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Axisymmetrical elliptical solution of the couple Vlasov plus Poisson equations

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In 1942 S. Chandrasekhar published a book entitled «Principles of Stellar Dynamics» (Dover Pub.) containing the first attempts to describe sets of stars as self–gravitating non–collisional systems, through the couple Vlasov plus Poisson equations.
In the seventies I extended this approach to galaxies, using more compact techniques inspired by the well known book of S.Chapman and T.G. Cowling «The mathematical theory of non uniform gases». This gave quite promising results. But the models showed infinite mass. Now, we know the solution. Galaxies are imbedded in repellent negative mass, so that a model of a galaxy, can be expected as an exact solution of two Vlasov equations, coupled by Poisson.
This is completely out of the mainstream approach for galactic dynamics. But, as one says :

– Let the best win !

To be continued

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EXTRAIT

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ELLIPTICAL SOLUTIONS OF THE VLASOV EQUATION

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I) Introduction : A historic problem has been the search for elliptic solutions of the Vlasov equation.First,Boltzmann derived the maxwellian solution.However,although in that case the solution does exist,it disagrees with any real self-consistent stellar ystem.Later Eddington returned to the general solution,and wrote the equations in curvilinear coordinates,associated with the principal directions of the velocity ellipsoïd.He demonstrated that the principal velocity surfaces should be confocal quadrics, and derived the spherically symmetric solution.In 1928 Oort partially solved the general problem,by deriving expressions for the velocity dispersions and the differential motion.In order to achieve the solution,two remaining equations,involving th stellar density and the gravitational potential,had to be coupled to Poisson's equation,and solved.Although the partial results of Oort gave some encouraging agreement with the observations,the mathematical existence of the solution was not proved.In 1941 Camm attacked the problem.He first showed that the potential should be of the form :

$$\psi = \frac{F(\xi) - G(\eta)}{\xi - \eta}$$

ξ and η being the confocal coordinates.Camm checked the validity of the solution requiring there not to be an infinite mass for the system.Following this criterium he found that no self consistent elliptic function could satisfy the Vlasov equation.

II) Preliminary remark : Is it desirable to pay such an attention to the solution at the infinity ? We can remark that :

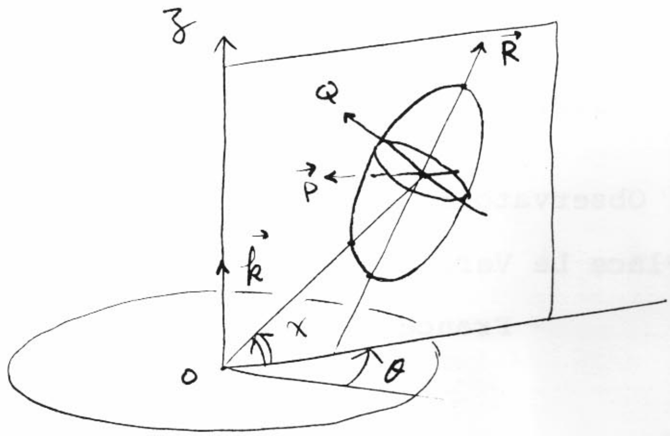
. The Vlasov equation does not take account on the non Euclidian expansion of the Universe.

.. A galaxy is from being an isolated object in space,and ar targe radii its field must be modified (shielded) by the neighbours.

In the following we intend to reconsider the problem,without any requirement with respect to the total mass.

III) Building a solution :

We start with geometric assumptions about the velocity ellipsoïd. Then we derive the macroscopic quantities. At the end, when trying to solve the potential equations, we pay attention to the existence problem. Consider an elliptical fonction in which two axis of the velocity ellipsoïd lie in the ($\vec{\rho}$, $\vec{z}$) plane :



Such that :

$$f = \mathcal{F}\left\{ \varphi - \frac{mC^2}{2kH} + a(\vec{C}\cdot\vec{r})^2 + \alpha\left[\vec{C}\cdot\vec{k}\times\vec{r}\right]^2 + \beta(\vec{C}\cdot\vec{r})\left[\vec{C}\cdot\vec{r}\times(\vec{k}\times\vec{r})\right]\right\} \quad (1)$$

where $\vec{C}$ is the residual velocity, $\vec{k}$ the unit vector (o,o,1), k the constant of Boltzmann, m the mass of a star, n the density, φ the normalization factor and, in general H , a , $a, \alpha, \beta, \varphi$ are unknown time and space dependent functions, to be determined. For reason of symmetry β vanishes for z = o and for $\rho = 0$
We will introduce the particular solution :

$$f_* = \mathcal{F}\left\{ \varphi - \frac{mC^2}{2kH} + a(\vec{C}\cdot\vec{r})^2 + \alpha\left[\vec{C}\cdot\vec{k}\times\vec{r}\right]^2 \right\} \quad (2)$$

For which one of the axis of the ellipsoïd points, in all space, towards the center of the system. First we concentrate on this solution (2). Introducing (2) into the Vlasov equation, written in terms of residual velocity, we first derive the dispersions of the velocity :

$$\left\{ \begin{array}{l} \sigma_R = \sqrt{\frac{kT_0}{m}} \; ; \; \sigma_P = \sqrt{\frac{kT_0}{m}}\left(1 + \frac{2(a-\alpha)kT_0}{m}\rho^2 + \frac{2akT_0}{m}z^2\right)^{-\frac{1}{2}} \\ \sigma_Q = \sqrt{\frac{kT_0}{m}}\left(1 + \frac{2akT_0}{m}(\rho^2 + z^2)\right)^{-\frac{1}{2}} \quad \text{with } T_0 = T_0(t) \\ \qquad \overrightarrow{\frac{\partial a}{\partial r}} = \overrightarrow{\frac{\partial \alpha}{\partial r}} = 0 \end{array} \right. \quad (3)$$

In steady state, according to Eddington's analyzis, the princi-pas axis of the velocity ellipsoïd, associated to solution (1) must be tangent to confocal surfaces. In the plane (z=o) :

$$\beta = 0 \; ; \; \partial\beta/\partial z = 0 \qquad \text{(by symmetry)}$$

Such as the solutions (1) and (2) must be identical in that part of space. Thus, there we have :

$$\frac{\sigma_z}{\sigma_\rho} = \frac{1}{\left(1 + \frac{2akT_0}{m}\rho^2\right)^{\frac{1}{2}}} \tag{4}$$

At the vicinity of the focus there is no continuous variation of the principal directions of the velocity ellipsoïd except if : $\sigma_\rho = \sigma_z$. This contredicts (4) , thus : $\mathcal{F}$ identifies with zero

In steady state, the general axlsymmetric elliptic solution, if it exists, has an ellipsoïd of velocity whose principal axis points towards the center of the system.

Now the identification on the second order terms gives the field of the macroscopic velocity :

$$\left\{ \vec{C_0} = -\frac{1}{2T_0}\frac{dT_0}{dt}\vec{r} + \frac{\omega_0(t)\,(\vec{k}\times\vec{r})}{1 + \frac{2(a-\alpha)kT_0}{m}\rho^2 + \frac{2akT_0}{m}z^2} \right. \quad \begin{array}{|l} \frac{d\alpha}{dt} = 0 \\ \quad (5) \\ \frac{d}{dt}\left(\frac{a}{T_0}\right) = 0 \end{array}$$

When steady state, the results (3) and (4) identify with Oort's particular solution for which the ellipsoïd becomes a sphere at the center. Thus, this can be considered as a non steady extension of Oort's work.

Now we examine the first order equations. With $\phi = \log\frac{1}{\sqrt{T_0}}$ and for axially symmetric systems, we find :

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial r}\left[\varphi + \frac{m}{kT_0}\Psi + \frac{m}{kT_0}(\ddot\phi + \dot\phi^2)\frac{r^2}{2} + \Psi_1\right] = 0 & (6) \\ \frac{\partial}{\partial \chi}\left[\varphi + \frac{m}{kT_0}\left[\Psi + (\ddot\phi + \dot\phi^2)\frac{r^2}{2}\right]\left(1 + \frac{2akT_0}{m}r^2\right) + \Psi_1\right] = 0 & (7) \\ \frac{\partial\omega}{\partial t} + 2\dot\phi\omega + 2\dot\phi\left[\rho^2\frac{\partial\omega}{\partial(\rho^2)} + z^2\frac{\partial\omega}{\partial(z^2)}\right] = 0 & (8) \end{array} \right.$$

Ψ_1 being a known funtion of a, T_0, ω_0, α and r and χ : the spherical coordinates.

(6) abd (7) give immediatly :

$$\boxed{\Psi + (\ddot\phi + \dot\phi^2)\frac{r^2}{2} = \frac{F(r,t) - G(\chi,t)}{r^2}} \tag{9}$$

Which is an extension of Camm's work to non steady systems. Now we introduce the angular velocity as given by (5) into equation (8). It can be satisfied only for three cases :

$$\begin{cases} \text{(a) : steady state solution with differential motion.} \\ \text{(b) : non steady solution with } \partial\omega/\partial z \equiv 0 \\ \text{(c) : non steady elliptical solution with spherical symmetry (extension of Eddington's work).} \end{cases}$$

the cases (b) and (c) will be examined at the end of the paper. Now we concentrate on solution (a) and deal with the existence problem.

IV) The existence of an elliptical solution with differential ~~wx~~ motion :

Now, in this steady state case (a), we take a special form of the functional $\mathcal{F}$ which is an exponential. Such as (2) becomes a particular Schwarzschild's solution :

$$\boxed{\log f = \log An - \frac{mc^2}{2kH} + a(\vec{C}.\vec{z})^2 + \alpha[\vec{C}.\vec{K}\times\vec{z}]^2} \quad (10)$$

The zeroth order equation is :

$$\vec{C_0} \cdot \overrightarrow{\frac{\partial}{\partial z} \log An} \quad (11)$$

$$A \equiv \left(1 + \frac{3(a-\alpha)kT_0}{m}\rho^2 + \frac{2akT_0}{m}z^2\right)^{\frac{1}{2}}\left(1+\frac{2akT_0}{m}(\rho^2+z^2)\right)^{\frac{1}{2}}$$

it is obviously satisfied for axially symmetric solution. Introduce Poisson's equation and the following adimensional quantities :

$$r = \rho\left(\frac{2akT_0}{m}\right)^{\frac{1}{2}} ; \quad \frac{m\psi}{kT_0} = \Psi ; \quad \frac{a-\alpha}{a} = \lambda ; \quad \mu^2 = \frac{\omega_0^2}{2a}\left(\frac{m}{kT_0}\right)^2$$

The existence of the solution is found to depend on the solvability of the following system :

$$\begin{cases} \dfrac{r^2F'' - 2rF' + 2F - 2G - G'\operatorname{tg}\chi + G''}{r^4} = \dfrac{\exp\left[G - \dfrac{F+G}{r^2} - \dfrac{\mu^2 r^2\cos^2\chi}{1+\lambda^2 r^2\cos^2\chi + r^2\sin^2\chi}\right]}{(1+r^2)^{\frac{1}{2}}(1+\lambda^2 r^2\cos^2\chi + r^2\sin^2\chi)} & (12) \\[2ex] \dfrac{\partial F}{\partial \chi} = 0 & (13) \\[2ex] \dfrac{\partial G}{\partial r} = 0 & (14) \end{cases}$$

If we can find a certain part of space where those three equations are satisfied, we shall say that, in that part of space, the solution does exist.

Let us expand into a series for $\chi \ll 1$; in the vicinity of the equatorial plane. We get, with $G(0)=0$

$$\left\{ \frac{r^2F''-2rF'-2F+G''(0)-2\chi^2G''(0)}{r^4} = \frac{\exp\left[G''(0)\frac{\chi^2}{2}\left(\frac{1+r^2}{r^2}\right)-\frac{F}{r^2}+\frac{1}{2}\frac{\mu^2r^2(1-\chi^2)}{1+\lambda^2r^2(1-\chi^2)+r^2\chi^2}\right]}{(1+r^2)^{\frac{1}{2}}\left(1+\lambda^2r^2(1-\chi^2)+r^2\chi^2\right)^{\frac{1}{2}}} \right. \qquad (15)$$

For $\chi \neq 0$ F is found to be : $F = F(r,\chi^2)$

Such as the solution does not exist in all space. But, in the equatorial plahe :

$$\frac{\partial F}{\partial \chi} = 0$$

Thus, in that particular part of space which is the equatorial plane, the solution does exist.

It is more convenient to return to $(r^2\psi = F)$ and then :

$$\boxed{\psi'' + \frac{2}{r}\psi' - \frac{4}{r^2}\psi + \frac{G''(0)}{r^4} = \frac{\exp\left[-\psi + \frac{\mu^2r^2}{2(1+\lambda^2r^2)}\right]}{(1+r^2)^{1/2}(1+\lambda^2r^2)^{1/2}}} \qquad (16)$$

V) The problem of the regularity at the center of the system :

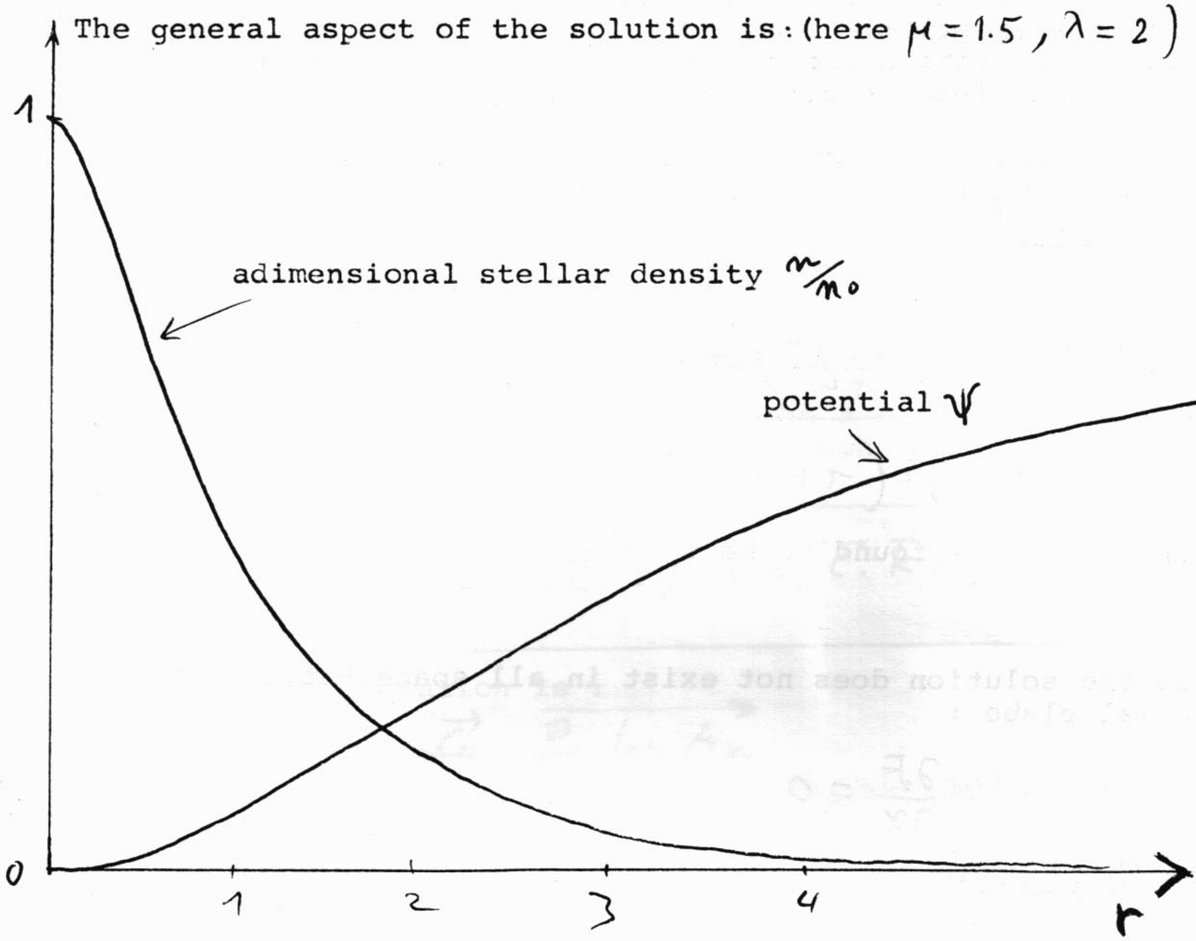
The solution shows some irregularity at the center. We know that our schwarzschild's solution becomes a maxwellian when r tends to zero. Such as, say for $r=0.1$ we can replace f by the maxwellian f°, which does exist and is regular at the center A convenient adjustment of the parameters will bring a continuous link between the two solutions at $r=0.1$

VI) Numerical results:

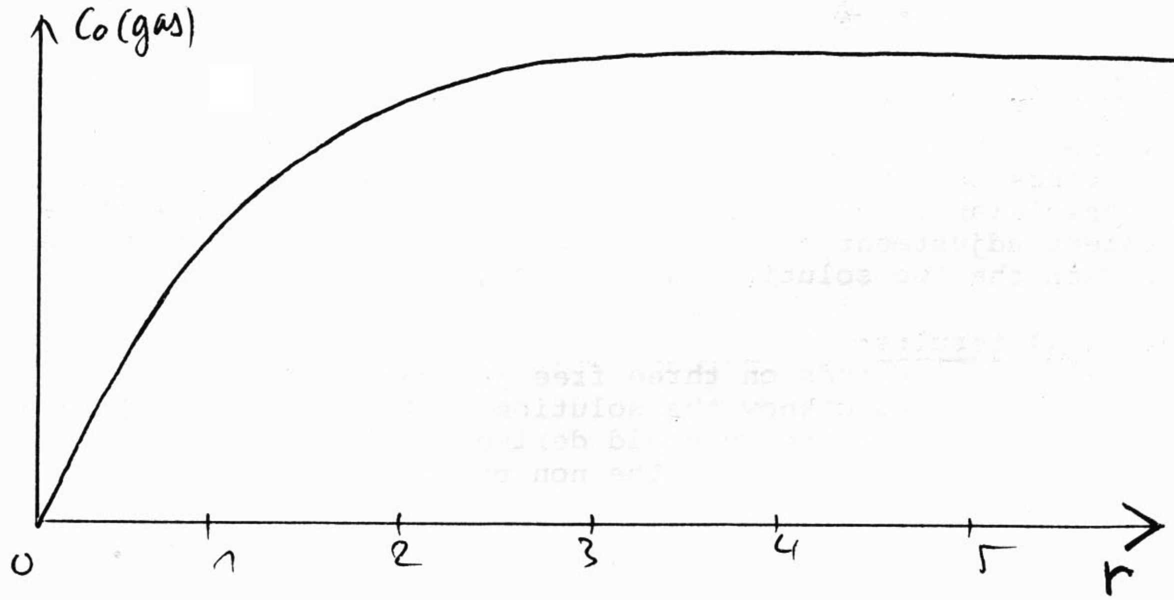
The solution (16) depends on three free parameters. They are not independent. If we could know the solution in the immediate vicinity of the equatorial plane we could derive a relation between them This is not possible because of the non existence of the solution for $\chi \neq 0$.

The boundary conditions are ; $\Psi(0)=0$; $\Psi'(0)=0$.

The general aspect of the solution is : (here $\mu = 1.5$, $\lambda = 2$)



The stellar velocity and the evolution of the dispersions of the velocity are wellknown and corresponds to Oort's. But we can derive a "gas" velocity (circular velocity of a test particle, in the gravitational potential due to the stars). We get :



VII) <u>The meaning of this solution</u> :

It is not proved yet that an elliptic functional can exist in all space.If it does it is not a Schwarzschild's functional.However we have found a Schwarzschild's functional that does exist in the plane z=o .Perhaps it is a part of a more sophisticated functional,that becomes a Schwarzschid's functional in the plane z = o ,and that exist everywhere.

VIII) <u>The problem of the integrals of the motion</u> :

The particular Schwarzschild's solution derived here obeys

$$\log f = \log A_n - \frac{mC^2}{2kH} + a(\vec{C}.\vec{z})^2 + \alpha[\vec{C}.\vec{k}\times\vec{z}]^2$$

$$H = \frac{T_o}{1 + \frac{2akT_o}{m}z^2}$$

$$\log A_n + \frac{m\Psi}{kT_o} - \frac{m\rho^2\omega_o^2}{2kT_o\left(1 + \frac{2(a-\alpha)kT_o}{m}\rho^2\right)} = cst.$$

Introducing the absolute velocity $\vec{W}$ we find :

$$\boxed{\log f = -\frac{mI_1}{kT_o} - (a-\alpha)(I_2)^2 + \frac{m}{kT_o}\omega_o I_2}$$

where :

$$I_1 = \frac{1}{2}mW^2 + \Psi \qquad \text{Energy}$$

$$I_2 = \rho W_\theta \qquad \text{momentum}$$

The distribution function thus can be put in terms of the two wellknown integrals of the motion.

IX) Non steady solutions :

They correspond to the cases (b) and (c) introduced in the section III .

(b) non steady maxwellian solution :

$$f = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mC^2}{2kT}}$$

$$\frac{\overrightarrow{\partial T}}{\partial r} = 0 \quad ; \quad \sigma_r = \sigma_p = \sigma_q = cst. \qquad \text{isothermy}$$

$$\vec{C_0} = -\frac{1}{2T}\frac{dT}{dt}\vec{r} + \vec{\omega}\times\vec{r} \quad ; \quad \frac{\partial \omega}{\partial r} = 0$$

$$\frac{d}{dt}\left(\frac{T}{\omega}\right) = 0 \qquad \text{compatibility equation}$$

$$\left(\frac{\partial}{\partial t} + \vec{C_0}\cdot\frac{\vec{\partial}}{\partial r}\right)\log\frac{n}{T^{3/2}} = 0 \qquad \text{isentropy}$$

$$\text{with } \phi = \log\frac{1}{\sqrt{T}} \quad ; \quad \nu^2 = cst \times T^{3/2}$$

$$\frac{\partial^2\Psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} + \frac{\partial^2\Psi}{\partial z^2} = \nu^2\exp -\frac{m}{kT}\left[\Psi + (\ddot{\phi} + \dot{\phi}^2)\frac{r^2}{2} - \frac{1}{2}\rho^2\omega^2\right]$$

(c) non steady elliptic solution with spherical symmetry :

$$f = n\left(\frac{m}{2\pi kT_0}\right)^{3/2}\left(1 + \frac{2akT_0}{m}r^2\right)$$

$$\exp -\frac{m}{kT_0}\left[C_r^2 + \left(1 + \frac{2akT_0}{m}r^2\right)\left(C_p^2 + C_q^2\right)\right]$$

$$\frac{\overrightarrow{\partial a}}{\partial r} = 0 \qquad \frac{\overrightarrow{\partial T_0}}{\partial r} = 0 \qquad \frac{2akT_0}{m} = \frac{1}{r_0^2}$$

$$\vec{C_0} = -\frac{1}{2T_0}\frac{dT_0}{dt}\vec{r} \quad ; \quad \nu^2 = Cst \times T_0^{3/2}$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} = \frac{\nu^2}{1+\frac{r^2}{r_0^2}} \exp - \frac{m}{kT_0}\left[\psi + (\ddot{\phi}+\dot{\phi}^2)\frac{r^2}{2}\right]$$

$$\left(\frac{\partial}{\partial t} + \vec{C_0}\cdot\frac{\vec{\partial}}{\partial r}\right)\log\left(1+\frac{2akT_0}{m}r^2\right)\frac{n}{T_0^{3/2}} = 0$$

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