principle of superposition no longer holds in realms in which the value of $\varphi$ is close to 1. In fact, the potentials of the effective forces are trigonometric functions of quantities that obey this principle. For sufficiently concentrated charge, it may happen that this charge is enclosed in surface $S$ upon which $\varphi$ reaches the value $\pi/2$ and hence $\Phi$ and $\sqrt{f}$ become infinite. Since according to (24), $hf$ remains finite on this "border to the outer world", the spatial line element becomes $d\sigma^2 = 0$; thus $S$ turns out to be without extent as measured by the invariant line element. — Our result can hardly be used to understand the processes in the atom; after all, discrepancies between the field of the electron charge $e$ and the field as determined by the classical theory without gravity, are only noticeable for distances of the order $a \sim 10^{-33}$cm!

In the canonical coördinate system, the spherically symmetric point mass appears as a circular disc of radius $a'$ on which the electricity is distributed as it is for a charged metal plate in standard electrostatics.

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