# NEWTONIAN UNIVERSES AND THE CURVATURE OF SPACE

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THE present paper investigates the relationship between the universes of relativistic cosmology and the universes that can be constructed using only Newtonian dynamics, Newtonian gravitation, and Newtonian relativity. It is shown that the governing differential equations are identical in form in the two cases, and that the locally observable phenomena predicted on the two theories are indistinguishable. It is further shown that a space of positive curvature corresponds to a Newtonian universe in which every particle has a velocity less than the parabolic velocity of escape from the observer, a space of negative curvature to one in which every particle has a velocity greater than the parabolic velocity. Thus universes of positive, zero, and negative curvatures correspond to elliptic, parabolic, and hyperbolic Newtonian universes respectively. These results are applied to obtain a number of new results concerning Doppler effects and accelerations in relativistic universes. Finally an investigation due to Lemaître is briefly discussed.

1. The results obtained in the foregoing paper can be extended as follows to the case in which, on Newtonian mechanics, the velocity $v$ is not necessarily equal to the parabolic velocity of escape. Let $v$, the velocity of a particle at distance $r$ from the observer at time $t$, be radial in direction, and a function of $r$ and $t$. The equation of motion,

$$\frac{Dv}{Dt} = -\frac{GM(r)}{r^2}, \tag{1}$$

may be written

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r} = -\tfrac{4}{3}\pi G\rho r, \tag{2}$$

where $\rho$ is a function of $t$ only. The equation of continuity may be written in the form

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v) = 0. \tag{3}$$

Hence, here,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v)$$

is a function of $t$ only, independent of $r$. Put

$$\frac{1}{\rho}\frac{d\rho}{dt} = -3F(t); \tag{4}$$

then

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v) = 3F(t),$$

which integrates in the form

$$r^2 v = r^3 F(t) + G(t),$$

i.e.

$$v = rF(t) + \frac{G(t)}{r^2}. \tag{5}$$

Insert this in (2). Then, since $\rho$ is a function of $t$ only,

$$\frac{1}{r}\left[rF'(t) + \frac{G'(t)}{r^2} + \left(rF(t) + \frac{G(t)}{r^2}\right)\left(F(t) - \frac{2G(t)}{r^3}\right)\right]$$

must be a function of $t$ only. This requires $G(t) \equiv 0$. Hence by (5),

$$v = rF(t). \tag{6}$$

Inserting this in (2) as before, we have

$$F'(t) + [F(t)]^2 = -\tfrac{4}{3}\pi G\rho. \tag{7}$$

Writing (6) in the form

$$\frac{1}{r}\frac{dr}{dt} = F(t),$$

and integrating it following the motion, we have

$$r = fR(t), \tag{8}$$

where $f$ defines the particle considered and $R(t)$ is a universal function of $t$ satisfying

$$\frac{1}{R}\frac{dR}{dt} = F(t) = -\frac{1}{3\rho}\frac{d\rho}{dt}. \tag{9}$$

Hence

$$\rho = B/R^3, \tag{10}$$

where $B$ is a constant. Introducing (9) and (10) in (7), we find

$$\frac{1}{R}\frac{d^2R}{dt^2} = -\frac{\tfrac{4}{3}\pi GB}{R^3}, \tag{11}$$

of which the integral is

$$\left(\frac{dR}{dt}\right)^2 = \frac{\tfrac{8}{3}\pi GB}{R} + K, \tag{12}$$

where $K$ is a constant. Hence, by (9),

$$F(t) = \left[\frac{\tfrac{8}{3}\pi GB}{R^3} + \frac{K}{R^2}\right]^{\frac{1}{2}}$$

$$= [\tfrac{8}{3}\pi G\rho + A\rho^{\frac{2}{3}}]^{\frac{1}{2}}, \tag{13}$$

where $$K = AB^{\frac{2}{3}}. \tag{14}$$

Accordingly by (6), $$v = r[\tfrac{8}{3}\pi G\rho + A\rho^{\frac{2}{3}}]^{\frac{1}{2}}. \tag{15}$$

This is simply the Newtonian integral of motion, for since

$$M(r) = \tfrac{4}{3}\pi\rho r^3,$$

it may be written

$$\tfrac{1}{2}v^2 = \frac{GM(r)}{r} + \frac{A}{2}\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}[M(r)]^{\frac{2}{3}}, \tag{16}$$

and of course $M(r)$ is constant following the motion.

It follows that the particle, and so every particle, possesses the elliptic, parabolic, or hyperbolic velocity according as $A \lesseqgtr 0$. The constant $A$ is the same for all particles. By (6) or (15), $v$ obeys a velocity-distance proportionality at any one epoch; and (15) gives explicitly the connexion between the mean density $\rho$ and the coefficient in the velocity-distance proportionality.

Differentiating (16) following the motion, we see that $v$ obeys (1), so that (16) is an actual solution. Moreover, it is clear that the solution satisfies Einstein's cosmological principle. The density $\rho$ is obtained as a function of $t$ by integrating (12) in the form

$$t = \int\limits_0^R \frac{R^{\frac{1}{2}}\,dR}{(\frac{8}{3}\pi GB + KR)^{\frac{1}{2}}} = \int\limits_0^\theta \frac{\theta^{\frac{1}{2}}\,d\theta}{(\frac{8}{3}\pi G + A\theta)^{\frac{1}{2}}}, \tag{17}$$

where $\rho = 1/\theta^3$.

## 2. Comparison with the equations of relativistic cosmology.

By (10), equation (11) may be written

$$\frac{2}{R}\frac{d^2R}{c^2\,dt^2} = -\tfrac{1}{3}\kappa\rho, \tag{18}$$

where $$\kappa = 8\pi G/c^2.$$

Similarly, (12) may be written

$$\left(\frac{1}{R}\frac{dR}{c\,dt}\right)^2 + \frac{k}{R^2} = \tfrac{1}{3}\kappa\rho, \tag{19}$$

where $$k = -\frac{K}{c^2} = -\frac{AB^{\frac{2}{3}}}{c^2}. \tag{20}$$

Adding (18) and (19) we have

$$\frac{2}{R}\frac{d^2R}{c^2\,dt^2} + \left(\frac{1}{R}\frac{dR}{c\,dt}\right)^2 + \frac{k}{R^2} = 0. \tag{19'}$$

Equations (19) and (19′) are formally identical with the equations of relativistic cosmology* for an expanding universe of 'radius' $R$, curvature $k/R^2$, with $\lambda = 0$, $p = 0$. Thus the Newtonian distance $r$ is the same function of the Newtonian time $t$ as the 'distance' $r$ is of 'cosmic time' $t$ in the relativistic solutions. Further, we see that an expanding space of positive curvature ($k > 0$) corresponds to a Newtonian universe with elliptic velocities ($A < 0$), an expanding space of negative curvature ($k < 0$) to a Newtonian universe with hyperbolic velocities ($A > 0$).† By choice of a multiplying factor for $R$, which is equivalent to choice of $B$, $k$ may be reduced to $\pm 1$. The constant $B$ has no physical significance; it disappears from all formulae relating observable quantities. The constant $A$ alone is of physical significance.‡

It follows that the *local* properties of the universes in expanding spaces of positive, zero or negative curvatures are observationally the same as in Newtonian universes with velocities respectively less than, equal to, or greater than the parabolic velocity of escape. This gives great insight into the physical significance of expanding curved space.

3. The 'cosmological' terms involving the cosmical constant $\lambda$ are at once obtained, if we superimpose on Newtonian gravitation a repulsive force proportional to distance. By a simple application of the triangle of forces the equation of motion (1) becomes modified to

$$\frac{Dv}{Dt} = -\frac{GM(r)}{r^2} + \tfrac{1}{3}c^2\lambda r, \tag{21}$$

and the equation of continuity is unaffected. The integrations can now be carried out as before,§ and we are led to equations for $R$ of the relativistic form with the usual $\lambda$-terms. Einstein's cosmological principle is still satisfied. But from this point of view the introduction of $\lambda$-terms is somewhat artificial. Their introduction is permissible simply because they are the only new type of action at a distance (in addition to Newton's law) compatible with the satisfaction of the cosmological principle. This last point may be worthy of remark as throwing light on the nature of a 'law of gravitation'.

* e.g. H. P. Robertson, *Reviews of Modern Physics*, 5 (1933), 62–90, equations (3.2). It is clear that (18) and (19) are formally identical with the Newtonian equation of motion and its first integral, and that they imply the equation of continuity.

† Cf. de Sitter, *Univ. California Pub. Math.* 2 (1933), 171.

‡ e.g. the curvature $k/R^2$ is simply $-A\rho^{2/3}/c^2$.

§ The velocity-distance proportionality becomes $v = r[\frac{8}{3}\pi G\rho + A\rho^{2/3} + \frac{1}{3}c^2\lambda]^{1/2}$.

**4.** Omitting further consideration of the $\lambda$-terms, we may put, in (17), $\theta = \frac{8}{3}\pi G\phi/|A|$, when it becomes

$$t = \frac{\frac{8}{3}\pi G}{|A|^{\frac{3}{2}}} \int\limits_0^{\phi} \frac{\phi^{\frac{1}{2}}\,d\phi}{\left[1+\frac{A}{|A|}\phi\right]^{\frac{1}{2}}}. \tag{22}$$

If $A > 0$ (hyperbolic case), as $t$ increases, $\phi$ or $\theta$ steadily increases, and $\rho$ steadily decreases to zero. If $A < 0$ (elliptic case) $\theta$ can never exceed $\frac{8}{3}\pi G/|A|$, and the density $\rho$ has a lower limit, after which it increases again. We then have an oscillating universe. It should be noted that in all cases $Dv/Dt$ is always negative, so that in the Newtonian case each particle is steadily decelerated.

In the hyperbolic case $v/r \to 0$ following any particle, and $v \to$ const., though to a different constant for each particle corresponding to a different value of the parameter $f$. Also, since $\rho \to 0$, the asymptotic form corresponds to the 'hydrodynamic' solution previously obtained* by one of us from kinematic principles only. This has been previously shown by the methods of general relativity,† but owing to the formal identity of the various relations with the corresponding Newtonian ones it holds in the Newtonian case as well. In particular, in this asymptotic case we have $v = r/t$ for every particle. We can easily show, still of course omitting cosmical repulsion, that in all other cases

$$v < r/t. \tag{23}$$

For the integral (12) may be written

$$\frac{dR}{dt} = \left(\frac{C}{R}+K\right)^{\frac{1}{2}}, \tag{24}$$

where $C = \frac{8}{3}\pi GB = \frac{1}{3}\kappa c^2 B > 0$ since the density is positive. Hence for small $R$ we have

$$t \sim \tfrac{2}{3}C^{-\frac{1}{2}}R^{\frac{3}{2}}, \tag{25}$$

where we choose $t = 0$ for $R = 0$. Also from (6) and (9),

$$v = rR'/R.$$

Therefore we shall have $v < r/t$, provided

$$t < R/R'. \tag{26}$$

* E. A. Milne, *Zeits. für Astrophys.* 6 (1933), 1–95, 71 et seq.

† W. O. Kermack and W. H. McCrea, *M.N.R.A.S.* 93 (1933), 519–29; also H. P. Robertson, *Zeits. für Astrophys.* 7 (1933), Heft 3. In the present paper we leave aside the question of the wider significance of the kinematic solution.

But from (24),

$$\frac{d}{dt}\left(\frac{R}{R'}\right) = \frac{d}{dt}\left(\frac{R}{(CR^{-1}+K)^{\frac{1}{2}}}\right) = \frac{\frac{3}{2}CR^{-1}+K}{(CR^{-1}+K)^{\frac{3}{2}}}R' = \frac{\frac{3}{2}CR^{-1}+K}{CR^{-1}+K}, \quad (27)$$

and this exceeds unity for $K > 0$. Thus in the hyperbolic case $R/R'$ increases faster than $t$, and, from (25), $R/R' = 0$ when $t = 0$, so that the inequality (26) holds for all $t > 0$. Thus (23) is established in the hyperbolic case. In the elliptic case a similar proof holds so long as $R' > 0$, and after that the inequality is true *a fortiori*. This shows in a simple manner the retarding effect of gravitation on the scattering of the particles.

It is of some interest to integrate (17), or equivalently (24); (17) is more significant, since the meaningless multiplier $B$ is absent. We find, putting $\frac{8}{3}\pi G = \alpha$,

$$t = -\frac{\theta^{\frac{1}{2}}(\alpha-A'\theta)^{\frac{1}{2}}}{A'}+\frac{\alpha}{A'^{\frac{3}{2}}}\sin^{-1}\left(\frac{A'\theta}{\alpha}\right)^{\frac{1}{2}} \qquad (A = -A' < 0,\, k > 0),$$

$$t = \frac{2}{3}\frac{\theta^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}} \qquad (A = 0, \quad k = 0),$$

$$t = \frac{\theta^{\frac{1}{2}}(\alpha+A\theta)^{\frac{1}{2}}}{A}-\frac{\alpha}{A^{\frac{3}{2}}}\sinh^{-1}\left(\frac{A\theta}{\alpha}\right)^{\frac{1}{2}} \qquad (A > 0, \quad k < 0),$$

where it may be recalled that $\rho = 1/\theta^3$.

**5. Doppler effect in curved universes.** The general features of the motions studied in this paper suggest that in a relativistic universe the Doppler effect for any given particle will decrease as the epoch of observation advances, and for negative curvatures tend to a constant limit, different for different particles. We shall now show that this actually is the case.

We take the metric for the 'expanding-space' universe in the form*

$$ds^2 = c^2\,dt^2 - R^2\,du^2,$$

so that the particles have fixed coordinates in the space represented by $du^2$. Hence if $t$ is the 'cosmic time' of departure of a light-signal, $t_2$ its time of arrival at the observer, we have

$$\int_t^{t_2} \frac{c\,dt}{R(t)} = \int du = \text{const.}, \quad (28)$$

* H. P. Robertson, loc. cit.

where $\int du$ is the coordinate distance along the light-track, for which, of course, $ds = 0$. Hence

$$\frac{dt_2}{R(t_2)} - \frac{dt}{R(t)} = 0. \tag{29}$$

If then $\lambda$ is the emitted wave-length, $\lambda_2$ the observed wave-length, then

$$\frac{\lambda_2}{\lambda} = \frac{dt_2}{dt} = \frac{R(t_2)}{R(t)} > 1, \tag{30}$$

since $R(t)$ is an increasing function of $t$, and $t_2 > t$. This gives the red-shift. Consequently, this shift will decrease as the time of observation $t_2$ advances, if

$$\frac{d}{dt_2}\left(\frac{R(t_2)}{R(t)}\right) < 0,$$

that is, if

$$\frac{R'(t_2)}{R(t_2)}\,dt_2 - \frac{R'(t)}{R(t)}\,dt < 0,$$

that is, using (29), if

$$R'(t_2) < R'(t), \tag{31}$$

that is, if $R'(t)$ is a decreasing function of $t$. But we have proved, (cf. equation (8)), that $R(t)$ is the same function of $t$, apart from a constant multiplier, as the Newtonian distance $r$ is of Newtonian time $t$, in the corresponding Newtonian universe. Now in this universe every particle is *decelerated*. Hence $d^2r/dt^2 < 0$, or $d^2R/dt^2 < 0$. Hence $dR/dt$ is a decreasing function of $t$. Therefore *the Doppler effect decreases with advancing epoch of observation*. Also, in the asymptotic form of the hyperbolic case, $R(t) = t \times \text{const.}$, and so

$$\lambda_2/\lambda = t_2/t = \text{const.},$$

from (28).*

These results are worthy of note, since it is sometimes supposed that particles necessarily undergo acceleration and not deceleration. As a matter of fact acceleration away from the observer is possible only when cosmical repulsion predominates over gravitational attraction.

**6. Lemaître's theory of condensations.** A further instance of the applicability of the same physical interpretation of the equations of general relativity as is given in this paper is provided by Lemaître's theory of the formation of condensations.†

* Kermack and McCrea, loc. cit. equation (21).

† G. Lemaître, *Comptes Rendus*, 196 (1933), 903–4, 1085–7.

He studies the motion of a distribution of matter having spherical symmetry about the origin, and obtains for any particle an 'equation of motion'

$$\left(\frac{\partial r}{\partial t}\right)^2 = -c^2\sin^2\chi+2G\frac{m(\chi,t)}{r}+\tfrac{1}{3}\lambda c^2r^2, \tag{32}$$

where $r$ is its 'distance' from the origin, $(\chi, \theta, \phi)$ are the coordinates of the particle in the space in which it is fixed, and

$$m(\chi,t) = \int\limits_0^{r(\chi,t)} 4\pi\rho r^2\,dr$$

gives a measure of the mass inside radius $r$. With the addition of the cosmical constant $\lambda$, (32) is the analogy for Lemaître's case of our equation (16).

He proceeds to treat the case where $m(\chi,t)$ is such a function of $\chi$ alone that there exists a real value $\chi_0$ such that the positive values of $r$ for which $\partial r/\partial t = 0$ are imaginary, coincident, or real, according as $\chi > \chi_0$, $\chi = \chi_0$, $\chi < \chi_0$. It follows that, if $r$ is small when $t$ is small, then, if $\chi > \chi_0$, $r$ varies from zero to infinity; if $\chi = \chi_0$, $r$ tends asymptotically to a value $r_0 > 0$; if $\chi < \chi_0$, $r$ increases up to the first positive root of $\partial r/\partial t = 0$, and then decreases again to zero. That is to say, the matter outside $\chi = \chi_0$ continues to spread away from the centre, while the matter inside $\chi = \chi_0$ ultimately falls back upon itself. This, then, is the mode of formation of condensations, for example, the extra-galactic nebulae, contemplated by Lemaître. He points out that it is necessary for the form of this theory that $\lambda \neq 0$.

Without going further into the implications of Lemaître's assumptions, it is now clear that his work would be interpreted in Newtonian language by saying that he is dealing with such a distribution of matter and motion that *all the particles outside a certain shell are endowed with a 'hyperbolic' radial velocity of escape from the matter between themselves and the centre, while all the particles inside this shell have an 'elliptic' radial velocity, and so must fall back on the centre.**

* *Added in proof:* In a very recent paper (*Proc. Nat. Acad. Sci.* 20 (Jan. 1934), 12–17), seen after the above was written, Lemaître has himself alluded to the classical analogy to his work, without however following it out in detail or showing the connexion with the equation of continuity.