International Journal of Modern Physics D, Vol. 8, No. 3 (1999) 271–289

SCALE INVARIANT COSMOLOGY

PIERRE MIDY*
C.R.I., Université Paris XI,
Centre d'Orsay, F 91405 Orsay, France

JEAN-PIERRE PETIT†
Observatoire de Marseille, 2, Place Le Verrier,
F 13248 Marseille Cedex 4, France

Received 15 January 1999
Communicated by A. Dolgov

An attempt is made here to extend to the microscopic domain the scale invariant character of gravitation — which amounts to consider expansion as applying to any physical scale. Surprisingly, this hypothesis does not prevent the redshift from being obtained. It leads to strong restrictions concerning the choice between the presently available cosmological models and to new considerations about the notion of time. Moreover, there is no horizon problem and resorting to inflation is not necessary.

1. Introduction

Since Hubble's discovery of the recession of galaxies, obtaining the variation with cosmic time of the scale parameter (or "radius") $R(t)$ which describes the expansion of the universe is the basic problem a cosmological theory has to deal with. The closely related question of deciding relative to what the expansion takes place, or where it stops, is much more seldom explicitly stated.[1] It is commonly admitted there is no expansion at distances smaller than the size of clusters of galaxies, so that this size, or the size of our galaxy, or of the solar system, ... can be used as a reference scale. This may seem to be sensible at present, but what about the past, when $R(t)$ was smaller than these objects? The existence and observability of permanent reference scales for time and length intervals thus appear to be assumed. It is also commonly admitted — but this argument has been questioned[2,3] — that the scale parameter cannot be applied to all length scales in the universe, on the grounds that this would amount to getting no expansion at all, so that the universe would be scale invariant and there would be no redshift. On the other hand, if the inflation theory eliminates the causality problem for the presently observed regions

*E-mail: pierre.midy@cri.u-psud.fr
†E-mail: perso@jp-petit.com
For scientific correspondence: Villa Jean-Christophe, Chemin de la Montagnère, F 84120 Pertuis.

of the universe, it does not in the future .... In order to solve these problems, L. Nottale[1] introduces a transition from scale independence to scale dependence in the general frame of a theory of scale relativity.

Already Laplace[1,4] had already pointed out the scale invariance of Newton's theory of gravitation, concluding that "the universe reduced to the smallest imaginable space would always present the same appearance to observers" and that "the laws of nature only permit us to observe relative dimensions." Poincaré made similar observations as concerns comparing time intervals as well as lengths at two different instants (and this is, of course, the only way to proceed for time intervals): he notes "we have no direct intuition of the equality of two lapses of time" — for instance those "between noon and one o'clock and between two and three o'clock."[5] So it should be possible to allow a general variation of all lengths[2] and time intervals as long as one keeps in mind that this kind of variation cannot be physically observed by comparisons necessarily performed at the same instant: there would be no permanent observable reference scale; since this looks as an unnecessary complication, the possibility for expansion to act at any scale has seldom been considered seriously. However, Hoyle and Narlikar,[6] as an alternative to expansion, used a somewhat similar view in their variable mass theory, where all masses are supposed to increase with cosmic time, which implies decreasing wavelengths for atomic radiations and decreasing sizes for atoms. Indeed, it may be observed that, as opposed to the case of pure gravitation, where the size and period of a Keplerian orbit can be changed arbitrarily as long as Kepler's laws are respected, without any modification of Newton's constant nor of the masses of the bodies, physical constants must be allowed to change if the expansion is considered to apply, for instance, to structures the size of which is directly linked to atomic sizes. This is clear from the expression of the Bohr radius,

$$a_0 = \frac{h^2 \epsilon_0}{\pi m_e e^2} = \frac{\hbar}{m_e \alpha c}\,, \tag{1}$$

which involves physical constants only: supposing the absence of permanent observable reference scales for length and time intervals implies some physical constants, among which c, can vary with cosmic time.

So, since c also appears in the expression of the metric tensor, we shall suppose it to be variable and we shall look for a function $R(t)$ which resulted from one of the classical cosmological theories: such a function should not depend on hypotheses which can be made about the reference scale relative to which $R(t)$ is defined. This leads to a highly constrained problem which cannot be solved realistically (i.e., with nonzero pressure and density) unless more than four dimensions are introduced in the description of spacetime. This idea, initially due to Einstein and recently reintroduced by P. Wesson and his co-workers,[7,8] aims at obtaining the properties of matter in our four-dimensional spacetime in a purely geometrical way, starting from Einstein's equations

$$R_{\mu\nu} = 0 \tag{2}$$

written for five dimensions or more with no matter-energy terms on the right-hand side to find the usual four-dimensional equations

$$R^{(1+3)}_{\mu\nu} = -\frac{8\pi G}{c^2} S_{\mu\nu} = -\frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\ \lambda}\right), \tag{2'}$$

where $T_{\mu\nu}$ is the energy–momentum tensor of a perfect fluid, and we shall try it together with the hypothesis of a variable c.

2. The Einstein Equations

We write Einstein's equations for an arbitrary number n of space dimensions and with a variable c in a maximally symmetric space, hence with a metric which, as concerns space, generalizes to n dimensions the metric of Robertson–Walker. Choosing such a metric with n greater than 3 is obviously surprising — usually a different scale factor is associated with space dimensions beyond the third one — and this will be discussed later in Secs. 5 and 6.

It is to be noted that, as a consequence of our hypotheses, the geometrical spacetime coordinates — defined with respect to permanent reference scales — will appear to be distinct from the physically observable ones. This is obvious for lengths: expansion applying to spatial domains of arbitrary size, physical distances here are the comobile ones; the case of time intervals will be dealt with in Secs. 3 and 4. So as to make this distinction easier, geometrical coordinates or intervals will generally be written using Latin characters, and physical coordinates or intervals using Greek ones.

Designing by 0 the index corresponding to the chronological coordinate x^0, with $dx^0 = c(t)dt$, and by i, $j = 1, 2, \ldots, n$ the indices associated with the spatial coordinates, the components of the metric tensor are

$$g_{00} = -1\,, \tag{3}$$

$$g_{i0} = 0\,, \tag{4}$$

$$g_{ij} = R^2(t)\tilde{g}_{ij}\,, \quad \tilde{g}_{ij} = 0 \quad (i \neq j)\,, \tag{5}$$

where the $\tilde{g}_{ij}$ depend on the space coordinates only. Choosing x^0 rather than t and taking into account the time-dependence of c at the end of the calculations allow one to get a constant g_{00} and thus a smaller number of nonzero components for the affine connection. Starting from

$$\Gamma^{\beta}_{\gamma\alpha} = \frac{1}{2} g^{\beta\eta}[\partial_\gamma g_{\alpha\eta} + \partial_\alpha g_{\eta\gamma} - \partial_\eta g_{\gamma\alpha}]\,, \tag{6}$$

(where the four indices run from 0 to n), these components are found to be (the $'$ designing a derivation with respect to x^0)

$$\Gamma^{0}_{ki} = RR'\tilde{g}_{ki}\,, \tag{7}$$

$$\Gamma^{j}_{0i} = \Gamma^{j}_{i0} = \frac{R'}{R}\delta^{j}_{i}\,, \tag{8}$$

$$\Gamma^j_{ki} = \tilde{\Gamma}^j_{ki}\,, \tag{9}$$

so the Γ^j_{ki} are time-independent.

This yields for the Ricci tensor

$$R_{\alpha\beta} = \partial_\beta \Gamma^\eta_{\eta\alpha} - \partial_\eta \Gamma^\eta_{\alpha\beta} + \Gamma^\lambda_{\alpha\eta}\Gamma^\eta_{\beta\lambda} - \Gamma^\eta_{\eta\lambda}\Gamma^\lambda_{\alpha\beta}\,, \tag{10}$$

the nonvanishing components

$$R_{00} = n\frac{R''}{R}\,, \tag{11}$$

$$R_{ij} = \tilde{R}_{ij} - \big(RR'' + (n-1)R'^2\big)\tilde{g}_{ij}\,, \tag{12}$$

where,[9] $k = 0, \pm 1$ being the curvature index,

$$\tilde{R}_{ij} = -(n-1)k\tilde{g}_{ij}\,, \tag{13}$$

hence

$$R_{ij} = -[RR'' + (n-1)R'^2 + (n-1)k]\tilde{g}_{ij}\,. \tag{14}$$

Taking into account the hypothesis of a variable c and introducing the time variable t gives

$$R' = \frac{dR}{dx^0}\,, \tag{15}$$

$$R'' = \frac{d^2R}{{dx^0}^2} = \frac{1}{c^2}\frac{d^2R}{dt^2} - \frac{1}{c^3}\frac{dc}{dt}\frac{dR}{dt}\,, \tag{16}$$

so that (11) and (14) become

$$R_{00} = \frac{n}{Rc^2}\left(R'' - \frac{c'}{c}R'\right) \tag{17}$$

and

$$R_{ij} = -\left[\frac{R}{c^2}\left(R'' - \frac{c'}{c}R'\right) + (n-1)\left(\frac{R'^2}{c^2} + k\right)\right]\tilde{g}_{ij}\,, \tag{18}$$

where the prime now stands for the derivation relative to t.

Of course, the same result might have been obtained using t instead of x^0 from the beginning, getting

$$g_{tt} = -c^2(t)$$

instead of (3), a nonzero

$$\Gamma^t_{tt} = \frac{c'}{c}$$

and

$$\Gamma^t_{ki} = \frac{RR'}{c^2}\tilde{g}_{ki}$$

instead of (7).

Applying (2) to (17) and (18) immediately entails

$$k = -1 \tag{19}$$

if the density is to be nonzero, so that the n-space curvature is negative, and

$$R'^2 = c^2\,, \tag{20}$$

hence

$$R' = \epsilon c \quad (\epsilon = \pm 1)\,, \tag{21}$$

$$R'' = \epsilon c' \tag{22}$$

and

$$\frac{c'}{c} = \frac{R''}{R'}\,. \tag{23}$$

Noting

$$R_{00}^{(F;1+3)} = \frac{3R''}{Rc^2} \tag{24}$$

and

$$R_{ij}^{(F;1+3)} = -\frac{1}{c^2}(RR'' + 2R'^2)\tilde{g}_{ij} \quad (i,j = 1,3) \tag{24$'$}$$

the parts of the Ricci tensor components which correspond to the Friedmann model with zero curvature and using Eqs. (19) to (23), the time–time (17) and space–space (18) components of the Ricci tensor can be written

$$R_{00} = R_{00}^{(F;1+3)} - \frac{3\epsilon c'}{Rc^2}\,, \tag{25}$$

$$R_{ij} = R_{ij}^{(F;1+3)} + \left(2 + \epsilon\frac{Rc'}{c^2}\right)\tilde{g}_{ij} \quad (i,j = 1,3)\,. \tag{25$'$}$$

Identifying (2) with (2$'$) where the source term is

$$S_{00} = \frac{1}{2}(\rho + \frac{3p}{c^2})\,, \tag{26}$$

$$S_{i0} = 0\,, \tag{27}$$

$$S_{ij} = \frac{1}{2}(\rho - \frac{p}{c^2})R^2\tilde{g}_{ij} \tag{28}$$

gives

$$\frac{4\pi G}{c^2}\left(\rho + \frac{3p}{c^2}\right) = -\frac{3\epsilon c'}{Rc^2}\,, \tag{29}$$

$$\frac{4\pi G}{c^2}\left(\rho - \frac{p}{c^2}\right) = \left(2 + \epsilon\frac{Rc'}{c^2}\right)\frac{1}{R^2}\,. \tag{30}$$

Of course, for this identification to be possible, the negative curvature n-space has to contain an Euclidean three-dimensional variety: an example of this situation will be given later in Sec. 5.

Both R and c depending on t, there must be a relation between them. Writing, with K constant,

$$\phi(R, c) = K \tag{31}$$

hence

$$\frac{\partial\phi}{\partial R}R' + \frac{\partial\phi}{\partial c}c' = 0 \tag{32}$$

and, using (21),

$$\frac{c'}{c} = -\epsilon\frac{\partial\phi/\partial R}{\partial\phi/\partial c} \tag{33}$$

(29) and (30) give

$$\frac{\rho c^2 + 3p}{\rho c^2 - p} = \frac{3\delta}{2-\delta}\,, \tag{34}$$

where

$$\delta = \frac{R}{c}\frac{\partial\phi/\partial R}{\partial\phi/\partial c}\,. \tag{35}$$

So, for an equation of state (34) with δ a constant,

$$\frac{dc}{dR} = -\frac{\partial\phi/\partial R}{\partial\phi/\partial c} = -\delta\frac{c}{R} \tag{36}$$

and

$$cR^\delta = K' \tag{37}$$

or, with $\gamma = 1/\delta$,

$$Rc^\gamma = K''\,, \tag{38}$$

where K' and K'' are constants.

The equation of state (34) can be given the form

$$p = \left(\frac{2}{\gamma} - 1\right)\frac{\rho c^2}{3} \tag{39}$$

and includes the two cases

$$p = 0 \tag{40}$$

and

$$p = \frac{\rho c^2}{3} \tag{40$'$}$$

of the dust universe and of the radiation era for $\gamma = 2$ and 1 respectively.

Whatever γ, (29) and (30) yield

$$\frac{8\pi G}{3}\rho = \frac{R'^2}{R^2}\,, \tag{41}$$

so that the density of matter ρ corresponds, as results from the elimination procedure leading to (41), to the curvature term in the expressions (18) or (25′) of R_{ij}. Introducing the Hubble constant

$$H = \frac{R'}{R}\,, \tag{42}$$

(41) writes

$$\rho G = \frac{3}{8\pi}H^2 \tag{43}$$

— so the density equals the critical density. This gives for the density of energy $\varepsilon = \rho c^2$

$$\varepsilon = \frac{3c^2}{8\pi G}H^2\,. \tag{43'}$$

Of course, the pressure is a function of γ

$$p = \frac{c^2}{8\pi G}\left(\frac{2}{\gamma} - 1\right)H^2\,. \tag{44}$$

The scale parameter $R(t)$ is easily obtained deriving (38) with respect to time and using (23) to get

$$R'' + \frac{R'^2}{\gamma R} = 0 \tag{45}$$

from which it can be seen that δ is but the deacceleration parameter q. The non-trivial solution of (45) which satisfies

$$R(0) = 0 \tag{46}$$

and

$$R(t_0) = R_0 \tag{46'}$$

is:

$$R(t) = R_0\left(\frac{t}{t_0}\right)^{\frac{\gamma}{\gamma+1}}\,, \tag{47}$$

as in the zero curvature Friedmann model for an equation of state of the type (39).

Thus for the radiation era ($\gamma = 1$),

$$R(t) = R_0\left(\frac{t}{t_0}\right)^{1/2} \tag{48}$$

and for the dust universe ($\gamma = 2$),

$$R(t) = R_0\left(\frac{t}{t_0}\right)^{2/3}\,. \tag{49}$$

3. Time and the Redshift

In the same way as lengths remain proportional to the cosmological scale factor $R(t)$, intervals of cosmic time vary as t itself: for instance, the period $T = l/c$ of a radiation of wavelength l varies as

$$T \approx \frac{R}{c} \approx R^{\frac{\gamma+1}{\gamma}} \approx t \tag{50}$$

and the same kind of variation will characterize the period of any circular motion, since its speed can be defined as a fraction of c, and thus varies as $R^{-1/\gamma}$, as also results for various examples (period of a pendulum, of planetary motions, ...) from the gauge relations described in Sec. 7: every time interval expands proportionally to cosmic time with respect to a permanent reference scale.

At a given t, a cosmic time interval dt can but be defined as proportional to t and to the number $d\vartheta$ of periods a reference clock completes during this interval dt

$$dt = ktd\vartheta\,, \tag{51}$$

where k is a constant, the dimension of which should be the inverse of a time if ϑ is to be a time also. Obviously, intervals of ϑ only (and of the associated proper time), not of t, can be measured.

Let us now state the nature of ϑ more precisely. The square of the $(1 + 3)$ spacetime interval with zero space curvature writes, following from now on the timelike convention

$$ds^2 = c^2(t)dt^2 - dl^2\,, \tag{52}$$

where dl is the space interval, or, introducing $R(t)$, which we suppose to have the dimension of a length

$$ds^2 = c^2(t)dt^2 - R^2(t)d\xi^2 \tag{52$'$}$$

and here $d\xi$ stands for the spatial distance in dimensionless comoving coordinates.

Computing $c(t)$ (with $\epsilon = 1$) from (21) and (47)

$$c(t) = \frac{\gamma}{\gamma+1}\frac{R_0}{t_0}\left(\frac{t}{t_0}\right)^{-\frac{1}{\gamma+1}} \tag{53}$$

and writing in (51)

$$k = \frac{\gamma+1}{\gamma}\frac{c_M}{R_M} \tag{54}$$

with

$$R(t) = R_M a(t)\,, \tag{55}$$

where R_M and c_M are constant and $a(t)$ is dimensionless, yields

$$ds^2 = a^2(t)\left[c_M^2 d\vartheta^2 - R_M^2 d\xi^2\right]$$

or

$$ds^2 = a^2(t)\left[c_M^2 d\vartheta^2 - d\lambda^2\right] \tag{56}$$

with

$$d\lambda = R_M d\xi \tag{57}$$

to give the spatial distance $d\lambda$ the dimension of a length. Thus ϑ can be identified with the conformal time, which gives the Robertson–Walker metric an expression of the conformal type, and c_M can be interpreted as the constant light velocity in the Minkowski spacetime which is measured when using coordinates ϑ and λ. It may be noted that, up to a scaling factor, a unique definition of the conformal time is obtained here for the dust universe as well as the for the radiation one — which does not happen in the classical theory.

So as to study the formation of the redshift, let us consider, as in Ref. 10, two events occurring at the same point in space and separated by a time interval corresponding to (56)

$$\Delta s_1 = c_M a(t_1)\Delta\vartheta\,, \tag{58}$$

where this time interval is small as compared to time t_1. If these two events consist in the emission of two light signals which will be perceived at another point in space, the time interval between the two instants of reception will be the same in conformal time as in (58), that is, $\Delta\vartheta$. This results directly from (56): light propagation is described by

$$d\lambda = \pm c_M d\vartheta \tag{59}$$

so that, for a propagation along the radial coordinate χ,

$$d\lambda^2 = R_M^2 d\chi^2\,, \tag{60}$$

$$d\vartheta = \pm\frac{R_M}{c_M}d\chi \tag{61}$$

hence

$$\vartheta = \pm\frac{R_M}{c_M}\chi + \mathcal{C}\,, \tag{62}$$

where $\mathcal{C}$ is constant: the conformal time needed to go from the emission point to the reception one depends on the difference of the radial coordinates of these points only and not on the instant of emission, so both signals will arrive at the (same) reception point at two instants separated by the same conformal time interval $\Delta\vartheta$ as at their emission. To this $\Delta\vartheta$ corresponds at the reception time t_2 an interval

$$\Delta s_2 = c_M a(t_2)\Delta\vartheta\,, \tag{63}$$

hence

$$\frac{\Delta s_2}{\Delta s_1} = \frac{a(t_2)}{a(t_1)}\,, \tag{64}$$

or, after (55),

$$\frac{\Delta s_2}{\Delta s_1} = \frac{R(t_2)}{R(t_1)}\,. \tag{65}$$

This conclusion is valid whatever the type of coordinates used, either (t, l) or (ϑ, λ).

The remainder of the argumentation relies entirely on the definition of proper time

$$d\tau = \frac{1}{c} ds \tag{66}$$

in general relativity:

(a) with coordinates (ϑ, λ), c is constant and equals c_M, and proper time $d\tau_\vartheta$ is defined by

$$ds = c_M d\tau_\vartheta\,, \tag{67}$$

so that

$$\frac{\Delta\tau_{\vartheta_2}}{\Delta\tau_{\vartheta_1}} = \frac{R(t_2)}{R(t_1)}\ : \tag{68}$$

the redshift is obtained;

(b) with coordinates (t, l), proper time $d\tau$ is defined by

$$ds = c(t) d\tau \tag{69}$$

hence

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \frac{\Delta s_2}{\Delta s_1}\frac{c(t_1)}{c(t_2)} = \frac{R(t_2)}{R(t_1)}\frac{c(t_1)}{c(t_2)} \tag{70}$$

and, if γ has the same value at t_1 and t_2,

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \left[\frac{R(t_2)}{R(t_1)}\right]^{\frac{\gamma+1}{\gamma}} = \frac{t_2}{t_1}\,. \tag{71}$$

In the last relation, $\Delta\tau_1$ is the period of the radiation emitted at t_1 and $\Delta\tau_2$ the period of this radiation when received at t_2; now, the redshift is defined by the ratio between $\Delta\tau_2$ and the emission period $\Delta\tau_{e_2}$ of the same radiation at the reception place at t_2 and, as results from (50),

$$\frac{\Delta\tau_{e_2}}{\Delta\tau_1} = \frac{t_2}{t_1}\,, \tag{72}$$

so that

$$\frac{\Delta\tau_2}{\Delta\tau_{e_2}} = 1\ : \tag{73}$$

there would be no redshift if cosmic time intervals were accessible to measurement. In the present model they are not, and the redshift is obtained with conformal time intervals (68). This is a logical consequence of the absence of an observable permanent reference scale for time: t can be regarded as a purely geometrical variable which might be measured from outside the universe, but not inside it (and, by definition, there is no outside). The same can be said about the associated space variables l, and the actual physical variables are ϑ and λ, physical time being the proper time associated with coordinate time ϑ through

$$d\tau_\vartheta = a(t)d\vartheta\,. \tag{74}$$

Of course, at a given t, intervals $\Delta\vartheta$ of conformal time only can be measured. As concerns cosmic time and space variables, only relations established without using nonmeasurable quantities (such as in (72)) are physically meaningful. For example, horizon calculations can be performed with either type of variables:

(a) in geometrical coordinates, they involve ratios of quantities taken at the same time only, the event and the particle horizons being defined at time t_1 as

$$\varrho_e = \int_{t_1}^{\infty} \frac{c(t)dt}{R(t)} \tag{75}$$

and

$$\varrho_p = \int_{0}^{t_1} \frac{c(t)dt}{R(t)}\,; \tag{76}$$

now, from (47) and (53),

$$\frac{c(t)}{R(t)} \propto \frac{1}{t} \tag{77}$$

whatever γ: there is no particle nor event horizon, and no need for inflation.

(b) the same result is obvious in conformal coordinates (ϑ, λ) (here the lower limit of integration in ϱ_p should be replaced by $-\infty$, since conformal time extends to infinity, as will be seen hereafter).

4. Relation Between Times t and ϑ

Integrating (51) between t_0 and t, one gets

$$\vartheta(t_0, t) = \frac{1}{k}\int_{t_0}^{t} \frac{dt}{t} = \frac{1}{k}\ln\frac{t}{t_0} \tag{78}$$

for the conformal time interval corresponding to the interval (t_0, t) of cosmic time. This implies, introducing a third instant t_1

$$\vartheta(t_0, t_1) = \frac{1}{k}\ln\frac{t_1}{t_0} \tag{79}$$

and

$$\vartheta(t_1, t) = \frac{1}{k}\ln\frac{t}{t_1}\,, \tag{80}$$

hence

$$\vartheta(t_0, t) = \vartheta(t_0, t_1) + \vartheta(t_1, t)\ : \tag{81}$$

conformal time is additive.

A cosmic time interval can be defined as an ordered pair (t_1, t_2) to be associated with the ratio t_2/t_1 (or even $R(t_2)/R(t_1)$) — as in (71) for instance

$$\phi(t_1, t_2) = \frac{t_2}{t_1}\,. \tag{82}$$

This defines on cosmic time intervals a multiplicative group law

$$\phi(t_0, t_2) = \phi(t_0, t_1)\phi(t_1, t_2) \tag{83}$$

isomorphic to the additive one specified by (78) and (81). And indeed, a theorem from group theory states that every one — parameter connected differentiable group is isomorphic to an additive one and that the additive parameter is unique:[11,12] so ϑ only is additive, and it can be identified with the linear time θ introduced by Misner[13] as

$$\theta \sim \ln\frac{R(t)}{R(t_0)}\,. \tag{84}$$

Thus, the three possible time notions (cosmic, conformal and linear) of Lévy-Leblond[11] reduce here to two ones only. As already seen, one of them (conformal time ϑ) can be measured and thus appears to coincide with physical time, whereas, by the lack of observability of its reference scale, cosmic time t cannot. However in (50) and (51), t shows up as the dilation parameter of intervals of cosmic time, in the same way as $R(t)$ is the dilation parameter of intervals of length, so that not only space, but spacetime itself, is in expansion. It may be noted that this interpretation of t already seems to be possible for Kepler's third law: if the larger axis of an orbit is multiplied by a factor R, the period must be multiplied by a factor t such that t^2/R^3 is constant, hence the possibility of interpreting the relation $R \sim t^{2/3}$ as a relation between the scale parameters for space and time dimensions. Indeed, the third law expresses the invariance of Kepler's problem in the inhomogeneous dilation generated by the infinitesimal operator

$$X = 3t\frac{\partial}{\partial t} + 2x^i\frac{\partial}{\partial x^i} \tag{85}$$

(see Ref. 14).

One notion of time only being classically used in physics, it seems interesting to approximate the relation between ϑ and t by a linear function, i.e., to replace the function $\vartheta(t)$ by its tangent in the neighbourhood of an instant t, thus shifting from t to ϑ through a change of scale and origin which does not modify the equations of physics.

Substituting $t_1 + \Delta t$ for t in (78), one gets

$$\vartheta = \frac{1}{k}\ln\frac{t_1 + \Delta t}{t_0}\,, \tag{86}$$

hence

$$\vartheta \simeq \frac{1}{k}\left(\ln\frac{t_1}{t_0} + \frac{\Delta t}{t_1}\right) \tag{87}$$

or

$$\vartheta \simeq \frac{1}{k}(\ln\frac{t_1}{t_0} + \frac{t}{t_1} - 1)\,. \tag{87'}$$

Now, inverting (78),

$$t = t_0 \exp(k\vartheta)\,, \tag{88}$$

gives in (47)

$$R(t) = R(t_0)\exp(\frac{\gamma}{\gamma+1}k\vartheta)\,; \tag{89}$$

this expression can be approximated using (87) by

$$R(t) \simeq R(t_1)\left(1 + \frac{\gamma}{\gamma+1}\frac{\Delta t}{t_1}\right) \tag{90}$$

(which might have been obtained directly from (47)): if t_1 is large, (for instance corresponding to present time, so that t_1 is the "age of the universe"), this approximation is satisfactory even for large values of Δt as soon as they are small relative to t_1; if t_1 is small, in the neighbourhood of 0, the same linear approximation for ϑ, in the form (87′), yields

$$R(t) \simeq R(t_1)\exp\left[\frac{\gamma}{\gamma+1}\left(\frac{t}{t_1} - 1\right)\right] \tag{91}$$

or

$$R(t) \sim \mathcal{C}\exp\left(\frac{\gamma}{\gamma+1}\frac{t}{t_1}\right), \tag{92}$$

a result which is valid for t in the vicinity of t_1 and which gives, when extended to larger values of t, the same variation of R with cosmic time as in inflation theory (take for instance $t_1 = 10^{-n}$ s in (92)).

As a last remark, the relation between cosmic and conformal times might help understand — if this is confirmed — why the oldest stars seem to be older than the universe itself. It may be noted that, defining t_M by

$$t_M = \frac{\gamma}{\gamma+1}\frac{R_M}{c_M}\,, \tag{93}$$

(51) can be given the form

$$dt = \frac{t}{t_M}d\vartheta \tag{94}$$

so that $dt = d\vartheta$ at $t = t_M$: the cosmic and the conformal times are in coincidence (have the same scale) for $t = t_M$. Equation (79) for example becomes with this notation

$$\vartheta(t_0, t_M) = t_M \ln\frac{t_M}{t_0}\,. \tag{95}$$

Now, the age of a star leaving the main sequence of the H–R diagram is currently estimated from the time it has spent on this sequence, which is the longest one in its lifespan. This time it is determined by dividing the total nuclear energy available on the main sequence by the amount of energy used per time unit, and of course is computed as a conformal time interval. The relation between this interval and the cosmic times t_M and t_0 which respectively represent the age of the universe and the time the star entered the main sequence is given by (95) for a star which leaves the main sequence presently; it can be rewritten as

$$t_0 = t_M \exp\left(- \frac{\vartheta(t_0, t_M)}{t_M} \right). \tag{96}$$

Applying (96) to the results of Pierce *et al.*,[15] which imply $t_M \simeq 7.3\,10^9$ years for $\Omega = 1$, gives, with $\vartheta(t_0, t_M) \simeq 16.5\,10^9$ years for the age of the oldest star clusters

$$\frac{t_0}{t_M} \simeq \exp(-2.26) \simeq 0.1\,, \tag{97}$$

thus reducing to about $7\,10^8$ years the cosmic time at which these were formed.

5. Euclidian Subspaces in a Constant Negative Curvature Manifold

A well-known example of such a situation is that of 3-space in the spacetime of steady-state cosmology.[9] Schrödinger,[16] who remarked that curvature depends on the frame, had already studied such a case as the Lemaître–Robertson frame of the de Sitter universe, represented as a one-shell hyperboloid $\mathcal{H}_1$

$$x^2 + u^2 + v^2 + y^2 - z^2 = \mathcal{R}^2 \tag{98}$$

embedded in a five-dimensional space with a $(1, 4)$ Lorentzian metric

$$d\sigma^2 = -dx^2 - du^2 - dv^2 - dy^2 + dz^2\,. \tag{99}$$

What we need here — so as to have a negative curvature space and not spacetime — is a two-shell hyperboloid $\mathcal{H}_2$

$$x^2 + u^2 + v^2 + y^2 - z^2 = -\mathcal{R}^2 \tag{100}$$

embedded in the same Lorentz space, with the difference that this imbedding space is completely fictitious here, so that, in particular, no coordinate system in it has to be interpreted as including time as one of its components.

The Lemaître transformation leading to Euclidean subspaces of $\mathcal{H}_1$ reads

$$\bar{x} = \frac{\mathcal{R}x}{y+z}\,, \tag{101}$$

$$\bar{u} = \frac{\mathcal{R}u}{y+z}\,, \tag{102}$$

$$\bar{v} = \frac{\mathcal{R}v}{y+z}\,, \tag{103}$$

$$\bar{\theta} = \ln \frac{y+z}{\mathcal{R}}\,. \tag{104}$$

With these new variables, the $d\sigma^2$ in (99) becomes

$$d\sigma^2 = -\exp(2\bar{\theta})(d\bar{x}^2 + d\bar{u}^2 + d\bar{v}^2) + \mathcal{R}^2 d\bar{\theta}^2 \tag{105}$$

as can be shown using the following relations deduced from (101)–(104)

$$x = \bar{x}\exp(\bar{\theta})\,, \quad u = \bar{u}\exp(\bar{\theta})\,, \quad v = \bar{v}\exp(\bar{\theta})\,, \tag{106}$$

$$y + z = \mathcal{R}\exp(\bar{\theta})\,, \tag{107}$$

and

$$y - z = \mathcal{R}\exp(-\bar{\theta}) - \frac{\bar{r}^2}{\mathcal{R}}\exp(\bar{\theta})\,, \tag{108}$$

where

$$\bar{r}^2 = \bar{x}^2 + \bar{u}^2 + \bar{v}^2$$

and $y - z$ has been computed from

$$y - z = \frac{\mathcal{R}^2 - x^2 - u^2 - v^2}{y + z}\,.$$

It is clear from (105) that the subspaces $(\bar{x}, \bar{u}, \bar{v})$ are Euclidean.

The same Lemaître transformation leads to Euclidean subspaces of $\mathcal{H}_2$ with the only differences that now

$$y - z = \frac{-\mathcal{R}^2 - x^2 - u^2 - v^2}{y + z} = -\mathcal{R}\exp(-\bar{\theta}) - \frac{\bar{r}^2}{\mathcal{R}}\exp(\bar{\theta}) \tag{109}$$

and

$$d\sigma^2 = -\exp(2\bar{\theta})(d\bar{x}^2 + d\bar{u}^2 + d\bar{v}^2) - \mathcal{R}^2 d\bar{\theta}^2\,. \tag{110}$$

The Euclidean subspaces $(\bar{x}, \bar{u}, \bar{v})$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ are the three-dimensional generalizations of the parabolae traced on their restrictions to the dimensions (x, y, z) by planes $y + z = \mathcal{C}$, a constant, and it may be noted (as concerns $\mathcal{H}_2$) that when $\mathcal{R}$ varies with time (expansion), $\bar{\theta}$ remains constant, so that these subspaces are preserved.

Now, depending on the value of $\mathcal{C}$, there is an infinity of such subspaces and it seems to be logical, so as to keep (as supposed from the beginning) a zero four-dimensional density, to associate to each subspace with density ρ another one with negative density $-\rho$ in a twofold structure with

$$\rho > 0\,, \qquad G > 0 \quad \text{on the 1}^{\text{st}}\text{ sheet}$$

and

$$\rho < 0\,, \qquad G < 0 \quad \text{on the 2}^{\text{nd}}\text{ sheet}$$

both of which possibilities being allowed as solutions of (43) (it may be noted that ρG also, and not G alone, appears in the Einstein equations). ρ is an active gravitational mass density, so that two masses should attract when in the same sheet and repel when not if one keeps on identifying passive gravitational mass and inertial mass. Such a negative density sheet has already been proposed as a candidate for dark matter in first simulations of the large scale structure of the universe.[17,18]

6. The Case of Non-Euclidean Subspaces

Such subspaces obviously exist (for $k = -1$, take for instance y^2 constant in (100) and for $k = 1$, z^2 constant and larger than $\mathcal{R}^2$). However, supposing $k = -1$ in steps (24)–(30) implies a zero density (hence the necessity to take $n > 3$), and $k = 1$ yields $\gamma = 1/2$ and 1 for the radiation and the matter dominated eras respectively, that is no correspondence with previous theories. Hence the Friedmann solution with $k = 0$ is the only one which can be found following the present procedure.

7. Gauge Relations

Since, in geometrical coordinates, c has been assumed to vary with time, this should be the same for other physical constants also. This implies gauge relations which have been proposed in Refs. 19 and 20 for the case $\gamma = 2$ (Eq. (38)). Similar relations can be obtained here, without a special treatment being required for Schrödinger's equation. Assuming every mass varies as M and the Bohr radius as R, as already supposed, and using (38) yields for instance, if α (which is dimensionless) remains constant,

$$\hbar \approx MR^{1-\frac{1}{\gamma}}\,. \tag{111}$$

In the same way, starting from Newton's formula for the two-body problem and still using (38) gives for the Einstein constant

$$\frac{G}{c^2} \approx \frac{R}{M} \tag{112}$$

whatever the value of γ is. With no precision about the variance of masses, such relations are purely kinematical — being derived from the variances of length and time intervals. One more hypothesis has to be made to go further; for example, if G/c^2 is to be a constant, this implies

$$M \approx R \tag{113}$$

hence

$$\hbar \approx R^{2-\frac{1}{\gamma}} \tag{114}$$

and

$$G \approx R^{-\frac{2}{\gamma}}\,, \tag{115}$$

and so on, ...

None of these variations can be detected in the laboratory since, as in the case of length and time intervals, no permanent reference scale can be used for that. So, in the physical world, constants are but constants, as light velocity c_M in (56).

However, the above assumption about G/c^2 is important: it allows the divergenceless character of the energy–momentum tensor to be kept in Einstein's equations (2′). Moreover, substituting (47) into (41) yields for the energy density corresponding to the equation of state (39)

$$\varepsilon = \frac{3c^2}{8\pi G}\left(\frac{\gamma}{\gamma+1}\right)t_0^{-2}\left(\frac{R}{R_0}\right)^{-2\frac{\gamma+1}{\gamma}} \tag{116}$$

so that a constant value of G/c^2 implies

$$\varepsilon \propto R^{-2\frac{\gamma+1}{\gamma}} \tag{117}$$

and for the energy contents of a covolume V,

$$E \propto R^{1-\frac{2}{\gamma}} \tag{118}$$

so that during the radiation era ($\gamma = 1$),

$$\varepsilon \propto R^{-4} \tag{119}$$

and

$$E \propto R^{-1} \tag{120}$$

— hence, using the Planck formula, the same variation with R of the temperature of the radiation gas, as also results from the classical treatment — and during the matter-dominated era ($\gamma = 2$),

$$\varepsilon \propto R^{-3} \tag{121}$$

hence a constant energy for the massive particle contents of V.

8. Discussions and Conclusion

In order to express the hypothesis of nonobservability of permanent reference scales for time and length intervals, we have assumed the scale parameter $R(t)$ applies to any scale in the universe — which implies the possibility of a variation with cosmic time of the constants of physics. Thus a clear distinction is made between the universe and the permanent scales with respect to which its time and space charactistics (its age and scale parameter) are defined, so that these permanent scales are to be considered as external to the universe. As opposed to what can be expected, this does not prevent the redshift from being obtained in the resulting theory if one notes that cosmic time (and length) intervals — for which permanent reference scales have to exist as soon as one writes down expressions (3)–(5) or (52) for the metric — are no longer measurable quantities. The physical quantities, measurable from inside the universe, are from this viewpoint the proper time and the proper distances associated with conformal time ϑ and comobile coordinates

λ in Eq. (56) (as ϑ only, and its associated proper time, can be measured, one may wonder why not drop t: however, no permanent reference scale is supposed to exist for ϑ, and if time is to be geometrized, it seems it should be endowed with this property). Thus time and space are dealt with in the same way, as contrasts with the classical theory: as well as space, cosmic time expands, and a time arrow arises naturally. The cosmological and the gravitational redshifts both originate in the variations of the g_{00} coefficient of the metric, the former with time, the latter with position in space — although in both cases, the shift owes its existence in the phenomenon (the emission of a radiation) being produced and observed as two distant events in spacetime.

In physical (i.e., conformal) time, the origin of the universe is shifted to $-\infty$, corroborating the ideas developed by Lévy-Leblond and Misner in Refs. 11 and 13, and this is linked with the logarithmic dependence of ϑ on t as results from (51). This removes the question of knowing what existed before the big bang and might help to solve the problem raised by the age of the oldest star clusters. Other interesting aspects of the model are the absence of horizons and the fact it yields the Euclidean character of one of the Friedmann models, in full agreement with many observations.

As regards the interpretation, the fact that intervals of ϑ and λ only are physically measurable implies the impossibility of directly observing any expansion — hence there is no problem about where it stops; however the metric (56) is not stationary, hence the redshift. Concerning the matter-dominated era, only experiments can help to decide whether scale invariance holds: structures such as atoms or galaxies may well withstand expansion owing to their internal forces; as for the radiation era, the absence of horizons is an interesting characteristic which may supply possibilities for an alternative to the scalar field which is evoked to justify inflation at the microscopic level.

Interestingly, as resulted from the resolution of the system — (29) and (30), the density corresponds to the curvature of n-space; the derivative of c with respect to cosmic time appears in the pressure term only.

The possibility of an objection to the model might be found in the identification of terms involving a variable c with the energy–matter terms of another one (the Friedmann model) which does not suppose this variation. In fact, it may be noted that if both sides of (2′) are multiplied by c^2, so that $R_{\mu\nu}$ is replaced everywhere by $c^2 R_{\mu\nu}$, the Ricci tensor $c^2 R_{\mu\nu}^{(F;1+3)}$ (Eqs. (24) and (24′)) of the zero curvature Friedmann model (and of this one only) does not depend on c. And the classical equations of this model write

$$2\frac{R''}{R} + \frac{R'^2}{R^2} = -\frac{8\pi G}{c^2} p \tag{122}$$

and

$$3\frac{R'^2}{R^2} = \frac{8\pi G}{c^2} \varepsilon \,, \tag{123}$$

which can also be derived from (29) and (30) above: they involve G/c^2 only when expressed in terms of the density of energy ε. With G/c^2 constant as indicated above, there is no contradiction in identifying the classical terms with those of the present model.

As a last remark, it may be noted that the basic hypotheses we have started from involve both general relativity and, through the expression of the Bohr radius, a fundamental result of quantum mechanics — which are both, at an elementary level, connected with an inverse square law.

Acknowledgments

We wish to thank Drs. Guy Oliver and Ivan Brissaud and Profs. Michel Mizony and Léon Brenig for numerous fruitful discussions.

References

1. L. Nottale, in *Fractal Spacetime and Microphysics* (World Scientific, Singapore, 1993).
2. S. K. Blau, *Am. J. Phys.* **63**, 779 (1995).
3. C. Callender and R. Weingard, *Am. J. Phys.* **63**, 780 (1995).
4. P. S. de Laplace, in *Œuvres Complètes* (Gauthier-Villars, Paris, 1878).
5. H. Poincaré, in *La Valeur de la Science* (Flammarion, Paris, 1973).
6. F. Hoyle, in *Astronomy and Cosmology* (Freeman, San Francisco, 1975).
7. P. Wesson, *Mod. Phys. Lett.* **A7**, 921 (1992).
8. J. Ponce de Leon and P. S. Wesson, *J. Math. Phys.* **34**, 4080 (1993).
9. S. Weinberg, in *Gravitation and Cosmology* (Wiley, New York, 1972).
10. L. D. Landau and E. M. Lifschitz, in *Théorie du Champ* (Mir, Moscou, 1962).
11. J. M. Lévy-Leblond, *Am. J. Phys.* **58**, 156 (1990).
12. M. Hamermesh, in *Group Theory* (Addison-Wesley, Reading, 1964).
13. Ch. W. Misner, *Phys. Rev.* **186**, 1328 (1969).
14. N. H. Ibragimov, *The Math. Intelligencer* **16**, 20 (1994).
15. M. J. Pierce, D. L. Welch, R. D. Mc Clure, S. van den Bergh, R. Racine and P. B. Stetson, *Nature* **371**, 385 (1994).
16. E. Schrödinger, in *Expanding Universes* (Cambridge University Press, 1957).
17. J. P. Petit, *Il Nuovo Cimento* **B109**, 697 (1994).
18. J. P. Petit, *Astrophys. Space Sci.* **226**, 273 (1995).
19. J. P. Petit, *Mod. Phys. Lett.* **A3**, 1527 (1988).
20. J. P. Petit, *Mod. Phys. Lett.* **A3**, 1733 (1988).