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The Expanding Universe†

by the Abbé Georges Lemaître

INTRODUCTION AND SUMMARY

In this paper we do not intend to discuss the hypotheses on which the theory of the expansion of the Universe is based, or the value of the astronomical evidence which supports it. Such a discussion seems to us at present premature and it certainly could not arrive at definitive conclusions in the present state of the theory and the observations.

The theory can be developed in two ways: by the study of exact solutions of the gravitational equations, providing simplified models, or by approximate expansion of the solution of more complex problems. It seems to us useful not to mix these two methods, and in this paper we will be concerned only with mathematically exact solutions. When we want to apply these to real problems, we will have to appeal to physical intuition in order to reduce an overcomplicated problem to a simplified model for which we have a solution. Many of our results seem to be able to serve as starting points for the methods of expansion in series which we hope to treat in a later paper.

In the first two sections, we give in detail the tensor calculations which we shall need, and which we summarize in Section 3, in the course of introducing the notation which makes manifest the analogy between the relativistic results and the classical formulae.

We then introduce the concept of a quasi-static field which immediately allows us to generalize the known static solutions by allowing adia-

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batic variations in them. We give a solution, probably new, for the case of a sphere with constant radial pressure, and we use it to highlight Schwarzschild's paradox and to prove that the very severe limitation on the radius of a given mass which is introduced by the solution of the interior problem vanishes when one does not impose on the matter the condition of being in the fluid state.[1] We describe putting stress into the Einstein universe, assumed to be fluid, so that the rest mass[2] of the universe decreases without changing the volume or disturbing the equilibrium.

In Section 6, we summarize and complete the results obtained in our doctoral thesis (unpublished), presented in 1927 at the Massachusetts Institute of Technology, which concern a modification of the Schwarzschild interior problem proposed by Edddington.[3]

Section 7 concerns the influence of the formation of local condensations on the breakdown of the equilibrium of an Einstein universe: we recover our result (*Monthly Notices* 91, 490 (1931)) that the pressure at the neutral zone is the determining factor in the breakdown, while eliminating the technical complications which cluttered up our original proof.

In Section 8 we study the evolution of spherical condensations in an expanding universe under the hypothesis that pressure is negligible, and rediscover as a particular case the Friedmann universe.

We then integrate the Friedmann equation by Weierstrass's elliptic functions in Section 9 and put the equations in a form adapted to numerical calculations.

In Section 10, we introduce the hypothesis that clusters of galaxies are in equilibrium. This hypothesis can be checked by observation, and the result is favourable. One obtains 7×10^8 solar masses as the average mass of the nebulae and 13 as the expansion coefficient of the universe.[4]

We indicate how this new hypothesis can give cosmological significance to the relative frequency of clusters and isolated galaxies, and so remove the

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[1] In this paper Lemaître uses the term 'fluid' to mean a fluid with isotropic stresses, i.e. a perfect fluid — *Transl.*

[2] In the original, 'masse propre'.

[3] This thesis was: Lemaître, Georges (1927). "The Gravitational Field in a Fluid Sphere of Uniform Invariant Density, according to the Theory of Relativity." Ph. D. Thesis, Massachusetts Institute of Technology — *Transl.*

[4] By this, Lemaître means the ratio of the current length scale to the length scale of an Einstein universe with the same mass. He refers to the latter universe as in 'equilibrium', and its length scale as the 'radius of equilibrium'. He appears to have in mind throughout the paper the models of the universe which start or end in the Einstein static universe, or the cases, now known as Lemaître models, in which there is a 'coasting' phase close to this model, and he seems always to take the cosmological constant to be positive — *Transl.*

uncertainty which remains in the expansion law. We then calculate, under various hypotheses, the expansion time and the radius of the universe.

The hypothesis of equilibrium of the nebulae seems to exclude the critical case for which the equilibrium radius would greatly exceed a billion light-years. We establish the result that in the critical case the distance at the moment of equilibrium of the most widely separated points which can exchange light during expansion is still several billion light-years.

In Section 11, we remove an apparent contradiction between Friedmann's theory and the solution of Schwarzschild's exterior problem. In the latter, a mass such as that of the universe cannot have a radius less than a billion light years. We show that the singularity of the Schwarzschild exterior is an apparent singularity due to the fact that one has imposed a static solution and that it can be eliminated by a change of coordinates.[5]

In Section 12, we discuss the possibility of the universe reaching the theoretical zero of radius.

With the help of an anisotropic model of the universe suggested to us by Einstein, we show that anisotropy only precipitates contraction. Analysing the various forces which could stop contraction of a universe whose radius is decreasing to zero, we arrive at the conclusion that only the non-Maxwellian forces which prevent the interpenetration of the fundamental particles of matter seem to be capable of putting an end to the contraction, when the radius of the universe is reduced to the size of the solar system.

We thus conclude that the origin of the earth is later than such an event and this forces us to discard the solutions where the radius of the universe is much smaller than the equilibrium radius and in particular discard the quasi-periodic solutions.

1. CALCULATION OF THE RIEMANN TENSOR

We take as our starting point the study of the gravitational equations in the very general case of a quadratic form

$$ds^2 = a_1^2\, dx_1^2 + a_2^2\, dx_2^2 + a_3^2\, dx_3^2 + a_4^2\, dx_4^2 = a_\mu^2\, dx_\mu^2\,, \tag{1.1}$$

where a_1, a_2, a_3, and a_4 are functions of four coordinates, and we aim to write explicitly the gravitational equations

$$\kappa T_\mu^\upsilon + \lambda g_\mu^\upsilon = -R_\mu^\upsilon + \tfrac{1}{2} g_\mu^\upsilon R, \tag{1.2}$$

[5] Here Lemaître is referring to the apparent singularity at $r = 2m$ in the usual coordinates — *Transl.*

where

$$R_{\mu\upsilon} = g_{\mu\sigma} R^{\sigma}_{\upsilon} = -\frac{\partial \Gamma^{\alpha}_{\mu\upsilon}}{\partial x_{\alpha}} + \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x_{\upsilon}} - \Gamma^{\alpha}_{\mu\upsilon}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\upsilon\beta} . \qquad (1.3)$$

The calculations simplify considerably if one notices that the a_{μ}, which are not tensors under a general coordinate transformation, are nevertheless covariants of the first order for the special transformations of the form

$$x'_{\mu} = c_{\mu} x_{\mu} ,$$

where c_1, c_2, c_3, and c_4 are constants.

For these special transformations, the expressions

$$\alpha_{ik} = \frac{1}{a_i a_k} \frac{\partial a_i}{\partial x_k} \qquad (1.4)$$

and

$$\alpha_{ik\ell} = \frac{1}{a_i a_k a_\ell} \frac{\partial^2 a_i}{\partial x_k \partial x_\ell} \qquad (1.5)$$

are invariants. We must thus expect that the derivatives only enter the expression for R^i_i (without summation) through the α_{ik} and $\alpha_{ik\ell}$, since R^i_i is invariant under the special transformations.

In what follows, we suspend the usual summation convention for indices denoted by Latin letters.

The Christoffel symbols which are not identically zero are ($i \neq k$)

$$\left.\begin{aligned} \Gamma^{i}_{ii} &= \frac{1}{a_i}\frac{\partial a_i}{\partial x_i} = a_i \alpha_{ii} \\ \Gamma^{k}_{ii} &= -\frac{a_i}{a_k^2}\frac{\partial a_i}{\partial x_k} = -\frac{a_i^2}{a_k}\alpha_{ik} \\ \Gamma^{k}_{ik} &= \frac{1}{a_k}\frac{\partial a_k}{\partial x_i} = a_i \alpha_{ki} . \end{aligned}\right\} \qquad (1.6)$$

We calculate first of all the contracted Riemann tensor R_{ii}, $\mu = \upsilon = i$. In the summations, we make explicit the summation index values equal to i, and those k, ℓ different from i and from one another, and replace the Christoffel symbols by their values (1.6).

We thus obtain

$$\begin{aligned} R^i_i = {} & \frac{1}{a_i^2}\frac{\partial}{\partial x_k}\left[\frac{a_i}{a_k^2}\frac{\partial a_i}{\partial x_k}\right] + \frac{1}{a_i^2}\frac{\partial}{\partial x_i}\left[\frac{1}{a_k}\frac{\partial a_k}{\partial x_i}\right] \\ & - \alpha_{ii}\alpha_{ii} - \alpha_{ii}\alpha_{ki} + \alpha_{ik}\alpha_{ik} + \alpha_{ik}\alpha_{kk} + \alpha_{i\ell}\alpha_{k\ell} \\ & + \alpha_{ii}\alpha_{ii} - 2\alpha_{ik}\alpha_{ik} + \alpha_{ki}\alpha_{ki} . \end{aligned}$$

Carrying out the differentiations and substituting from (1.4) and (1.5) gives

$$R_i^i = \alpha_{ikk} + \alpha_{kii} - \alpha_{ii}\alpha_{ki} - \alpha_{ik}\alpha_{kk} + \alpha_{i\ell}\alpha_{k\ell} \tag{1.7}$$

an expression taken to be summed over k and ℓ different from i and each other.

This expression can be written

$$R_i^i = \sum_k \beta_{ik} \tag{1.8}$$

where the β_{ik} are taken to be zero for $i = k$ and for which the expressions for $i \neq k$ can, from (1.4) and (1.5), be written

$$\beta_{ik} = \frac{1}{a_i a_k}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{a_k}\frac{\partial a_i}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{a_i}\frac{\partial a_k}{\partial x_i}\right) + \sum_\ell \frac{\partial a_i}{\partial x_\ell}\frac{\partial a_k}{\partial x_\ell}\right] \tag{1.9}$$

the sum over ℓ being taken over values different from i and k.

The completely contracted scalar R is obtained by making the further sum over i. It contains each β_{ik} twice and we can write

$$\tfrac{1}{2}R = \textstyle\sum_{i<k} \beta_{ik} \, . \tag{1.10}$$

The gravitational equations (1.1) are thus written, for $\mu = \upsilon = i$,

$$\kappa T_i^i + \lambda = \sum \beta_{kl} \tag{1.11}$$

where the summation is taken without repetition ($k < \ell$) and for values k and ℓ different from i, i.e. explicitly,

$$\left.\begin{aligned} \kappa T_1^1 + \lambda &= \beta_{23} + \beta_{24} + \beta_{34} \\ \kappa T_2^2 + \lambda &= \beta_{13} + \beta_{14} + \beta_{34} \\ \kappa T_3^3 + \lambda &= \beta_{12} + \beta_{14} + \beta_{24} \\ \kappa T_4^4 + \lambda &= \beta_{12} + \beta_{13} + \beta_{23} \, . \end{aligned}\right\} \tag{1.12}$$

It remains for us to calculate the components R_{ik} for $i \neq k$. Using the same method, we obtain

$$\begin{aligned} \frac{R_{ik}}{a_i a_k} = \frac{1}{a_i a_k}\Bigg[& - \frac{\partial}{\partial x_i}\left(\frac{1}{a_i}\frac{\partial a_i}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(\frac{1}{a_k}\frac{\partial a_k}{\partial x_i}\right) \\ & + \frac{\partial}{\partial x_k}\left(\frac{1}{a_i}\frac{\partial a_i}{\partial x_i}\right) + \frac{\partial}{\partial x_k}\left(\frac{1}{a_k}\frac{\partial a_k}{\partial x_i}\right) + \frac{\partial}{\partial x_k}\left(\frac{1}{a_\ell}\frac{\partial a_\ell}{\partial x_i}\right)\Bigg] \\ & - \alpha_{ik}\alpha_{ii} - \alpha_{ik}\alpha_{ki} - \alpha_{ik}\alpha_{\ell i} \\ & - \alpha_{ki}\alpha_{kk} - \alpha_{ki}\alpha_{ik} - \alpha_{ki}\alpha_{\ell k} \\ & + \alpha_{ii}\alpha_{ik} + \alpha_{kk}\alpha_{ki} + 2\alpha_{ik}\alpha_{ki} + \alpha_{\ell i}\alpha_{\ell k} \end{aligned}$$

that is, on carrying out the differentiations and substituting from (1.4) and (1.5),

$$\frac{R_{ik}}{a_i a_k} = \alpha_{\ell ik} - \alpha_{\ell i}\alpha_{ik} - \alpha_{\ell k}\alpha_{ki}$$

or by (1.1)

$$-\kappa T_{ik} = \sum_{\ell} \frac{1}{a_\ell}\left[\frac{\partial^2 a_\ell}{\partial x_i \partial x_k} - \frac{1}{a_i}\frac{\partial a_\ell}{\partial x_i}\frac{\partial a_i}{\partial x_k} - \frac{1}{a_k}\frac{\partial a_\ell}{\partial x_k}\frac{\partial a_k}{\partial x_i}\right] \tag{1.13}$$

the sum over ℓ being taken for values different from i and k.

2. SPHERICAL SYMMETRY

By spherical symmetry, we mean the case where two of the coordinates x_2 and x_3 appear in the ds^2 only through the expression

$$dx_2^2 + \sin^2 x_2\, dx_3^2 \tag{2.1}$$

or an equivalent expression.

The ds^2 is thus invariant under the transformations of x_2 and x_3 which leave this expression invariant and which form the group of rotations of a unit sphere about its centre.

In this case a_1, a_2, and a_4 are functions of x_1 and x_4 only, and

$$a_3 = a_2 \sin x_2\,. \tag{2.2}$$

All the derivatives with respect to x_3 are thus zero, as are the derivatives with respect to x_2, except for the first derivatives

$$\frac{\partial a_3}{\partial x_2} = a_2 \cos x_2\,. \tag{2.3}$$

For the second derivatives one has in particular

$$\frac{1}{a_3 a_2}\frac{\partial}{\partial x_2}\left(\frac{1}{a_2}\frac{\partial a_3}{\partial x_2}\right) = -\frac{1}{a_2^2}\,. \tag{2.4}$$

The equations (1.9) thus become

$$\left.\begin{aligned}
\beta_{23} &= \frac{1}{a_2^2}\left[-1 + \frac{1}{a_1^2}\left(\frac{\partial a_2}{\partial x_1}\right)^2 + \frac{1}{a_4^2}\left(\frac{\partial a_2}{\partial x_4}\right)^2\right] \\
\beta_{12} = \beta_{13} &= \frac{1}{a_1 a_2}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{a_1}\frac{\partial a_2}{\partial x_1}\right) + \frac{1}{a_4^2}\frac{\partial a_1}{\partial x_4}\frac{\partial a_2}{\partial x_4}\right] \\
\beta_{24} = \beta_{34} &= \frac{1}{a_2 a_4}\left[\frac{\partial}{\partial x_4}\left(\frac{1}{a_4}\frac{\partial a_2}{\partial x_4}\right) + \frac{1}{a_1^2}\frac{\partial a_4}{\partial x_1}\frac{\partial a_2}{\partial x_1}\right] \\
\beta_{14} &= \frac{1}{a_1 a_4}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{a_1}\frac{\partial a_4}{\partial x_1}\right) + \frac{\partial}{\partial x_4}\left(\frac{1}{a_4}\frac{\partial a_1}{\partial x_4}\right)\right]
\end{aligned}\right\} \tag{2.5}$$

while (1.13) gives

$$-\kappa T_{14} = \frac{2}{a_2^2}\left[\frac{\partial^2 a_2}{\partial x_1 \partial x_4} - \frac{1}{a_1}\frac{\partial a_2}{\partial x_1}\frac{\partial a_1}{\partial x_4} - \frac{1}{a_4}\frac{\partial a_2}{\partial x_4}\frac{\partial a_4}{\partial x_1}\right] \tag{2.6}$$

$$T_{12} = T_{13} = T_{23} = T_{24} = T_{34} = 0\,.$$

The coordinates x_1 and x_4 are so far chosen arbitrarily.

When the matter tensor is non-zero, there is a natural split into space and time imposed by the matter; one can in fact determine the worldlines such that if one chooses x_1 constant along those lines one has $T_{14} = 0$. The curves of constant x_4 are then the orthogonal trajectories of the curves of constant x_1.

In what follows, we will stick to the study of the field when the coordinates have been thus chosen.

It is important to note that this way of working does not reduce the generality of the results obtained at all.

In certain cases, the choice of coordinates may be more or less indeterminate. It can also happen that the introduction of these coordinates produces analytical singularities which demand special study.

For the coordinates such that $T_{14} = 0$ it is convenient to make use of the conservation theorem

$$T^{\upsilon}_{\mu,\upsilon} = 0$$

which gives the two relations

$$\frac{\partial T_1^1}{\partial x_1} + \frac{2}{a_2}\frac{\partial a_2}{\partial x_1}\left(T_1^1 - T_2^2\right) + \frac{1}{a_4}\frac{\partial a_4}{\partial x_1}\left(T_1^1 - T_4^4\right) = 0 \tag{2.7}$$

$$\frac{\partial T_4^4}{\partial x_4} + \frac{2}{a_2}\frac{\partial a_2}{\partial x_4}\left(T_4^4 - T_2^2\right) + \frac{1}{a_1}\frac{\partial a_1}{\partial x_4}\left(T_4^4 - T_1^1\right) = 0, \tag{2.8}$$

expressing the theorem of energy conservation and the balance equation (zero momentum).

Eliminating T_2^2 between these two equations and grouping the terms in T_1^1 and T_4^4 gives

$$\frac{\partial a_2}{\partial x_4}\frac{\partial T_1^1}{\partial x_1} + \frac{2}{a_2}\frac{\partial a_2}{\partial x_1}\frac{\partial a_2}{\partial x_4}T_1^1 + \left(\frac{1}{a_4}\frac{\partial a_2}{\partial x_4}\frac{\partial a_4}{\partial x_1} + \frac{1}{a_1}\frac{\partial a_2}{\partial x_1}\frac{\partial a_1}{\partial x_4}\right)T_1^1$$

$$= \frac{\partial a_2}{\partial x_1}\frac{\partial T_4^4}{\partial x_4} + \frac{2}{a_2}\frac{\partial a_2}{\partial x_1}\frac{\partial a_2}{\partial x_4}T_4^4 + \left(\frac{1}{a_4}\frac{\partial a_2}{\partial x_4}\frac{\partial a_4}{\partial x_1} + \frac{1}{a_1}\frac{\partial a_2}{\partial x_1}\frac{\partial a_1}{\partial x_4}\right)T_4^4$$

and using (2.6) with $T_{14} = 0$ and multiplying by a_2^2,

$$\frac{\partial}{\partial x_1}\left[T_1^1 a_2^2 \frac{\partial a_2}{\partial x_4}\right] = \frac{\partial}{\partial x_4}\left[T_4^4 a_2^2 \frac{\partial a_2}{\partial x_1}\right]. \tag{2.9}$$

This leads us to consider whether there exists an expression Φ in the a and their derivatives such that

$$T_1^1 a_2^2 \frac{\partial a_2}{\partial x_4} = \frac{\partial \Phi}{\partial x_4} \tag{2.10}$$

$$T_4^4 a_2^2 \frac{\partial a_2}{\partial x_1} = \frac{\partial \Phi}{\partial x_1}. \tag{2.11}$$

Because of the symmetry between the indices 1 and 4 which remains in our formulae, it is sufficient to prove this for one of the two cases, for example for (2.11).

We have, by (1.12) and (2.5),

$$\begin{aligned}(\kappa T_4^4 + \lambda) a_2^2 \frac{\partial a_2}{\partial x_1} &= (\beta_{23} + 2\beta_{12}) a_2^2 \frac{\partial a_2}{\partial x_1} \\ &= -\frac{\partial a_2}{\partial x_1} + \frac{1}{a_1^2}\left(\frac{\partial a_2}{\partial x_1}\right)^3 + \frac{2a_2}{a_1}\frac{\partial a_2}{\partial x_1}\frac{\partial}{\partial x_1}\left(\frac{1}{a_1}\frac{\partial a_2}{\partial x_1}\right) \\ &\quad + \frac{1}{a_4^2}\left(\frac{\partial a_2}{\partial x_4}\right)^2 \frac{\partial a_2}{\partial x_1} + \frac{2a_2}{a_1 a_4^2}\frac{\partial a_2}{\partial x_1}\frac{\partial a_1}{\partial x_4}\frac{\partial a_2}{\partial x_4}.\end{aligned}$$

Taking into account (2.6) ($T_{14} = 0$), the last term can be written

$$\begin{aligned}&\frac{2a_2}{a_4^2}\left[\frac{\partial^2 a_2}{\partial x_1 \partial x_4} - \frac{1}{a_4}\frac{\partial a_2}{\partial x_4}\frac{\partial a_4}{\partial x_1}\right]\frac{\partial a_2}{\partial x_4} \\ &\quad = \frac{2a_2}{a_4}\frac{\partial a_2}{\partial x_4}\frac{\partial}{\partial x_1}\left[\frac{1}{a_4}\frac{\partial a_2}{\partial x_4}\right] = a_2 \frac{\partial}{\partial x_1}\left[\frac{1}{a_4^2}\left(\frac{\partial a_2}{\partial x_4}\right)^2\right].\end{aligned}$$

So this gives

$$(\kappa T_4^4 + \lambda) a_2^2 \frac{\partial a_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left\{a_2\left[-1 + \frac{1}{a_1^2}\left(\frac{\partial a_2}{\partial x_1}\right)^2 + \frac{1}{a_4^2}\left(\frac{\partial a_2}{\partial x_4}\right)^2\right]\right\}$$

which justifies the relation (2.11) with

$$\Phi = \frac{a_2}{\kappa}\left[-1 + \frac{1}{a_1^2}\left(\frac{\partial a_2}{\partial x_1}\right)^2 + \frac{1}{a_4^2}\left(\frac{\partial a_2}{\partial x_4}\right)^2 - \frac{\lambda a_2^2}{3}\right]. \tag{2.12}$$

3. SUMMARY OF RESULTS FOR SPHERICAL SYMMETRY

Before discussing the equations we have just obtained and showing their significance and the analogies they provide with the formulae of classical mechanics, we must go back through them using notation better adapted to applications.

Let us consider a ds^2 of the form

$$ds^2 = -a^2 d\chi^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + c^2 dt^2. \tag{3.1}$$

$-a^2$, $-r^2$, and $-c^2$ are the functions of $x_1 = \chi$ and $x_4 = t$ previously denoted by a_1^2, a_2^2, and a_4^2. We also write

$$T_4^4 = \rho, \qquad T_1^1 = -p, \qquad T_2^2 = T_3^3 = -\tau. \tag{3.2}$$

Einstein's constant is

$$\kappa = \frac{8\pi K}{c_0^2}$$

where K is the gravitational constant and c_0 the speed of light. In place of Φ we introduce a function $m = -4\pi i\Phi$.

The equations (2.10) and (2.11) are written in this manner

$$4\pi\rho r^2 \frac{\partial r}{\partial\chi} = \frac{\partial m}{\partial\chi} \tag{3.3}$$

$$4\pi p r^2 \frac{\partial r}{\partial t} = -\frac{\partial m}{\partial t}. \tag{3.4}$$

The first of these is the classical equation between distance, density and mass.

The equation (2.12) can be written

$$\frac{c_0^2}{c^2}\left(\frac{\partial r}{\partial t}\right)^2 = -c_0^2\left[1 - \frac{1}{a^2}\left(\frac{\partial r}{\partial\chi}\right)^2\right] + \frac{2Km}{r} + \frac{\lambda c_0^2}{3} r^2. \tag{3.5}$$

It is analogous to the classical equation for energy under the action of various forces, among which one recognizes the Newtonian gravitational force.

The equation (2.6) ($T_{14} = 0$) can be written

$$\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial r}{\partial\chi}\right) = \frac{1}{ac}\frac{\partial r}{\partial t}\frac{\partial c}{\partial\chi}. \tag{3.6}$$

Differentiating (3.5) and taking into account (3.4) and (3.6) one obtains, after removing a factor $2\partial r/\partial t$,

$$\frac{c_0}{c}\frac{\partial}{\partial t}\left(\frac{c_0}{c}\frac{\partial r}{\partial t}\right) = \frac{c_0^2}{ca^2}\frac{\partial r}{\partial \chi}\frac{\partial c}{\partial \chi} - 4\pi K pr - \frac{Km}{r^2} + \frac{\lambda c_0^2}{3} r. \tag{3.7}$$

This equation is particularly useful when $\partial r/\partial t$ vanishes, in which case the equation (3.4) becomes empty. It is easy to show directly that (3.7) still applies in this case.

Finally, the conservation theorems (2.7), (2.8) are written

$$\frac{\partial p}{\partial \chi} + \frac{2}{r}\frac{\partial r}{\partial \chi}(p - \tau) + \frac{1}{c}\frac{\partial c}{\partial \chi}(p + \rho) = 0, \tag{3.8}$$

$$\frac{\partial \rho}{\partial t} + \frac{2}{r}\frac{\partial r}{\partial t}(p + \tau) + \frac{1}{a}\frac{\partial a}{\partial t}(p + \rho) = 0\,. \tag{3.9}$$

In this form, the equations become remarkably intuitive. The coordinate χ is attached to the matter and plays the role of initial values of coordinates in classical hydrodynamics. r is analogous to the distance variable from the origin; in fact, r is the distance that can be evaluated starting from the normal measures of a radius vector. Equations (3.5) and (3.7) are therefore the equations of motion of the matter, m corresponding to the mass within a moving material sphere of radius χ.

The equation (3.8) is analogous to the balance equation, $(1/c)(\partial c/\partial \chi)$ playing the role of the gravitational force remaining after the removal of the effect of the moving frame.[6]

4. QUASI-STATIC FIELDS

Let us consider the case where

$$\frac{\partial r}{\partial t} \equiv 0,$$

where the matter therefore is in equilibrium. We then have, by (3.6),

$$\frac{\partial a}{\partial t} \equiv 0,$$

[6] The French 'réaction d'entrainement' refers to the 'fictitious forces' in a non-inertial frame — *Transl.*

and, by (3.4),

$$\frac{\partial m}{\partial t} \equiv 0,$$

and thus by (3.3) or (3.9)

$$\frac{\partial \rho}{\partial t} \equiv 0\,.$$

However c is not necessarily time-independent. It is for this reason that we give this case the name quasi-static, in contrast to the static fields where c is time-independent or can be made independent of time by a change of variables.

One has, by (3.5),

$$a^2 d\chi^2 = \frac{dr^2}{1 - (2Km/c_0^2 r) - (\lambda/3)r^2}\,, \tag{4.1}$$

with, by (3.3),

$$4\pi\rho r^2 = \frac{dm}{dr}\,, \tag{4.2}$$

Equation (3.7) becomes

$$\frac{4\pi K}{c_0^2} p + \frac{Km}{c_0^2 r^3} - \frac{\lambda}{3} = \left(1 - \frac{2Km}{c_0^2 r} - \frac{\lambda}{3} r^2\right) \frac{1}{cr} \frac{\partial c}{\partial r}\,, \tag{4.3}$$

while (3.8) is written as

$$\frac{\partial p}{\partial r} + \frac{2}{r}(p - \tau) + \frac{1}{c}\frac{\partial c}{\partial r}(p + \rho) = 0\,. \tag{4.4}$$

Naturally these equations concern only the mechanical part of the problem which can only be determined when we have some information on the nature of the matter with which we are dealing. We have available 4 equations between 6 variables a, ρ, p, τ, m and c ; we require two supplementary conditions. For example we could consider a fluid

$$p = \tau$$

with a given distribution of matter ρ as a function of r.

5. UNIFORM ENERGY DENSITY

Let us consider in particular the case where ρ is independent not only of t but also of χ. One can then, by a change of variable, make a constant, and choose the value of that constant. We take

$$\frac{1}{a^2} = \frac{8\pi K}{3c_0^2}\rho + \frac{\lambda}{3} \tag{5.1}$$

and obtain, by (4.1) and (3.3),

$$r = a \sin\chi \tag{5.2}$$

and (4.3) becomes

$$\frac{4\pi K}{c_0^2}p + \frac{1}{2a^2} - \frac{\lambda}{2} = \frac{\cot\chi}{a^2 c}\frac{\partial c}{\partial\chi}. \tag{5.3}$$

For a fluid, (4.4) becomes

$$\frac{\partial p}{\partial\chi} + \frac{1}{c}\frac{\partial c}{\partial\chi}(p + \rho) = 0, \tag{5.4}$$

whence, since ρ is constant,

$$\frac{4\pi K}{c_0^2}(p + \rho) = \frac{f_1(t)}{ca^2}.$$

Substituting into (5.3), taking into account (5.1), and integrating, gives

$$c = f_1(t) - f_2(t)\cos\chi. \tag{5.5}$$

We thus obtain

$$ds^2 = -a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] + [f_1(t) - f_2(t)\cos\chi]^2 dt^2 \tag{5.6}$$

with

$$3\kappa p = \frac{3\kappa\rho f_2(t)\cos\chi - (\kappa\rho - 2\lambda)f_1(t)}{f_1(t) - f_2(t)\cos\chi}. \tag{5.7}$$

The pressure can be zero at

$$\cos\chi_1 = \frac{(\kappa\rho - 2\lambda)f_1(t)}{3\kappa\rho f_2(t)} \tag{5.8}$$

and infinite at

$$\cos\chi_0 = \frac{f_1(t)}{f_2(t)}. \tag{5.9}$$

When the functions $f_1(t)$ and $f_2(t)$, or at least their ratio, are reduced to a constant, one recovers Schwarzschild's known results.

For $f_2(t) = 0$ and $\kappa\rho = 2\lambda$, we obtain the Einstein universe. If we make $f_2(t)$ vary, we obtain a progressive loading of the universe, the pressure varying according to the law

$$p = \frac{\rho f_2(t)\cos\chi}{f_1(t) - f_2(t)\cos\chi}. \tag{5.10}$$

One can imagine this pressure to be exerted at the origin $\chi = 0$, and distributing itself throughout the incompressible fluid while maintaining the equilibrium. The pressure decreases outwards from the centre, and vanishes at the polar plane of the centre, $\chi = \pi/2$.

Things happen differently for an Einstein universe of the simple elliptic form or for a universe with distinct antipodal points. In the latter case, χ varies from 0 to π and the pressure is different in the two parts separated by the plane $\chi = \pi/2$; it is negative in the other half of the space, and different in absolute value at corresponding points.

These results are naturally without direct interest in the study of the real universe, which can never be compared with an incompressible fluid. They have however the interest of showing how the universe could stay in equilibrium even though its rest mass varies.

This itself is easily calculated; one has

$$M(\chi) = \int_0^{\chi} 4\pi a^3(\rho - 3p)\sin^2\chi\, d\chi,$$

where p is given by (5.10).

Setting

$$\sin\beta = \frac{f_2(t)}{f_1(t)},$$

one finds

$$M(\chi) = 4\pi a^3\rho\left\{\left(2 - \frac{3}{\sin^2\beta}\right)\chi - \sin 2\chi - \frac{3\sin\chi}{\sin\beta} + 6\frac{\cos\beta}{\sin^2\beta}\operatorname{arc\,tg}\left[\operatorname{tg}\frac{\chi}{2}\operatorname{tg}\left(\frac{\pi}{4} + \frac{\beta}{2}\right)\right]\right\}$$

and for the rest mass of the universe with distinct antipodal points

$$M(\pi) = 2\pi^2 a^3 \rho\left(1 - 3\,\mathrm{tg}^2\,\frac{\beta}{2}\right).$$

For $f_1 = 0$ and $\rho = 0$ we obtain the de Sitter universe.

A consequence of the Schwarzschild interior solution is that it appears to impose, for the minimum radius of a sphere of given mass, a more severe limit than that imposed by the exterior solution.

This limit is obtained for

$$f_1(t) = f_2(t),$$

in which case the pressure is infinite at the centre.

One has then (for $\lambda = 0$) by (5.8)

$$\cos\chi_1 = \tfrac{1}{3},$$

whence, for the corresponding radius,

$$r = a\sin\chi_1 = a\sqrt{8/9},$$

while the exterior problem allows a radius of whatever multiple of a one likes.

This limitation holds only because one has supposed the matter to be a fluid.

Let us consider, in fact, matter maintaining itself like an arch under the action of transversal forces. The radial pressure p can be zero or more generally constant.

In this case, the equation (5.8) can still be integrated and gives

$$c = f_3(t)[\cos\chi]^{-(1/2)(1-\lambda a^2+\kappa p a^2)},$$

while the balance equation (3.8) gives

$$\tau - p = \frac{\mathrm{tg}\,\chi}{2c}(p+\rho)\frac{\partial c}{\partial\chi},$$

that is

$$\tau - p = \frac{\mathrm{tg}^2\,\chi}{4}(\rho + p)(1 - \lambda a^2 + \kappa p a^2).$$

In particular for $p = 0$ and $\lambda = 0$, one has

$$ds^2 = -a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] + f_3^2(t)\frac{dt^2}{\cos\chi} \tag{5.11}$$

with

$$\tau = \frac{\rho}{4}\,\mathrm{tg}^2\chi \quad \text{and} \quad \kappa\rho = \frac{3}{a^2}. \tag{5.12}$$

One can thus construct a sphere supporting itself by transversal tensions and filling space as completely as one wishes.

The lower limit of radius for a given mass is thus determined by the exterior field and not by the solution of the interior problem, if one does not impose the condition that the matter is fluid. The two solutions can be combined. One can imagine a liquid, water for example, part of which is frozen and forms concentric spheres of ice which are self-supporting, independently of one another, by normal tensions. These spheres are then adiabatically melted starting from the centre, giving the Schwarzschild fluid. The Schwarzschild solution can at each instant be related to the solution $p = 0$ by choosing suitably the values of the functions $f_1(t)$, $f_2(t)$, $f_3(t)$. One can thus progressively increase the radius of the melted region until the central pressure becomes infinite and the Schwarzschild problem has no solution. This shows clearly the really paradoxical nature of Schwarzschild's result.

6. EDDINGTON'S PROBLEM

Eddington has suggested that one could more naturally consider for the problem of the homogeneous fluid sphere the case where the density of rest mass

$$\delta = T = T_4^4 + 3T_1^1 = \rho - 3p, \tag{6.1}$$

and not ρ, is considered constant.

The equations of the problem are, eliminating c between (4.3) and (4.4),

$$\frac{4\pi K}{c_0^2}p + \frac{Km}{c_0^2 r^3} - \frac{\lambda}{3} = -\left(1 - \frac{2Km}{c_0^2 r} - \frac{\lambda}{3}r^2\right)\frac{1}{\delta + 4p}\frac{\partial p}{r\partial r} \tag{6.2}$$

$$4\pi(\delta + 3p)r^2 = \frac{dm}{dr}$$

where the two unknown functions are p and m.

It is convenient to use, in place of m, the average pressure q defined by

$$q = \frac{3}{r^3}\int_0^r pr^2 dr.$$

One then has

$$m = \frac{4\pi r^3}{3}(\delta + 3q).$$

If we put

$$\frac{\kappa p - \lambda}{x} = \frac{\kappa q - \lambda}{y} = \frac{\kappa\delta + 4\lambda}{12} = \frac{u}{r^2}, \tag{6.3}$$

the equations become

$$\frac{dx}{du} + \frac{(x + y + 4)(x + 3)}{1 - (y + 4)u} = 0 \tag{6.4}$$

$$\frac{dy}{du} + \frac{3(y - x)}{2u} = 0. \tag{6.5}$$

The solutions $x = y = -2$, and $x = y = -3$ correspond respectively to the Einstein and de Sitter universes.

It is equally easy to study the behaviour of x and y for large values of these variables. One can then neglect the numerical terms added to x or y.

Setting

$$x = \frac{X}{u}, \qquad y = \frac{Y}{u} \tag{6.6}$$

one can eliminate u and find

$$\frac{dY}{dX} = \frac{3X - Y}{1 - X - 2Y}\,\frac{1 - Y}{2X}. \tag{6.7}$$

The solution of this equation corresponding to finite initial values of x and y is the particular solution passing through the origin. It is easy to discuss the behaviour of this solution and show that, starting from the origin at an angle of 45°, it winds round in an anticlockwise[7] spiral and tends asymptotically to the point

$$X = \tfrac{1}{7}, \qquad Y = \tfrac{3}{7}.$$

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[7] In the original, 'sens direct'.

From this it follows that X passes successively through a maximum X_1, a minimum X_2, etc., and that the curves of x are successively tangent to hyperbolae

$$x = \frac{X_k}{u}.$$

When one varies the initial values, the points of contact are displaced and the hyperbolae form so many envelopes of the curves of x.

One may expect that these general characteristics survive in the form of the solutions even when x and y are no longer small.

In fact, it follows from the numerical calculations which were the subject of an unpublished thesis, presented in 1927 to the Massachusetts Institute of Technology, that the first envelope can be represented up to values of x close to -2 by the formula

$$x = \frac{0.220}{u} - 2.65$$

while the asymptotic limit can be expanded in a series

$$x = \frac{1}{7u} - 2.8571 + 0.168u + 0.22u^2 + \ldots$$

From this it follows that when one increases the central pressure, the radius ($p = 0$) at first increases, passes through a maximum on the first envelope, then decreases to the second envelope, then increases again and tends in an oscillatory manner to a limit point on the limit of the envelopes.

For $\lambda = 0$, the first maximum takes place at

$$u = 0.083$$

and the limiting point is at

$$u = 0.05.$$

One can easily enough give an account of the mechanism of this apparently paradoxical result.

When the central pressure increases, one naturally tends to increase the radius, but at the same time one increases the energy content of the matter

$$\rho = \delta + 3p.$$

The gravitational effect of this energy eventually compensates the effect of the pressure and the two influences take turns to prevail.

In other words, under Eddington's hypothesis there is no longer any question of adiabatic variations; one cannot increase the pressure without adding energy to the exterior and the gravitational effect of this additional energy eventually dominates.

For certain radii, there exist several equilibrium configurations; it is not clear whether these configurations are unstable, except for that of minimum energy.

7. INSTABILITY OF THE EINSTEIN UNIVERSE

Having studied the quasi-static spherical fields, we intend to examine how the breakdown of equilibrium of a quasi-static field can be produced and in particular the breakdown of the equilibrium of the Einstein universe.

We imagine that by a process which we try to keep as general as possible one modifies either the equation of state of the matter or its distribution. We suppose that at the moment of the breakdown of equilibrium one has still

$$\frac{\partial r}{\partial t} = 0 \tag{7.1}$$

and consequently

$$\frac{\partial a}{\partial t} = \frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t} = 0 \tag{7.2}$$

as for the quasi-static fields; but these relations are no longer maintained as identities. We go back to (3.7) for the acceleration, and, taking account of the relations (7.1) and (7.2), we see that the breakdown of equilibrium can only come about through a modification of p or $\partial c/\partial \chi$.

We have seen above examples of such modifications, but then these modifications were adjusted so as not to upset the equilibrium.

It is clear that if p and $\partial c/\partial \chi$ do not change, it is impossible to break equilibrium, and that is true even if p and $\partial c/\partial \chi$ change at points other than the one under consideration. If one sets the interior region in motion, for example, taking care to preserve the spherical symmetry, that will have no effect on the exterior region, since the pressure and the force of gravity $\partial c/\partial \chi$ would not be modified there.

The condition

$$\frac{\partial c}{\partial \chi} = 0$$

can still be considered as the condition that the worldlines with constant χ defined by the matter are geodesics.

To study the breakdown of the equilibrium of the Einstein universe due to the effect of the formation of local condensations distributed uniformly in space, we imagine a large number of centres of condensation distributed more or less uniformly. There is no way to suppose them distributed in a perfectly homogeneous manner for, in an elliptic space, there is no equivalent of the cubic lattices or the space-filling spheres[8] of Euclidean space. But statistically the distribution can be assumed uniform.

The condensation process is supposed to develop in a similar manner around each centre of condensation, and there naturally exists a network of surfaces, forming cells around the centres of condensation, which are the loci of points which are no more under the influence of one of the two condensations which they separate than the other. These cells form the neutral zone between the gravitational fields of the condensations.

By virtue of the global homogeneity which we have assumed, it is clear that all the cells behave in the same way; they are all in equilibrium, or they dilate or contract together. It thus suffices to consider just one of them in order to work out the equilibrium or motion of the whole universe.

Fixing our attention on one cell, the neutral zone of a particular condensation, we suppose that that condensation enjoys spherical symmetry, and that we can take account of the influence of neighbouring condensations by replacing them with a spherically symmetric distribution of matter. The neutral zone is then a sphere.

The points of this sphere enjoy the property that their worldlines are geodesics, or that the force of gravity still vanishes there, since neither the internal condensation nor the neighbouring condensations have a preponderant influence there. One must thus have at the neutral zone

$$\frac{\partial c}{\partial \chi} = 0,$$

and consequently the equilibrium can only be broken if the modifications introduced into the state of the matter have made p, the radial pressure at the neutral zone, vary.

Thus if we want to compare a universe which is globally homogeneous but contains a large number of uniformly distributed condensations with the perfectly homogeneous Einstein universe, we have to consider the network of cells formed by the neutral zones separating the condensations. The homogeneous universe must, so to say, be tangent at those points to the universe presenting the condensations, and the pressure normal to the neutral zones must be the pressure adopted for the homogeneous universe. Then the equilibrium, or the expansion, of the homogeneous universe gives us the equilibrium or expansion of the network of neutral zones.

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[8] In the original, ‘piles de boulets’

The two universes can have different masses or different volumes. One can conclude nothing from that, the determining factor being the pressure at the neutral zone.

The interest of this result is that it is completely independent of the particular process which the development of the condensations follows from. It provides the means, for any particular process, to foresee the effect of that process on the equilibrium of the universe.

In particular if the pressure is zero and remains zero in the neutral zones, the condensations do not affect the equilibrium. The radial pressure at the neutral zone is the energy density crossing that zone, and thus measures the intensity of the exchanges between condensations. We have called a reduction of such exchanges of energy a 'stagnation of the universe'. Only this process of stagnation can determine the breakdown of the equilibrium in the sense of expansion.

8. CONDENSATIONS IN THE EXPANDING UNIVERSE

In applications to the real universe the pressure is generally negligible compared with the density. In the case of equilibrium we have had to take it into account, because the study of a breakdown of equilibrium naturally depends on minimal forces, but for the study of the expansion of the universe and the development of condensations in the course of the expansion, we can neglect it.

In this case, the equation (3.4) tells us that m is a function only of χ, and equation (3.8), for $p = \tau = 0$, that c is a function of t alone.

By means of a change of variable, we can thus assume c constant and put

$$c = c_0 .$$

We then have, by (3.6)

$$\frac{1}{a}\frac{\partial r}{\partial \chi} = f(\chi),$$

and (3.1) becomes

$$ds^2 = -\left(\frac{\partial r}{\partial \chi}\right)^2 \frac{d\chi^2}{f^2(\chi)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + c^2 dt^2 \tag{8.1}$$

where r is a function of χ and of t satisfying (3.5).

$$\left(\frac{\partial r}{\partial t}\right)^2 = -c^2[1 - f^2(\chi)] + \frac{2Km}{r} + \frac{\lambda c^2}{3} r^2 \tag{8.2}$$

where by (3.3)

$$4\pi\rho r^2 \frac{\partial r}{\partial \chi} = \frac{dm}{d\chi}\,. \tag{8.3}$$

Finally, equation (3.7) becomes

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Km}{r^2} + \frac{\lambda c^2}{3} r. \tag{8.4}$$

The element of length at an instant t is, from (8.1),

$$d\sigma^2 = \frac{dr^2}{f^2(\chi)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

When $f(\chi) = 1$, the geometry is thus Euclidean. The equations then differ from the equations of classical mechanics only by the introduction of a cosmic repulsion and, in addition, by the fact that the constant energy in (8.2), which, from the classical point of view, can have an arbitrary value, is now zero.

In the general case, one still can consider r as the distance from the origin, and the energy constant at each material point, that is to say, at each value of χ, can be chosen arbitrarily. But the geometry is then not Euclidean. One can make a map of it in a Euclidean space where the lengths normal to the radius vector are represented at their real size. The lengths along the radial vector are then represented at a scale

$$\frac{dr}{d\sigma} = f(\chi).$$

The radial length scale depends only on χ, that is to say it stays the same for each material point throughout its motion, and it is linked to the energy constant in the equation of motion of that point from equation (8.2).

The coordinate χ can naturally be chosen arbitrarily. When $f(\chi)$ is less than or equal to one, one could choose the coordinate χ in such a way that

$$f(\chi) = \cos\chi.$$

Then (8.2) is written more simply,

$$\left(\frac{\partial r}{\partial t}\right)^2 = -c^2\sin^2\chi + \frac{2Km}{r} + \frac{\lambda c^2}{3} r^2. \tag{8.21}$$

This coordinate choice is convenient when the space is closed. For a space of the simple elliptic type, the whole space is described when χ varies from 0 to $\pi/2$.

It is important to note that m is not the real mass within a sphere χ, but rather the mass calculated starting from the density without taking the curvature of the space into account. The real mass is

$$M(\chi) = \int_0^{\chi} \frac{dm}{\cos\chi} \tag{8.5}$$

and, just like m, it is independent of time.

In the particular case where m is proportional to $\sin^3\chi$, we have

$$m = \frac{4}{3\pi} M \sin^3\chi \tag{8.6}$$

where $M = M(\pi/2)$ is the total mass of the (simply elliptic) universe.

In this case, one can write

$$r = R(t)\sin\chi \tag{8.7}$$

and one obtains the Friedmann universe

$$ds^2 = -R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] + c^2dt^2, \tag{8.8}$$

with

$$\left(\frac{dR}{dt}\right)^2 = -c^2 + \frac{8KM}{3\pi R} + \frac{\lambda c^2}{3}R^2. \tag{8.9}$$

Using the same method as in Section 7, we can study the development of a condensation in the expanding universe. We assume this condensation is spherically symmetric, and we replace the exterior condensations by an averaged density. This comes back to assuming that m is proportional to $\sin^3\chi$ outside the condensation, but it follows another law in the central region.

For the universe in the large, the trajectories of concentric material shells are homothetic (8.7). In the central region on the other hand, they could equally well approach or move away from each other, so marking the progress or attenuation of the condensation.

It may also happen that the trajectories corresponding to different values of χ come to cut one another. In this case our solution becomes inadmissible, for χ is a coordinate and thus cannot have two values at the same point. Physically this means that the hypothesis that we have introduced that the pressure is zero becomes inadmissible from a certain value of χ.

In particular, if the trajectories fall back to the centre, it will not be permissible to treat the problem without introducing pressure. Our aim is simply to study the tendency of condensations to develop, rather than to follow their final arrangement for which we can obviously no longer suppose c (the remaining gravitational potential) to be constant, nor neglect the rotational effects excluded by our hypothesis of spherical symmetry.

It is well known that Friedmann's equations admit the following types of solutions:

1. unlimited expansion from 0 to ∞, when the roots of the right hand side of (8.21) are imaginary:
2. the bounded case with coincident positive roots, r varying from zero to the equilibrium radius, or from that equilibrium distance to infinity;
3. the case of real roots:

(a) a branch bouncing from a minimum to infinity, with as a limiting case the de Sitter solution;
(b) a quasi-periodic branch from zero to a maximum.

These different eventualities arise according to whether

$$\frac{3Km\sqrt{\lambda}}{c^2 \sin^3 \chi}$$

is greater than, equal to, or less than one.

If, for example, m is proportional to $\sin^4 \chi$, the central region will be of the quasi-periodic type finally falling back into the centre, while the exterior region will be of the unlimited expansion type. Such a model thus allows us, subject to the remarks made above, to study the formation of condensations in a universe of the unlimited expansion type.

It is tempting to apply this model to the formation of the nebulae. It seems however preferable to await a further development of the theory which will free us from the hypothesis of spherical symmetry which is manifestly not realised by the spiral nebulae. This development goes outside the scope of this article which considers only exact solutions of the gravitational equations.

In the following section, we expand the Friedmann solution in terms of the Weierstrass elliptic functions. The problem is the same for the universe with condensations and for the homogeneous universe. We consider the first case, and the passage to the homogeneous universe is made by the equations (8.6) and (8.7), or more simply by putting $\chi = \pi/2$, $r = R$, $m = (4M/3\pi)$.

In the case of the homogeneous universe, there is a quantity

$$U = \int \frac{c\,dt}{R} \tag{8.10}$$

which has particular importance: it is the angular distance travelled by light. It can serve as a measure of time. Its meaning is not so immediate for the universe with condensations.

9. INTEGRATION OF THE FRIEDMANN EQUATION BY THE WEIERSTRASS ELLIPTIC FUNCTIONS

Equation (8.21) can be written, when we consider only variation with t,

$$\left(\frac{dr}{dt}\right)^2 = \frac{A^2}{r}(r + 2r_0)[r - r_0(1 - \eta)][r - r_0(1 + \eta)] \tag{9.1}$$

where

$$\left.\begin{aligned} A^2 &= \frac{\lambda c^2}{3} \\ A^2 r_0^2(3 + \eta^2) &= c^2 \sin^2 \chi \\ A^2 r_0^3(1 - \eta^2) &= Km. \end{aligned}\right\} \tag{9.2}$$

Introducing a Weierstrass function $\wp(u)$ having roots

$$e_1 = 6 - 2\eta^2, \qquad e_2 = -3 + 6\eta + \eta^2, \qquad e_3 = -3 - 6\eta + \eta^2, \tag{9.3}$$

and putting

$$\wp(u) = 3 + \eta^2 - 6(1 - \eta^2)\frac{r_0}{r}, \tag{9.4}$$

eq. (9.1) becomes

$$432(1 - \eta^2)^2\left(\frac{du}{dt}\right)^2 = -A^2[\wp(u) - 3 - \eta^2]^2. \tag{9.5}$$

Consider a value v such that

$$\wp(v) = 3 + \eta^2, \tag{9.6}$$

whence

$$[\wp'(v)]^2 = -432(1 - \eta^2)^2. \tag{9.7}$$

This gives

$$\pm A\frac{dt}{du} = \frac{\wp'(v)}{\wp(u) - \wp(v)} = 2\zeta(v) - \zeta(u+v) + \zeta(u-v), \qquad (9.8)$$

whence, on integrating,

$$\pm At = C + 2u\zeta(v) + \log\frac{\sigma(u-v)}{\sigma(u+v)}. \qquad (9.9)$$

Equations (9.4) and (9.9) provide a parametric representation of the motion.

The variable u is proportional to the quantity U introduced at the end of the preceding section; one has in fact

$$U^2 = -12(3+\eta^2)u^2. \qquad (9.9^1)$$

The period ω corresponding to e_1 is calculated from the following formulae:

$$\ell = \frac{\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_1 - e_2}}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}} = \frac{\sqrt[4]{(1+\eta)(3-\eta)} - \sqrt[4]{(1-\eta)(3+\eta)}}{\sqrt[4]{(1+\eta)(3-\eta)} + \sqrt[4]{(1-\eta)(3+\eta)}}. \qquad (9.10)$$

When η is imaginary $= i\bar{\eta}$, one puts

$$\operatorname{tg}\psi = \frac{2\bar{\eta}}{3+\bar{\eta}^2}, \qquad (9.11)$$

and obtains

$$\ell = i\operatorname{tg}\frac{\psi}{4}. \qquad (9.12)$$

Subsequently one has

$$q = \frac{\ell}{2} + 2\left(\frac{\ell}{2}\right)^5 + 15\left(\frac{\ell}{2}\right)^9 + \ldots \qquad (9.13)$$

and

$$\sqrt{\frac{\omega}{2\pi}} = \frac{1 + 2q^4 + 2q^{16} + \ldots}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}} = \frac{1}{\sqrt[4]{3}}\,\frac{1 + 2q^4 + 2q^{16} + \ldots}{\sqrt[4]{(1+\eta)(3-\eta)} + \sqrt[4]{(1-\eta)(3+\eta)}}$$
$$= \frac{1 + 2q^4 + \ldots}{2\cos(\psi/4)}\sqrt[4]{\frac{\cos\psi}{3(3+\bar{\eta}^2)}}. \qquad (9.14)$$

For practical calculations, we must replace $\wp$ and σ by their expressions in terms of the θ functions.

Putting

$$u = \frac{2\omega}{\pi}\alpha, \qquad \beta = \frac{\pi}{2} - \alpha \tag{9.15}$$

we have

$$\begin{aligned}\wp(u) &= e_1 + \sqrt{(e_1 - e_2)(e_1 - e_3)}\left[\frac{\theta_2(\alpha)}{\theta_1(\alpha)}\right]^2 \\ &= e_1 + \sqrt{(e_1 - e_2)(e_1 - e_3)}\left[\frac{\theta_1(\beta)}{\theta_2(\beta)}\right]^2\end{aligned} \tag{9.16}$$

and

$$\sigma(u) = \text{Const.}\, e^{(\pi^2/2\eta\omega)\alpha^2}\,\theta_1(\alpha).$$

One has

$$(e_1 - e_2)(e_1 - e_3) = 9(9 - \eta^2)(1 - \eta^2)$$

and

$$\wp(v) - e_1 = -3(1 - \eta^2).$$

Thus from (9.4) and (9.6) comes

$$\begin{aligned}\frac{2r_0}{r} = \frac{\wp(v) - \wp(u)}{3(1 - \eta^2)} &= -1 - \sqrt{\frac{9 - \eta^2}{1 - \eta^2}}\left[\frac{\theta_2(\alpha)}{\theta_1(\alpha)}\right]^2 \\ &= -1 - \sqrt{\frac{9 - \eta^2}{1 - \eta^2}}\left[\frac{\theta_1(\beta)}{\theta_2(\beta)}\right]^2.\end{aligned} \tag{9.17}$$

Denoting the values of α and β corresponding to $u = v$ by α_0 and β_0, gives, for t,

$$\begin{aligned}\pm At &= C_1 + \log\frac{\theta_1(\alpha + \alpha_0)}{\theta_1(\alpha - \alpha_0)} - 2\alpha\frac{\theta_1{}'(\alpha_0)}{\theta_1(\alpha_0)} \\ &= C_2 + \log\frac{\theta_2(\alpha - \beta_0)}{\theta_2(\alpha + \beta_0)} + 2\alpha\frac{\theta_2{}'(\beta_0)}{\theta_2(\beta_0)} \\ &= C_3 + \log\frac{\theta_1(\beta + \beta_0)}{\theta_1(\beta - \beta_0)} - 2\beta\frac{\theta_2{}'(\beta_0)}{\theta_2(\beta_0)}.\end{aligned} \tag{9.18}$$

One has

$$\frac{1}{q^{1/4}}\theta_1(\alpha) = \sin\alpha - q^2\sin 3\alpha + q^6\sin 5\alpha \ldots,$$

$$\frac{1}{q^{1/4}}\theta_2(\alpha) = \cos\alpha + q^2\cos 3\alpha + q^6\cos 5\alpha \ldots .$$

Of course α and β are imaginary.

In the case of real roots one easily sees that for a pure imaginary α, r is real and positive and starts from zero at $\alpha = 0$. That corresponds to the quasi-periodic universe. For pure imaginary β, r is infinite for $\beta = \beta_0$ and decreases when β increases its absolute value.

It remains for us to transform the imaginary trigonometric curves.

Towards this end, we put

$$x = e^{\alpha/i} \qquad y = e^{\beta/i}. \tag{9.19}$$

We obtain for the quasi-periodic universe

$$\frac{2r_0}{r} = -1 + \sqrt{\frac{9-\eta^2}{1-\eta^2}}\left[\frac{x + x^{-1} + q^2(x^3 + x^{-3}) + q^6(x^5 + x^{-5}) + \ldots}{x - x^{-1} - q^2(x^3 - x^{-3}) + q^6(x^5 - x^{-5}) + \ldots}\right]^2 \tag{9.20}$$

and for the bouncing universe

$$\frac{2r_0}{r} = -1 + \sqrt{\frac{9-\eta^2}{1-\eta^2}}\left[\frac{y - y^{-1} - q^2(y^3 - y^{-3}) + q^6(y^5 - y^{-5}) + \ldots}{y + y^{-1} + q^2(y^3 + y^{-3}) + q^6(y^5 + y^{-5}) + \ldots}\right]^2 \tag{9.21}$$

which, for $r = \infty$, gives the value y_0 corresponding to β_0.

We thence have for the quasi-periodic branch the equation

$$\begin{aligned}\pm At - C_2 &= \text{Log}\,\frac{xy_0^{-1} + x^{-1}y_0 + q^2(x^3y_0^{-3} + x^{-3}y_0^3) + q^6(x^5y_0^{-5} + x^{-5}y_0^5) + \ldots}{xy_0 + x^{-1}y_0^{-1} + q^2(x^3y_0^3 + x^{-3}y_0^{-3}) + q^6(x^5y_0^5 + x^{-5}y_0^{-5}) + \ldots}\\ &\quad + 2\frac{y_0 - y_0^{-1} + 3q^2(y_0^3 - y_0^{-3}) + 5q^6(y_0^5 - y_0^{-5}) + \ldots}{y_0 + y_0^{-1} + q^2(y_0^3 + y_0^{-3}) + q^6(y_0^5 + y_0^{-5}) + \ldots}\,\text{Log}\,x\end{aligned} \tag{9.22}$$

for t, and for the bouncing branch

$$\begin{aligned}\pm At - C_3 &= \text{Log}\,\frac{yy_0 - y^{-1}y_0^{-1} - q^2(y^3y_0^3 - y^{-3}y_0^{-3}) + q^6(y^5y_0^5 - y^{-5}y_0^{-5}) + \ldots}{yy_0^{-1} - y^{-1}y_0 - q^2(y^3y_0^{-3} - y^{-3}y_0^3) + q^6(y^5y_0^{-5} + y^{-5}y_0^5) + \ldots}\\ &\quad - 2\frac{y_0 - y_0^{-1} + 3q^2(y_0^3 - y_0^{-3}) + 5q^6(y_0^5 - y_0^{-5}) + \ldots}{y_0 + y_0^{-1} + q^2(y_0^3 + y_0^{-3}) + q^6(y_0^5 + y_0^{-5}) + \ldots}\,\text{Log}\,y.\end{aligned} \tag{9.23}$$

These formulae apply equally well to the case where the roots are imaginary. Then q is pure imaginary, but since it appears only squared, this results in a simple change of sign.

Besides, one can identify the two expressions by putting

$$y = \frac{i}{qx}. \tag{9.24}$$

The sign choices correspond and

$$C_3 - C_2 = -4 \operatorname{Log} y_0 + 2\frac{y_0 - y_0^{-1} + 3q^2(y_0^3 - y_0^{-3}) + \dots}{y_0 + y_0^{-1} + q^2(y_0^3 + y_0^{-3}) + \dots} \operatorname{Log} \frac{i}{q}. \tag{9.25}$$

It is advantageous to use the first formulae for x between 1 and $1/\sqrt{-qi}$, and the second for larger values of x.

For real q, the maximum or minimum of r occurs for x or y equal to $1/\sqrt{q}$.

When one is given r, the calculation of x or y can be carried out through the formulae ($\ell = \ell_1 i$)

$$\frac{\sqrt{1 + 2r_0/r} + \sqrt[4]{(9 - \eta^2)/(1 - \eta^2)}}{\sqrt{1 + 2r_0/r} - \sqrt[4]{(9 - \eta^2)/(1 - \eta^2)}} = \frac{\operatorname{tg} 2\varphi}{\ell_1} = \frac{\sin 2\theta}{\ell}, \tag{9.26}$$

where one of the angles φ and θ is real. One then has ($q = q_1 i$)

$$x^2 = \frac{\operatorname{tg}\varphi}{q_1}\left(1 + 4q_1^4 \frac{\operatorname{ctg} 2\varphi}{\sin 2\varphi} + \dots\right) = \frac{\operatorname{tg}\varphi}{q}\left(1 - 4q^4 \frac{\operatorname{ctg} 2\theta}{\sin 2\theta} + \dots\right) \tag{9.27}$$

$$y^2 = \frac{\operatorname{ctg}\varphi}{q_1}\left(1 - 4q_1^4 \frac{\operatorname{ctg} 2\varphi}{\sin 2\varphi} + \dots\right) = -\frac{\operatorname{ctg}\theta}{q}\left(1 + 4q^4 \frac{\operatorname{ctg} 2\theta}{\sin 2\theta} + \dots\right). \tag{9.28}$$

For real q, the quasi-periodic branch corresponds to values of θ between zero and 45^o, and the bouncing branch to values between 135 and 90 degrees. For imaginary q, the angle φ ranges between zero and 90 degrees.

10. THE CLUSTERS OF NEBULAE

One of the characteristics of the universe which is revealed to us by astronomical observations is that, while there exist isolated nebulae, there also are agglomerations of nebulae, the population of which varies from some tens up to hundreds of nebulae.

We intend to discuss the hypothesis under which the clusters of nebulae would be essentially in equilibrium and have the form of a part of the

Einstein universe. We prove in the sequel that some information about the expansion of the universe can be deduced from this hypothesis.

If the clusters are in equilibrium, the current radius of the universe is clearly greater than the equilibrium radius, in such a way that the Hubble ratio

$$\frac{r}{v} = 1.8 \times 10^{9} \text{ years} \qquad (10.1)$$

between the distance of the nebulae and their spectroscopic velocity of recession is a measure of the cosmological constant. Adopting

$$\lambda = 10^{-54},$$

we can calculate from the formula (8.4) at what distance r_e the cosmic repulsion and the force of gravitation due to a mass m come into equilibrium; one has

$$r_e^3 = \frac{3Km}{\lambda c^2} \qquad (10.2)$$

or

$$r_e = 80\sqrt[3]{m}, \qquad (10.3)$$

the distances being measured in light-years and the Sun's mass having been taken as the unit.

If the clusters of nebulae are in equilibrium, r_e must be the radius of the neutral zone corresponding to each nebula. The mean distance between nebulae must thus be $2r_e$.

If there are N nebulae distributed in a more or less spherical fashion, the volume of the cluster must be

$$\frac{4\pi}{3} N r_e^3$$

and its diameter

$$2r_e\sqrt[3]{N} = 160\sqrt[3]{Nm}.$$

We can estimate the distance D and the angular diameter d of the cluster; we must then have as the condition of equilibrium

$$Dd = 160\sqrt[3]{Nm}. \qquad (10.4)$$

If d is expressed in degress and D in megaparsecs, the diameter in light years is

$$\frac{Dd}{0.31 \times 57.3} \times 10^{6} = 160\sqrt[3]{Nm}.$$

whence

$$Nm\,10^{-9} = 0.043 D^3 d^3. \tag{10.5}$$

Hubble's estimates (Mount Wilson Contr. no. 427) allow us to calculate the mean mass of a galaxy under the hypothesis of equilibrium. For certain clusters, the data of table IX do not agree with the information in the text; we have then made the calculation for both values.

Cluster	N	D	d		$m\ 10^{-9}$	
Virgo	(500)	1.8	12°	11°	0.9	0.7
Pegasus	100	7.3	1		0.2	
Pisces	20	7	0.5	1	0.1	0.7
Cancer	150	9	1.5	1	0.7	0.2
Perseus	500	11	2.0		0.9	
Coma	800	14	1.7		0.7	
Ursa Major	300	22	0.7		0.5	
Leo	400	32	0.6		0.8	

These data are clearly of very unequal value. In particular, Shapley finds a very much larger distance and a smaller number of nebulae for the Virgo cluster. However, for the Virgo clusters A, B, C and D, Shapley finds diameters and numbers of nebulae of the same order of magnitude.

If one takes into account the uncertainty in the data on which we are basing the calculations, and in particular the irregular form of most clusters, one can consider the result as favourable to the hypothesis of the equilibrium of the clusters of nebulae.

The numerical value of the mass found for the nebulae is of the order of magnitude indicated by Hubble's research.

The data concerning the Coma cluster seem to be the most secure, all the more so as this cluster appears to have quite a globular form. We thus adopt as our estimate of the mean mass of nebulae

$$0.7\ 10^9\ \odot,$$

and thus, as the mean distance between the nebulae,

140 000 light-years.

Comparing this value with the mean distance of the isolated nebulae, estimated by Hubble as

1, 800, 000 light-years,

we have, as the coefficient of expansion of the universe

$$\frac{R}{R_e} = 13. \tag{10.6}$$

The hypothesis of cluster equilibrium thus seems to provide the means of making precise, and confirming, Hubble's estimates.

It has also the interest of providing a cosmological significance to the relative frequency of clusters and isolated nebulae.

Without us having so far developed a truly precise theory, it is clear that if the expansion is not much slowed down in the neighbourhood of the equilibrium position, it is almost impossible that the parts of the universe could have deviated in great numbers from the average motion at the moment when they were in equilibrium, and perhaps one could prove that if the expansion is too much slowed down in the neighbourhood of the equilibrium, the clusters would have to be more numerous and more important than they really are. Thus there is here a new line of attack which allows us to find the value of η^2, or at least exclude the neighbourhood of the two critical values $\eta^2 = -3$ and $\eta^2 = 0$.

This suffices to determine the order of magnitude of the radius of the universe and the expansion time.

We have in fact, by (9.2) and (10.2)

$$R_0^3(1 - \eta^2) = R_e^3 \tag{10.7}$$

$$A^2 R_0^2(3 + \eta^2) = c^2 \tag{10.8}$$

so that

$$R_0 = \frac{c}{A\sqrt{3 + \eta^2}} = \frac{10^{37}\ \text{cm}}{\sqrt{1 + \frac{1}{3}\eta^2}} = \frac{1}{\sqrt{1 + \frac{1}{3}\eta^2}}\, 10^9 \text{ light-years} \tag{10.9}$$

whence

$$R = 13R_e = 13\frac{\sqrt{1 - \eta^2}}{\sqrt{1 + \frac{1}{3}\eta^2}} \text{ light-years.}$$

If η^2 is not around -3, the order of magnitude of the radius of the universe is thus known.

It is the same for the expansion time.

The limiting case $\eta^2 = -3$ gives the exact solution

$$R = 2R_0 \operatorname{sh}^{2/3} \frac{3At}{2} \simeq R_0 \sqrt[3]{2}\, e^{At} \tag{10.10}$$

with

$$R_e = \sqrt[3]{4}\, R_0 ,$$

whence

$$2At = 2 \times 2.303 \log 13 \sqrt[3]{2} = 5.6. \tag{10.11}$$

As $2A \cong 10^{-9}$ years, the expansion time is 5.6×10^9 years.

For $\eta^2 = -0.1$ one finds by the formulae of the preceding section

$$2At = 8.437. \tag{10.12}$$

When η^2 tends to zero, one can easily find the asymptotic value of the expansion time from $R = 0$ up to a value greater than R_0. Putting

$$X^2 = \frac{R}{R + 2R_0} \tag{10.13}$$

one obtains

$$At = \text{Log} \frac{1 + X}{1 - X} + \frac{1}{\sqrt{3}} \text{Log} \frac{X\sqrt{3} - 1}{X\sqrt{3} + 1} + \frac{1}{\sqrt{3}} \text{Log} \frac{1}{q_1^2} - 2\,\text{Log}(2 + \sqrt{3}). \tag{10.14}$$

This equation shows how the solution tends to the limiting solution (R_0, ∞) when q tends to zero.

One has

$$q_1^2 = -\frac{\eta^2}{144} + \dots \tag{10.15}$$

and

$$\frac{\kappa M \sqrt{\lambda}}{2\pi^2} = 1 - \frac{3}{2}\eta^2 + \dots = 1 + \mu \tag{10.16}$$

where μ represents the accuracy with which the mass is adjusted to the cosmological constant in order to realise the position of equilibrium.

For the expansion coefficient = 13, one finds

$$2At = 5.93 + 2.66 \log_{10} \frac{1}{\mu}. \tag{10.17}$$

For the bouncing universe one similarly has

$$At = \text{Log} \frac{1 + X}{1 - X} + \frac{1}{\sqrt{3}} \text{Log} \frac{X\sqrt{3} - 1}{X\sqrt{3} + 1} + \frac{1}{\sqrt{3}} \text{Log} \frac{1}{q} - \text{Log}(2 + \sqrt{3}). \tag{10.18}$$

the time being measured from the minimum radius, that is

$$2At = 5.46 + 1.33 \log_{10} \frac{-1}{\mu}. \tag{10.19}$$

When η^2 tends to -3, the radius tends to infinity but U, the angular distance which light is capable of crossing during the expansion, tends to zero.

It is interesting to calculate $R_e U$, the distance at the moment of equilibrium of the most distant points which can transmit light to each other. One has, by (9.9^1), (10.7) and (10.8)

$$R_e U = 2\sqrt{3}\,\frac{u}{i}\,\sqrt[3]{1-\eta^2}\,\frac{c}{A} \tag{10.20}$$

and

$$\frac{u}{i} = \frac{\omega}{\pi} \log x^2.$$

One finds for $\eta^2 = -3$,

$$R_e U = 4.46\frac{c}{2A} = 4.46 \times 10^9 \text{ light-years}. \tag{10.21}$$

11. SCHWARZSCHILD'S EXTERIOR FIELD

The equations of the Friedmann universe admit solutions where the radius of the universe tends to zero for a non-zero mass. This contradicts the generally accepted result that a given mass cannot have a radius smaller than

$$\frac{2Km}{c^2}$$

or $2m$ in natural units ($K = c = 1$).

This result follows from the solution of Schwarzschild's exterior problem,

$$ds^2 = -\frac{dr^2}{1 - (2m/r) - (\lambda/3)r^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)d\tau^2. \tag{11.1}$$

We intend to prove that the singularity of the field is not real and arises simply because one wanted to use coordinates for which the field is static.

In vacuum, m is a constant. Let us consider the Euclidean case $f(\chi) = 1$ and put

$$r_0^3 = \frac{Km}{4A^2}, \qquad A^2 = \frac{\lambda c^2}{3}. \tag{11.2}$$

Equation (8.2) becomes

$$r\left(\frac{\partial r}{\partial t}\right)^2 = A^2(r^3 + 8r_0^3) \tag{11.3}$$

whence

$$r = 2r_0 \,\mathrm{Sh}^{2/3} \frac{3A}{2}(t - \chi). \tag{11.4}$$

We may write $F(\chi)$ in place of χ, but this does not introduce any more generality.

Since

$$\frac{\partial r}{\partial \chi} = -\frac{\partial r}{\partial t}$$

we then have

$$ds^2 = -A^2(r^3 + 8r_0^3)\frac{d\chi^2}{r} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + c^2 dt^2 \tag{11.5}$$

which is a solution for a vacuum field.

At each instant, space is Euclidean, and there is no singularity except for $r = 0$.

If we take r as a coordinate, there must be a means to define a coordinate τ in such a way as to put the field into Schwarzschild's form.

So one has

$$dr^2 = \frac{A^2}{r}(r^3 + 8r_0^3)(dt - d\chi)^2$$

whence

$$\frac{A^2}{r}(r^3 + 8r_0^3)d\chi^2 = dr^2 - \frac{A^2}{r}(r^3 + 8r_0^3)(dt^2 - 2d\chi\, dt)$$

and

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left[c^2 + \frac{A^2}{r}(r^3 + 8r_0^3)\right]dt^2 - \frac{2A^2}{r}(r^3 + 8r_0^3)d\chi\, dt$$

and, eliminating χ,

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left[c^2 - \frac{A^2}{r}(r^3 + 8r_0^3)\right]dt^2 + 2A\sqrt{\frac{r^3 + 8r_0^3}{r}}\, dr\, dt.$$

Setting

$$d\tau = dt + \frac{A\sqrt{\frac{r^3 + 8r_0^3}{r}}}{c^2 - \frac{A^2}{r}(r^3 + 8r_0^3)}\, dr, \tag{11.6}$$

gives

$$ds^2 = -\frac{dr^2}{1 - \frac{8A^2 r_0^3}{c^2 r} - \frac{A^2 r^2}{c^2}} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + c^2\left(1 - \frac{8A^2 r_0^3}{c^2 r} - \frac{A^2 r^2}{c^2}\right)d\tau^2, \tag{11.7}$$

which is Schwarzschild's form (11.1) for the field of a point mass.

The singularity is introduced because the expression which appears in the denominator of $d\tau$ (11.6) vanishes for sufficiently small values of r.

τ depends on an elliptic integral. In the particular case where λ tends to zero, the integration can be carried out. To simplify matters, let us take coordinates for which K and c are equal to one.

One has, at the limit where A tends to zero,

$$8A^2 r_0^3 = 2m \tag{11.8}$$

whence

$$d\tau = dt + \frac{\sqrt{\frac{2m}{r}}}{1 - \frac{2m}{r}}\, dr, \tag{11.9}$$

and, on integrating,

$$\tau = t + 2\sqrt{2mr} + 2m \operatorname{Log} \frac{\sqrt{r} - \sqrt{2m}}{\sqrt{r} + \sqrt{2m}}, \tag{11.10}$$

a transformation which is inadmissible for values of r less than $2m$. The equation (11.4) similarly becomes

$$\chi = t - \frac{2}{3}\frac{r^{2/3}}{\sqrt{2m}} \tag{11.11}$$

and the new form of the field is written without singularity

$$ds^2 = -2m\frac{d\chi^2}{r} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2, \tag{11.12}$$

where

$$r = [\tfrac{3}{2}\sqrt{2m}\,(t - \chi)]^{2/3}. \tag{11.13}$$

The singularity of the Schwarzschild field is thus a fictitious singularity, analogous to that which appears at the horizon of the centre in the original form of the de Sitter universe.

12. THE VANISHING OF SPACE

The radius of space may pass through zero. We intend to discuss this passage, and to examine in particular if there is a way of interpreting this zero value of the radius physically as simply representing a small quantity and, in this case, of fixing its order of magnitude.

For the study of the zero point, we may neglect the cosmological constant; setting

$$\frac{Km}{c^2} = a, \tag{12.1}$$

we then have

$$\frac{1}{c^2}\left(\frac{dR}{dt}\right)^2 = -1 + \frac{2a}{R}. \tag{12.2}$$

Introducing the angular distance U crossed by light during the time t,

$$dU = \frac{cdt}{R}, \tag{12.3}$$

we easily find Einstein's cycloidal universe

$$\begin{aligned} R &= a(1 - \cos U), \\ ct &= a(U - \sin U). \end{aligned} \tag{12.4}$$

When U varies from 0 to π, R returns to its initial zero value, and light just has time to go round the simply elliptic space.

The question is to know if there is a way to smooth out the cusp of the cycloid.

One can ask first of all, if one would not obtain this result if one took into account the effect of the pressure which need not necessarily be negligible. It is easy to see, going back to the equation (3.7), that the pressure only reinforces the gravitational action. Besides, the question has been treated in detail by Tolman.[9]

It is more important to examine the effect of a lack of isotropy in the distribution of tensions.

We intend to examine, following an idea which Einstein communicated to us, a universe defined by

$$ds^2 = -b_1^2 dx_1^2 - b_2^2 dx_2^2 - b_3^2 dx_3^2 + dx_4^2 \tag{12.5}$$

where b_1, b_2 and b_3 are functions of $x_4 = t$.

Such a universe is naturally inadmissible from many points of view, but it has the interest of introducing a marked and largely arbitrary anisotropy.

We can easily calculate the matter tensor by the formulae of Section 1.

We have, for k and i different from 4, by (1.9),

$$\beta_{ik} = \frac{b_i' b_k'}{b_i b_k} \tag{12.6}$$

the primes denoting derivatives with respect to t, and

$$\beta_{i4} = \frac{b_i''}{b_i}. \tag{12.7}$$

The components $T_{\mu\upsilon}$ $(\mu \neq \upsilon)$ vanish.

$$\sqrt{-g} = b_1 b_2 b_3 = R^3 \tag{12.8}$$

measures the volume occupied by a specified part of the matter. Here R is no longer the radius of the universe, since the space is Euclidean, but the volume of space tends to zero if R tends to zero.

We have

$$\frac{3R'}{R} = \frac{b_1'}{b_1} + \frac{b_2'}{b_2} + \frac{b_3'}{b_3}$$

and

$$3\left(\frac{R''}{R} - \frac{R'^2}{R^2}\right) = \frac{b_1''}{b_1} + \frac{b_2''}{b_2} + \frac{b_3''}{b_3} - \frac{b_1'^2}{b_1^2} - \frac{b_2'^2}{b_2^2} - \frac{b_3'^2}{b_3^2}.$$

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9 This reference is: Tolman, Richard C., and Ward, M. (1932). “On the Behavior of Non-Static Models of the Universe when the Cosmological Constant is Omitted.” Physical Review 39, 835-843 — *Transl.*

Setting

$$I^2 = \left(\frac{b_1'}{b_1} - \frac{b_2'}{b_2}\right)^2 + \left(\frac{b_2'}{b_2} - \frac{b_3'}{b_3}\right)^2 + \left(\frac{b_3'}{b_3} - \frac{b_1'}{b_1}\right)^2 \tag{12.9}$$

we obtain

$$3\frac{R''}{R} = \frac{b_1''}{b_1} + \frac{b_2''}{b_2} + \frac{b_3''}{b_3} - \frac{1}{3}I^2$$

or, by (1.12),

$$3\frac{R''}{R} = \frac{\kappa}{2}(T_1^1 + T_2^2 + T_3^3 - T_4^4) - \frac{1}{3}I^2. \tag{2.10}$$

In all reasonable applications, T_1^1, T_2^2 and T_3^3 will be negative, and in all cases less than $T_4^4 = \rho$ in absolute value. R'' will thus be essentially negative. If therefore at a certain instant R' is negative, R must attain the value zero and thus the volume vanishes.

One sees that anisotropy can no more prevent the vanishing of space than pressure can.

The above argument is not a formal proof of the impossibility of avoiding zero volume by anisotropy, since (12.5) is not the most general conceivable form, but it indicates all the same that in an already rather general case anisotropy acts in the opposite sense.

The matter has to find, though, a way of avoiding the vanishing of its volume.

As long as the matter is made up of stars, this is manifestly impossible.

When it is condensed into a single mass, it is clear that it must have acquired a high temperature much greater than the critical temperature of liquids and that nothing prevents it attaining a degree of concentration comparable to the interior of the companion of Sirius.

Even for a degenerate gas it seems that nothing could oppose the concentration, since the available energy M/R is unbounded.

When the distances between the atomic nuclei and the electrons become of the order of 10^{-12} cm, the non-Maxwellian forces which prevent the mutual interpenetration of elementary particles must become predominant and are without doubt capable of stopping the contraction. The universe would then be comparable to a colossal atomic nucleus. If the contraction is stopped, the process should continue in the opposite direction.

Adopting, following Eddington, 10^{78} as the number of protons in existence, we have, as the order of magnitude of the radius of the universe when reduced to its atomic state

$$10^{(78/3)-12} = 10^{14} \text{ cm},$$

which is about ten times the distance to the Sun.

We thus conclude that only the subatomic nuclear forces seem capable of stopping the contraction of the universe, when the radius of the universe is reduced to the dimensions of the solar system.

For the cosmological point of view, the zero of space must thus be treated as a beginning, in the sense that every astronomical structure with an earlier existence would have been completely destroyed there.

The epoch of this beginning, or, if one likes, of this recommencement, certainly dates from before the formation of the Earth's crust and the organization of the solar system, that is as a strict minimum from the study of radioactive rocks

$$1.6 \times 10^9 \text{ years.}$$

Comparing this value with Hubble's ratio

$$\frac{r}{v} = 1.8 \times 10^9 \text{ years,}$$

we conclude that all solutions in which the expansion speed has always been faster than it is now are excluded.

In particular, for Einstein's cycloidal universe (12.4) or the solution (10.10) for small R/R_0, one has

$$t = \frac{2}{3}\frac{r}{v} = 1.2 \times 10^9 \text{ years.}$$

We must thus exclude the solutions where the radius is less than the equilibrium radius and in particular the quasi-periodic solutions.

For a purely aesthetic point of view, one may perhaps regret this. Those solutions where the universe expands and contracts successively while periodically reducing itself to an atomic mass of the dimensions of the solar system, have an indisputable poetic charm and make one think of the phoenix of legend.

Translator's note.

French and English are comparatively closely related, and scientific papers use only very restricted forms of expression and vocabulary, so that in much of this paper I have been able to give a more or less direct literal translation, only recasting sentences where I felt it was really necessitated by differences of grammar or idiom between the languages. Where usage in scientific writing has changed, I have sometimes rendered words or phrases as their modern equivalents in order to turn Georges Lemaître's original French into the usual style of modern scientific papers in English (to take the first such case, 'paragraphe' became 'section'), but have on the whole tried to avoid anachronism. Similarly, the notation has not been changed, except for correction of a few obvious misprints. I have left historical or explanatory commentary to the editor, except where I felt Lemaître's meaning might otherwise be misunderstood by a modern reader. In a few cases, Lemaître's precise meaning was unclear not only to me but also to a native French speaker. I am grateful to Dr. Caroline Terquem for advice at those points, and have indicated them by giving the original phrase in a footnote. I am also grateful to Dr. Jean Eisenstaedt for drawing my attention to two typographical errors in the original, before (11.2) and in (11.11), and for the details of the references to Tolman's work and to Lemaître's thesis.