

ALTERNATIVE SPACE-TIME FOR THE POINT MASS

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ABSTRACT. Schwarzschild's actual exterior solution (g_S) is resurrected, and together with the manifold $M_0 = R^4 - \{r = 0\}$ is shown to constitute a space-time possessing all the properties historically thought to be required of a point mass. On the other hand, the metric (g_{DW}) that today is ascribed to Schwarzschild, but which was in fact first obtained by Droste and Weyl, is shown to give rise to a space-time that is neither equivalent to Schwarzschild's nor derivable from the "historical" properties of a point mass. Consequently, the point-mass interpretation of the Kruskal-Fronsdal space-time (M_W, g_{KF}) can no longer be justified on the basis that it is an extension of Droste and Weyl's space-time. If such an interpretation is to be maintained, it can only be done by showing that the properties of (M_W, g_{KF}) are more in accord with what a point-mass space-time should possess than those of (M_0, g_S). To do this, one must first explain away three seeming incongruities associated with (M_W, g_{KF}): its global nonstationarity, the two-dimensional nature of its singularity, and the fact that for a finite interval of time it has no singularity at all. Finally, some of the consequences of choosing (M_0, g_S) as a model of a point mass are discussed.

1. NOTATION AND INTRODUCTION

Let K denote the analytic manifold consisting of R^4 together with the single-chart atlas $\{R^4, Id\}$, and let the thereby-defined coordinates be denoted by (t, x, y, z) . Let L denote the line $x = y = z = 0$, and M_0 the submanifold $K - L$. Since it is essential to what follows, note that the dimensionality of any point set on a manifold is determined by the set's description in terms of the admissible coordinates, and thus has nothing to do with the subsequent choice of a metric. Thus, e.g., the assertions that L is a line, and that its intersection with $t = \text{const}$ is a point, are valid no matter what metric may be assigned to M_0 .

The properties that the metric (g) of a single, nonrotating, nonradiating, uncharged point mass¹ should possess were first stated by Einstein [1] in 1915, and subsequently employed by Schwarzschild [2], Droste [3], Weyl [4], Hilbert [5], etc., to derive the exact form of g . An explicit formulation of all these properties, including those that were only tacit (e.g., Lorentz-signature, global time coordinate), together with those of the associated

Published in Phys. Rev. D **20**, 2474 (1979).

(Received 11 March 1977; revised manuscript received 30 July 1979).

¹Henceforth, this description will be shortened to point mass.

manifold (M) was given by Finkelstein [6] in 1958. Specifically, M should be M_0 , while g should be analytic, Lorentz signature, and

- (a) a solution on M of Einstein's free-space field equations;
- (b) invariant under time translations;
- (c) invariant under spatial rotations;
- (d) (spatially) asymptotically flat;
- (e) inextendable to L ;
- (f) invariant under spatial reflections;
- (g) invariant under time reflection;
- (h) have a global time coordinate.

Contrary to popular opinion, an analytic, Lorentz signature metric (g_S) possessing properties (a)-(h) on M_0 does exist. Indeed, it has been available ever since 1916, when Schwarzschild [2] derived it. Introducing quasipolar coordinates via

$$(1) \quad x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

with

$$(2) \quad 0 < r, \quad 0 < \theta < \pi, \quad 0 \leq \phi < 2\pi,$$

the components of g_S in these coordinates² are given by

$$(3) \quad ds^2 = (1 - \alpha/f)dt^2 - (1 - \alpha/f)^{-1}f'^2 dr^2 - f^2 d\Omega^2,$$

where α is a positive constant,

$$(4) \quad d\Omega^2 \equiv d\theta^2 + d\phi^2 \sin^2 \theta,$$

$$(5) \quad f \equiv (r^3 + \alpha^3)^{1/3},$$

and the prime denotes differentiation with respect to r .

That the metric possesses properties (b), (c), (d), (f), (g), and (h) is evident by inspection. That it possesses property (a) can be verified by direct substitution into the field equations, or by examining Schwarzschild's derivation. Lastly, that it has property (e) follows from the violation³ of local flatness at every point of L . Clearly, (M_0, g_S) does not contain a black hole.

²Since quasipolar coordinates are inadmissible on the surface $x = y = 0$, it is to be understood here and afterwards that whenever a line element is displayed in terms of such coordinates, this is done solely for ease of recognition, and that the metric is actually defined by its components in terms of (t, x, y, z) . In the case of g_S these quasi-Cartesian components are analytic everywhere on M_0 .

³Let C_a denote the circle of "radius" a defined by $\{(t, r, \theta, \phi) | t = \text{const}, r = a, \theta = \pi/2\}$. The proper circumference of C_a is seen from (3) to be $2\pi f(a)$. As $a \rightarrow 0^+$, this tends to $2\pi\alpha$, rather than to zero as local flatness requires.

2. THE DROSTE-WEYL METRIC

In 1917, Droste [3] and, independently, Weyl, [4] derived a metric (g_{DW}) for the point mass by techniques similar to that used by Schwarzschild. Letting \overline{M}_0 denote a copy of M_0 with the same admissible coordinates (t, x, y, z) as before, and letting $(t, \bar{r}, \theta, \phi)$ denote the same quasipolar coordinates defined by (1) and (2), the components of g_{DW} are defined on that portion (\overline{M}_α) of \overline{M}_0 for which $\bar{r} > \alpha$ by

$$(6) \quad ds^2 = (1 - \alpha/\bar{r})dt^2 - (1 - \alpha/\bar{r})^{-1}d\bar{r}^2 - \bar{r}^2d\Omega^2,$$

and on the remainder of M_0 by analytic continuation. Note that the \bar{r} in (6), like the r in (3), is the ordinary quasipolar coordinate $(x^2 + y^2 + z^2)^{1/2}$, and thus its M_0 -spanning values satisfy $\bar{r} > 0$.

Believing that g_{DW} was “equivalent” to g_{S} , both Droste and Weyl credited their results to Schwarzschild. A few years later, Hilbert [5] opined that the form of g_{DW} was preferable to that of g_{S} , and ever since then the phrase “Schwarzschild solution” has been taken to mean g_{DW} rather than g_{S} .

3. INEQUIVALENCE OF THE SCHWARZSCHILD AND DROSTE-WEYL SPACE-TIMES

Let us relabel the r coordinate of points of M_0 by means of the coordinate transformation

$$(7) \quad T : \bar{r} = f(r) = (r^3 + \alpha^3)^{1/3}, \quad r > 0$$

⁽⁴⁾ so that the M_0 -spanning values of \bar{r} satisfy $\bar{r} > \alpha$.

Under T , g_{S} is carried into \bar{g}_{S} :

$$(8) \quad ds^2 = (1 - \alpha/\bar{r})dt^2 - (1 - \alpha/\bar{r})^{-1}d\bar{r}^2 - \bar{r}^2d\Omega^2.$$

In the early years of relativity, when “physical equivalence” was thought to be a relationship between metrics, the formal identity of (6) and (8) was taken to mean that \bar{g}_{S} (and *a fortiori* g_{S}) and g_{DW} describe the same physical phenomenon (which is the reason that both Droste and Weyl attributed their results to Schwarzschild). Today, however, it is recognized that physical equivalence is a relationship between space-times [7], and that for two space-times (M, g_1) and (M, g_2) defined on the *same* manifold to be equivalent, it is not only necessary that a coordinate transformation carry g_1 into g_2 , but also that the transformation be a diffeomorphism [8]. In the present case this latter requirement is not met, since T carries M_0 *into* \overline{M}_0 (viz., onto \overline{M}_α), rather than *onto* it, and is thus not a diffeomorphism. Consequently (M_0, g_{S}) and $(\overline{M}_0, g_{\text{DW}})$ are inequivalent space-times. This invalidates one of the two historical bases for regarding g_{DW} as the metric of a point mass.

⁴Here too, as in footnote 2, this equation is written in polar coordinates for simplicity. The actual transformation is defined by the corresponding relations in terms of the quasi-Cartesian coordinates (t, x, y, z) and $(t, \bar{x}, \bar{y}, \bar{z})$, obtainable from (7) by cubing both sides and then using (1) to eliminate the polar coordinates, e.g., $\bar{x}^3 = x^3 + \alpha^3 x^3 / (x^2 + y^2 + z^2)^{3/2}$, etc. These relations are defined everywhere on M_0 .

4. DERIVATION OF THE DROSTE-WEYL SPACE-TIME

The other historical basis for the point-mass interpretation of g_{DW} is that Droste [3], Weyl [4], Hilbert [5], etc., believed that they had derived it by explicitly tailoring the general metric to have properties (a)-(h). As will now be shown, their derivations produce not g_{DW} , but rather \bar{g}_{S} , the coordinate-transformed version of g_{S} , in which $\bar{r} > \alpha$ spans M_0 .

Consider a typical derivation⁵. Although the manifold is not specified, there is no basis for supposing that anything other than M_0 was contemplated, so that this will be assumed in what follows.

To begin with, properties (b), (c), (f), (g), and (h) are imposed on the metric, which as is well known [2, 3, 4, 5] restricts the line element to the form

$$(9) \quad ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)d\Omega^2$$

in terms of the quasipolar coordinates defined by (1) and (2). Note that property (h) requires that A be positive on M_0 , and that this together with the Lorentz-signature requirement compels B and C to be positive as well.

Next, the unknown C is eliminated by making the coordinate transformation

$$(10) \quad \bar{r} = [C(r)]^{1/2},$$

thereby transforming (9) into

$$(11) \quad ds^2 = \bar{A}(\bar{r})dt^2 - \bar{B}(\bar{r})d\bar{r}^2 - \bar{r}^2d\Omega^2.$$

Property (a) is now imposed, which leads to two ordinary differential equations for the unknown A and B . The solution of these equations determines both unknowns, apart from two constants of integration. Property (d) is used to eliminate one of them, whereupon the metric, hereinafter referred to as g'_{S} , takes the *form* of (6). Finally, a comparison is made with Newtonian physics to evaluate α - for our purposes, this step can be replaced by imposing property (e), which requires that α be nonzero.

As is evident from this brief sketch, in order for g'_{S} to be g_{DW} it is necessary that the M_0 -spanning values of \bar{r} in (11) be the same as those of the \bar{r} in (6), viz., $\bar{r} > 0$ [10]. But as can be seen from (10), the M_0 -spanning values of \bar{r} in (11) depend on the behavior of $C(r)$ as r varies over $(0, \infty)$. Thus the only way to determine the values in question is to substitute (9) directly into the empty-space field equations and solve for the most general $C(r)$ compatible with properties (d) and (e). This is done in the Appendix, and the result is

$$(12) \quad [C(r)]^{1/2} = \alpha/[1 - A(r)],$$

where α is a positive constant and A is any analytic, strictly monotonic increasing function of r which tends to zero as $r \rightarrow 0$ and behaves like

⁵Although what follows is really a generic version of the classical derivation, it is closely approximated by that in Ref. [9].

$1 - \alpha/r$ as $r \rightarrow \infty$. Together with (10), the just-mentioned properties of A show that the \bar{r} in (11) is a strictly monotonic increasing function of r which tends to α as $r \rightarrow 0$, and to infinity as $r \rightarrow \infty$. Consequently, no matter what choice of admissible A (and thus of admissible C) is made, the M_0 -spanning values of \bar{r} in (11), and *a fortiori* in g'_S , satisfy $\bar{r} > \alpha$, not $\bar{r} > 0$, and thus g'_S is not g_{DW} ⁶. [As a matter of fact, since g'_S has the same form as \bar{g}_S , and since the \bar{r} in (11) has the same M_0 -spanning values as the \bar{r} in (8), g'_S is \bar{g}_S .]

It is worth emphasizing that there is nothing wrong with making the transformation (10). An error arises only if it is assumed that the resulting \bar{r} is a “centered” radial coordinate - i.e., one whose M_0 -spanning values satisfy $\bar{r} > 0$ (whereas by “noncentered” is meant one whose M_0 -spanning values satisfy $\bar{r} > b > 0$).

To clarify this last point, consider Minkowski’s space-time (K, g_M) , where in polar coordinates the components of g_M are described by (see footnote 2)

$$(13) \quad ds^2 = dt^2 - dr^2 - r^2 d\Omega^2.$$

If one makes the coordinate transformation

$$(14) \quad \bar{r} = r + a \quad (a > 0),$$

then the transformed metric (\bar{g}_M) is given by

$$(15) \quad ds^2 = dt^2 - d\bar{r}^2 - (\bar{r} - a)^2 d\Omega^2 \quad (\bar{r} > a).$$

Again, there is nothing wrong with the use of (14), and (K, \bar{g}_M) is still Minkowski’s space-time, so long as one remembers that in (15), $\bar{r} = (x^2 + y^2 + z^2)^{1/2} + a$, so that its K -spanning values satisfy $\bar{r} \geq a$ - that is to say, if one remembers that values of $\bar{r} < a$ are meaningless, just as are values of $r < 0$ in (13). It is only if one decides to regard \bar{r} in (15) as the ordinary, centered radial coordinate $(x^2 + y^2 + z^2)^{1/2}$, whose K -spanning values satisfy $\bar{r} \geq 0$, that the interpretation of (K, \bar{g}_M) as Minkowski’s space-time is incorrect.

Similarly, it is the interpretation of \bar{r} in (6) as the ordinary, centered, quasipolar radial coordinate that invalidates the derivation of g_{DW} from (a)-(h), and thus deprives g_{DW} of the other of its two historical bases for being interpreted as the metric of a point mass.

⁶Droste’s derivation (Ref. [3]) differed from that described here, since he chose to define a new radial coordinate by setting $\bar{B} = 1$, rather than by setting $\bar{C} = \bar{r}^2$. This new radial coordinate is a more complicated function of A than the right-hand side of (12), and by suitable choice of an integration constant its M_0 -spanning values can be made to satisfy $\bar{r} > 0$. However, at the next-to-last step of his derivation he introduced another radial coordinate to bring the metric into the form of (6). The M_0 -spanning values of this last \bar{r} satisfy $\bar{r} > \alpha$, so that his final result is not g_{DW} , but g'_S .

5. COMPARISON OF THE SCHWARZSCHILD AND KRUSKAL-FRONSDAL SPACE-TIMES

An extension of $(\overline{M}_0, g_{\text{DW}})$ to a portion (M_W) of $R^2 \times S^2$ was found by Synge [11], Szekeres [12], Kruskal [13] and Fronsdal [14]; the latter two also showed that the extended space-time (M_W, g_{KF}) contains a black hole.

Because of the belief that g_{DW} was the metric of a point mass, and because heretofore there has been no viable⁷ challenger for the role, (M_W, g_{KF}) has come to be accepted as *the* space-time of a point mass. As seen in Secs. 3 and 4, however, neither of the two historical bases for regarding g_{DW} as the metric of a point mass is valid. Moreover, (M_0, g_{S}) was shown in Sec. 1 to be an eminently qualified challenger. Consequently, the point-mass interpretation of (M_W, g_{KF}) can no longer be based on the fact that it is an extension of Droste and Weyl's space-time. If such an interpretation is to be maintained, it can only be done on the basis of its own properties - more precisely, by comparing its properties with those of (M_0, g_{S}) , and articulating the reasons why the former are more in accord with a point mass than the latter. To facilitate this comparison, the following table describes the candidates' behavior with respect to several areas of interest:

Comparison of Schwarzschild and Kruskal-Fronsdal Space-Times

Property	Schwarzschild	Kruskal-Fronsdal
1. Topology	Euclidean	non-Euclidean
2. Type of singularity	point	two-surface [17]
3. Presence of singularity	all time	absent for a finite time [15]
4. Geometry	globally static	globally nonstationary [14]
5. Curvature invariant	finite as $r \rightarrow 0$	infinite as $v^2 - u^2 \rightarrow 1^-$.

Not all these properties are on the same footing as regards their decisiveness for the choice in question. In particular, few today would argue that the topology of a point-mass universe must be Euclidean - about the most that can be said is that "other things being equal", a simple model is preferable to a complicated one. Likewise, there is no absolute necessity for polynomial curvature invariants to tend to infinity as a singularity is approached [18] - all that can be said is that bounded values are the exception [19].

On the other hand, the behavior exhibited by Schwarzschild's space-time with regard to 2, 3, and 4 would seem to be inherent in the very concept of a point mass. Consequently, in order to choose (M_W, g_{KF}) over (M_0, g_{S}) ,

⁷A Euclidean-topology but nonanalytic model was constructed by A. Komar [15]; shortly thereafter Brans (Ref. [10]) showed that it was afflicted with a number of undesirable properties. A model proposed by A. Janis, E. Newman, and J. Winicour [16] was also nonanalytic, and its derivation involved a somewhat arbitrary choice for the limit of the coefficient of $d\Omega^2$ as $r \rightarrow \alpha^+$.

one should be prepared to explain, *first*, how it is possible for a static phenomenon to give rise to a time-varying geometry; *second*, how, in a theory in which matter is manifested by metric singularities, it is possible to represent a point mass for a finite time by a universe whose metric has no singularity whatever; and *third*, how it is possible for a mathematical point to be modeled as a two-dimensional surface. Although these contradictions have been pointed to in the past, so far as I am aware they have never been squarely faced by the proponents of the point-mass interpretation of (M_W, g_{KF}) . Now that the historical basis for this interpretation has been invalidated, it appears to be essential to do so.

6. CONCLUSIONS

Irrespective of which space-time is chosen, it should be clear from the analysis of Sec. 4 that use of (11) to represent a spherically symmetric phenomenon generally makes \bar{r} a noncentered radial coordinate, which if regarded as centered will give rise to an error similar to that which occurred in connection with g_{DW} . Since (11) (and its cylindrical counterpart) has been used to derive many of the exact solutions known today (e.g., the Reissner-Nordström metric⁸), the interpretation of the associated space-times is suspect, and might more appropriately be conferred on their “Schwarzschild-type” alternatives, obtained by substituting (9) directly into the field equations. The same suspicion also attaches to those exact solutions which have not been derived but merely “found”, and which for certain values of their parameters reduce to g_{DW} (e.g., the Boyer-Lindquist version [21] of Kerr’s metric).

If Schwarzschild’s space-time should ultimately prevail as a model for a point-mass, then two additional consequences arise:

First, of course, current ideas as to the nature of physics in the immediate neighborhood of a mass point will have to be revised, since until now this domain has been thought to be the inside of a black hole.

Second, analysis of gravitational equilibrium [22] or collapse [23] involves the choice of both an “interior” and an “exterior” metric. For the spherically symmetric case the exterior metric must be that of a point mass⁹, so that heretofore g_{DW} has been used. With (M_0, g_S) as the model for a point mass, the exterior metric must be g_S ¹⁰, so that all work done to date on spherically symmetric equilibrium or collapse would be invalid. In particular, with g_S as the exterior metric no black hole is formed no matter how far the collapse

⁸See, for example, the derivation given by R. Tolman [20].

⁹For the case of collapse to a point. In other cases, the fact that the exterior metric need not possess property (e) permits use of a wider class of metrics than just that of a point mass. See, e.g. K. Schwarzschild [24].

¹⁰For the case of collapse to a point. In other cases, g_S should be replaced by the generalization of the point-mass metric that is obtained by omitting property (e). As may be seen from Schwarzschild’s paper cited in Ref. [24], this generalization of g_S has the same form as (3), but with (5) replaced by $f = (r^3 + \rho^3)^{1/3}$, where ρ is independent of α . The corresponding generalization of (28) is similar - (28) is unaffected, but it is no longer necessary that A tend to zero as $r \rightarrow 0$.

proceeds - consequently, if uncharged, nonrotating, spherically symmetric black holes exist, they were not created by gravitational collapse, but are primordial.

ACKNOWLEDGMENTS

This paper would not have been written were it not for the encouragement and insistence of Ray Hemann of Lockheed California Company. Thanks are also due to F. Estabrook, R. Finkelstein, C. Fronsdal, Ralph Smith, and H. Wahlquist for helpful discussions, P. Bergmann, W. Bonnor, E. Newman and K. Thorne for critical comments on an earlier version, N. Rosen for support when most needed, and B. O'Neill and my son Howard for clarifying the differential geometry concepts involved.

APPENDIX: THE MOST GENERAL ANALYTIC, LORENTZ METRIC SATISFYING (a) – (h)

Although Schwarzschild [2] did not make use of (10), he did impose an extraneous condition on the solution, namely, that its determinant be -1 when expressed in terms of (t, x, y, z) . Since this requirement has nothing to do with the physics of a point mass, its imposition may eliminate some metrics that are consistent with Finkelstein's (a)-(h). By dropping this requirement and avoiding the use of (10), the most general solution is obtained.

To this end, let us suppose that A, B, C denote any analytic function of r such that (9) satisfies (a), (d), (e). Substituting (9) into Dingle's¹¹ expressions for T_j^i results in

$$(16) \quad -8\pi T_1^1 = -1/C + C'^2/4BC^2 + A'C'/2ABC = 0,$$

$$(17) \quad -8\pi T_2^2 = C''/2BC + A''/2AB - C'^2/4BC^2 - B'C'/4B^2C \\ - A'^2/4A^2B - A'B'/4AB^2 + A'C'/4ABC = 0,$$

$$(18) \quad T_3^3 = T_2^2,$$

$$(19) \quad -8\pi T_4^4 = C''/BC - 1/C - B'C'/2B^2C - C'^2/4BC^2 = 0,$$

with all other T_j^i identically zero.

Subtracting (19) from (16) leads to

$$(20) \quad 2C''/C' - C'/C - B'/B - A'/A = 0$$

which integrates to

$$(21) \quad C'^2 = JABC,$$

where J is an arbitrary constant. Since A, B, C are positive (see Sec. 4), it follows that $J \geq 0$. But if J were zero, then C' would be identically zero, which would make C constant and thus violate property (d). Hence $J > 0$, and C' never vanishes.

¹¹See Ref. [20], pp. 253-257.

Solving (21) for B and substituting therefrom for B into (16) yields, after some algebra

$$(22) \quad C'(C'/C + 2A'/A - 4C'/JAC) = 0.$$

Since $C' \neq 0$, we obtain from the other factor in (22)

$$(23) \quad C = \alpha^2/(4/J - A)^2,$$

where α is an arbitrary nonzero constant. Substituting for C and C' from (23) into (21) gives

$$(24) \quad B = 4\alpha^2 A'^2/JA(4/J - A)^4.$$

Substituting for B and C from (24) and (23) into (17) shows that the latter is satisfied identically for arbitrary A . Hence, (9) becomes

$$(25) \quad ds^2 = Adt^2 - [4\alpha^2 A'^2/JA(4/J - A)^4]dr^2 - [\alpha^2/(4/J - A)^2]d\Omega^2.$$

Applying property (d) to the coefficient of $d\Omega^2$ shows that

$$(26) \quad \alpha^2/(4/J - A)^2 \sim r^2 \text{ as } r \rightarrow \infty,$$

which reduces to

$$(27) \quad A \sim 4/J - \alpha/r \text{ as } r \rightarrow \infty.$$

Applying property (d) to the coefficient of dr^2 adds nothing to (27), but applying it to the coefficient of dt^2 shows that $J = 4$. Hence (25) becomes

$$(28) \quad ds^2 = Adt^2 - [\alpha^2 A'^2/A(1 - A)^4]dr^2 - [\alpha^2/(1 - A)^2]d\Omega^2.$$

A cannot be 1 since this would destroy the analyticity of the coefficient of $d\Omega^2$. This, together with the positivity of the coefficient of dr^2 , shows that A' cannot be zero. Together with the analyticity of A , this means that A' is either always negative or always positive. Ruling out the former on Newtonian grounds¹², it follows from (27) that $\alpha > 0$.

Finally, if $A \rightarrow a > 0$ as $r \rightarrow 0$, then the transformation $\bar{r} = A(r)$ would result in a diagonal set of \bar{g}_{ij} whose elements are nonzero, finite, and have a nonzero determinant as $r \rightarrow 0$, which would permit the metric to be extended to L , contrary to property (e). Hence, A must tend to zero as $r \rightarrow 0$. We conclude: The most general analytic, Lorentz-signature metric satisfying (a)-(h) (and having positive mass: see footnote 12) is given in quasipolar coordinates by (28), where α is an arbitrary positive constant, and $A(r)$ is an arbitrary analytic, strictly monotonic increasing function of r which tends to zero as $r \rightarrow 0$ and behaves like $1 - \alpha/r$ as $r \rightarrow \infty$.

Comparison of (28) and (3) shows that the $A(r)$ which gives rise to Schwarzschild's solution is

$$(29) \quad A_S(r) = 1 - \alpha/f(r),$$

¹² $A' < 0$ would require $\alpha < 0$, which in turn would give rise to negative mass when compared to the Newtonian expression for the potential at large values of r [see [25]]. There seems to be no way to rule this out on the basis of (a)-(h) alone.

which is easily seen to satisfy all of the above-mentioned requirements on A . Moreover, if A^* denotes any admissible choice of A , the resulting metric can always be transformed into g_S by means of the transformation

$$(30) \quad 1 - \alpha/f(\bar{r}) = A^*(r),$$

which converts (28) into (3) with r replaced by \bar{r} - and equally important, makes \bar{r} a strictly monotonic increasing function of r , with M_0 -spanning values satisfying $\bar{r} > 0$. Thus g_S can be regarded as a canonical form for the metric of a point mass.

On the other hand, comparison of (28) and (6) shows that the $A(r)$ which gives rise to the Droste-Weyl metric is

$$(31) \quad A_{\text{DW}}(r) = 1 - \alpha/r,$$

which violates the requirement that $A \rightarrow 0$ as $r \rightarrow 0$.

Finally, the simplest metric obtainable is that corresponding to $A(r) = r/(r + \alpha)$. This gives

$$(32) \quad ds^2 = [r/(r + \alpha)]dt^2 - [(r + \alpha)/r]dr^2 - (r + \alpha)^2 d\Omega^2.$$

ERRATUM [26]

1st and 3rd lines under Eq. (6): Replace M_0 by \overline{M}_0 .

9th line under Eq. (8): Insert “, when regarded as a mapping from M to M ,” between “transformation” and “be”.

2nd line under Eq. (11): Replace A by \overline{A} , and B by \overline{B} .

Also, the proof of the requirement $A(0^+) = 0$ is incorrect; however, the requirement is still valid [27].

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