

**Reconsidering Schwarzschild's original solution**

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We analyse the Schwarzschild solution in the context of the historical development of its present use, and explain the invariant definition of the Schwarzschild's radius as a singular surface, that can be applied to the Kerr-Newman solution too.

*Key words:* General relativity – Schwarzschild solution

**1. Introduction: Schwarzschild's solution and the "Schwarzschild" solution**

Nowadays simply talking about Schwarzschild's solution requires a preliminary reassessment of the historical record as *conditio sine qua non* for avoiding any misunderstanding. In fact, the present-day reader must be firstly made aware of this seemingly peculiar circumstance: Schwarzschild's spherically symmetric, static solution (Schwarzschild 1916) to the field equations of the version of the theory proposed by Einstein (1915a) at the beginning of November 1915 is different from the "Schwarzschild" solution that is quoted in all the textbooks and in all the research papers. The latter, that will be here always mentioned with quotation marks, was found by Droste (1917), Hilbert (1917) and Weyl (1917), who worked instead by starting from the last version of Einstein's theory (Einstein 1915b).<sup>1</sup> As far as the vacuum is concerned, the two versions have identical field equations; they differ only because of the supplementary condition

$$\det(g_{ik}) = -1 \tag{1}$$

that, in the theory of November 11th, limited the covariance to the group of unimodular coordinate transformations. Due to this fortuitous circumstance, Schwarzschild could not simplify his calculations by the choice of the radial coordinate made e.g. by Hilbert; he was instead forced to adopt "polar coordinates with determinant 1" that led him to a solution depending on two parameters instead of the single one found by Hilbert. Schwarzschild could then fix one of the parameters in such a way as to push in the "Nullpunkt" the singularity that is named after him. This move is impossible with Hilbert's choice, and this is the origin of the difference between Schwarzschild's solution and the "Schwarzschild" solution of the literature which, to put an end to the present confusion, should be more aptly named after Hilbert.

<sup>1</sup>Recent inspection of the relevant documents told that Schwarzschild contacted only Einstein about the question of the solution (Treder).

## 2. Solving Schwarzschild's problem without fixing the radial coordinate

In order to display the difference between the spherically symmetric, static solutions found with different choices for the radial coordinate, let us briefly recall the calculations done by Combridge (1923) and by Janne (1923) without fixing the radial coordinate at all. By following de Sitter (1916), they found that the line element of the most general static, spherically symmetric field can be reduced by choice of adapted coordinates to the form

$$ds^2 = -\exp \lambda dr^2 - \exp \mu [r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)] + \exp \nu dt^2 \quad (2)$$

in symmetry adapted coordinates. Here  $\lambda$ ,  $\mu$ ,  $\nu$  are functions of  $r$  only. The proof of the correctness of de Sitter's choice can be found e.g. in Eiesland's paper (1925). By availing of the field equations already made explicit by de Sitter (1916), the mentioned authors found that a solution that is Minkowskian at the spatial infinity could be expressed in terms of one arbitrary function  $f(r)$  and of its derivative  $f'(r)$  by setting:

$$\exp \lambda = \frac{f'^2}{1 - 2m/f}, \quad (3)$$

$$\exp \mu = \frac{f^2}{r^2}, \quad (4)$$

$$\exp \nu = 1 - 2m/f, \quad (5)$$

where  $m$  is a constant, while of course the arbitrary function  $f$  must have the appropriate behaviour as  $r \rightarrow \infty$ . It is now an easy matter to reproduce the results that one obtains when the most popular coordinate conditions are imposed. With the exception of Schwarzschild's case, the latter were adopted with the aim of making the calculations easier. However, a simple glance to the papers by Combridge and Janne shows that the calculations were not as *extra vires* as to be in urgent need of simplification.

Setting  $\lambda = 0$  provides Droste's initial way of fixing the radial coordinate, as given in his equation (4). Despite this initial choice, the "Schwarzschild" solution made its first appearance just in Droste's paper <sup>2</sup>.

Choosing  $\mu = 0$  leads instead straight to the "Schwarzschild" solution, i.e. to Hilbert's form (Hilbert 1917), given by equation (45) of his monumental paper. One notes that if one attributes to  $r$  the usual range  $0 \leq r < \infty$  equation (5) would only admit the value  $m = 0$  of the mass constant. Maybe this is the reason why Hilbert deviated from de Sitter's choice (2) for the line element and dropped the *a priori* condition of persistence of the sign for the components of the metric. The omission allows for a nontrivial solution, but also for the appearance of the well known singularity of the components of the metric at the "Schwarzschild" radius. The singularity separates an outer region  $r > 2m$ , that with the lapse of the years proved itself capable to precisely account for the workings of Nature, from an inner region whose occurrence soon evolved from a slightly annoying peculiarity into a major conundrum (Brillouin 1923).

Posing  $\lambda = \mu$  produces the solution expressed in isotropic coordinates, ingeniously found by Weyl through a coordinate transformation (Weyl 1917) by starting from his own "Schwarzschild" solution. Here again we get a singular behaviour, this time due to the vanishing of  $\det(g_{ik})$  when  $r = m/2$ . There is no track here of the tantalizing inner region of Hilbert. With isotropic coordinates Nature is well accounted for, maybe too well. There is in fact a little redundancy: only the outer part of Hilbert's solution is allowed to appear, but it appears twice, once for  $m/2 < r < \infty$ , and a second time in the range  $0 < r < m/2$ . The two replicas happen to be joined just at the "Schwarzschild" singularity. Confronted with such *embarras de richesse* Weyl flatly declared that in Nature only "ein Stück" of one of the two copies, not reaching the singularity, must be realised (Weyl 1917). Einstein and Rosen (1935) used the full structure to derive an argument for the mass to be positive. They felt one could postulate the necessity of hiding the curvature singularity by constructing such a "bridge" and identify it with a particle. Their hope to get a hint at some kind of quantisation was not met, but they found that only a positive mass allows to find that bridge.

Schwarzschild's true and authentic solution (Schwarzschild 1916), though written with the usual polar coordinates rather than with the original "polar coordinates of determinant 1", can instead be retrieved by imposing the condition  $\lambda + 2\mu + \nu = 0$ . Due to (3), (4) and (5)  $f$  must then fulfil the equation

$$\frac{f'^2 f^4}{r^4} = 1, \quad (6)$$

<sup>2</sup>Insisting with the coordinate condition  $\lambda = 0$  leads one to define  $f$  through the hardly solvable equation

$$r = f(1 - 2m/f)^{1/2} + m \ln \frac{f^{1/2} + (f - 2m)^{1/2}}{f^{1/2} - (f - 2m)^{1/2}} + \text{const.}$$

This is why Droste changed horses *in itinere* and eventually succeeded in becoming the forerunner of Hilbert.

i.e., with appropriate choice of sign,  $f' = r^2/f^2$ . The latter equation integrates to

$$f = (r^3 + \varrho)^{1/3} \quad (7)$$

where  $\varrho$  is a second integration constant, besides  $m$ . Had not he died just a few months after his discovery due to a rare illness contracted while at war on the Russian front, one could say that Schwarzschild, besides being very clever, was also a very lucky man. In fact the burden of the coordinate condition (1) he was *oborto collo* forced to confront by the version of Einstein's theory he was aware of turned out to be a blessing, when compared to the simplifying assumptions adopted by the later authors, that could enjoy the eventually conquered general covariance (Einstein 1915b), (Hilbert 1915). Although the move was later declared not recommendable by Hilbert with a footnote of devastating authority that decided the destiny of the true Schwarzschild's solution <sup>3</sup>, with his extra parameter  $\varrho$  Schwarzschild had the chance of imposing the continuity of the components of  $g_{ik}$  in the range  $0 < r < \infty$ . He did so by setting

$$\varrho = (2m)^3 \quad (8)$$

thereby dispatching in the "Nullpunkt" what the posterity would have unanimously called "the singularity at the Schwarzschild radius".

### 3. The role of hyperbolic motion in choosing Schwarzschild's solution

Schwarzschild's position (8) is sufficient for complying with de Sitter's prescription (2) for the line element, but it is by no means necessary. Larger values of  $\varrho$  fulfil the prescription as well; moreover, the solutions with  $\varrho \geq (2m)^3$  are different from each other, as the simple consideration of the maximum value attained by the scalar curvature in each of them immediately shows. One needs an additional postulate for fixing the value of  $\varrho$ . Abrams (1989) has shown that one can avail of an assumption that seems to be quite appropriate both from a geometrical and from a physical standpoint. Let us consider a test body whose four-velocity is  $u^i$ ; its acceleration four-vector is defined as

$$a^i \equiv \frac{Du^i}{ds} \equiv \frac{du^i}{ds} + \Gamma_{kl}^i u^k u^l, \quad (9)$$

where  $D/ds$  indicates the absolute derivative. From it one builds the scalar quantity

$$\alpha = (-a_i a^i)^{1/2}. \quad (10)$$

The motion of a test body kept at rest in the static, spherically symmetric field whose line element, in adapted coordinates, is given by equations (3), (4) and (5), is defined by postulating the constancy of  $r$ ,  $\vartheta$  and  $\varphi$ . At first glance, the definition of the world lines of *rest* seems to depend on the particular coordinates we use. However, we can identify the congruence of world-lines of our *particles at rest* through use of the Killing vectors of the metric. There is only one time-like Killing congruence that has not only the Killing property, but is hypersurface orthogonal ( $\xi_{[k,l}\xi_{m]} = 0$ ) too. It is the congruence we identify in our coordinates with  $r, \vartheta, \varphi$  constant. It obeys the differential equations

$$\frac{D}{ds} \left( \frac{a^i}{\alpha} \right) - \alpha u^i = 0, \quad \alpha = \text{const}; \quad (11)$$

therefore the test body under question describes an invariantly defined motion, that Rindler (1960) aptly called hyperbolic. The only nonvanishing component of  $a^i$  in a field with the line element (2) is

$$a^1 = \frac{1}{2} \nu' \exp(-\lambda). \quad (12)$$

For the solutions given by (3), (4) and (5) the constant  $\alpha$  has the expression

$$\alpha = \left[ \frac{m^2}{f^3(f - 2m)} \right]^{1/2} \quad (13)$$

in terms of the mass constant  $m$  and of the arbitrary function  $f$ . As noticed by Rindler, besides the geometrical meaning,  $\alpha$  has an immediate physical meaning. Let us consider a locally Minkowskian coordinate system whose spatial coordinates be centered at the position  $r = r_0$ ,  $\vartheta = \vartheta_0$ ,  $\varphi = \varphi_0$ ; the quantity  $\alpha$  equals the strength of the gravitational pull measured by a dynamometer that holds a unit mass at rest at the given position.

<sup>3</sup>See footnote 1 at page 71 of Hilbert (1917).

By substituting (7) in (13) one sees that, with a fixed value of the mass constant, the maximum value of the force that can be measured by the dynamometer is different for different values of  $\varrho$ . As already noticed, the solutions are geometrically and physically different, and we need some reason for choosing one of them. Abrams' argument is the following (Abrams 1989): since up to now no experimental evidence has been found for attributing a finite limiting value to  $\alpha$ , we cannot help mimicking in the case of general relativity the way out adopted in Newtonian physics. There the norm of the force exerted by the gravitational field of an ideal pointlike mass on a test body tends to infinity as the test body is brought nearer and nearer to the source of the field. According to equation (13), in the static spherically symmetric field defined by (3), (4) and (5)  $\alpha \rightarrow \infty$  only when  $f \rightarrow 2m$ , because  $f \rightarrow 0$  is prohibited by equation (5). If one chooses the arbitrary function  $f$  according to (7),  $\alpha$  is allowed to grow without limit at  $r = 0$  only when  $\varrho$  is chosen just in the way kept by Schwarzschild in his fundamental paper of 1916 <sup>4</sup>.

We add a remark on the Killing congruences that shows how to generalize the argument for the full Kerr-Newman-solution. The elements

$$\xi^k \frac{\partial}{\partial x^k} = \lambda \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \varphi} \tag{14}$$

of the Killing group ( $\xi_{k;l} + \xi_{l;k} = 0$ ) of the Kerr-Newman metric

$$\begin{aligned} ds^2 &= \frac{\varrho^2}{\Delta} dr^2 + \varrho^2 d\vartheta^2 + \frac{\sin^2 \vartheta}{\varrho^2} ((r^2 + J^2) d\varphi - J dt)^2 \\ &\quad - \frac{\Delta}{\varrho^2} (dt - J \sin^2 \vartheta d\varphi)^2 \\ \Delta &= r^2 + J^2 + Q^2 - 2Mr \\ \varrho^2 &= r^2 + J^2 \cos^2 \vartheta \end{aligned} \tag{15}$$

define invariantly a set of orbits. The acceleration on these orbits

$$\begin{aligned} \alpha^2 &= -g_{ij} a^i a^j = -g_{ij} \left( \frac{\xi^i}{N} \right)_{;k} \frac{\xi^k}{N} \left( \frac{\xi^j}{N} \right)_{;l} \frac{\xi^l}{N}, \\ N &= \sqrt{-g_{mn} \xi^m \xi^n} \end{aligned} \tag{16}$$

contains always the factor  $1/\sqrt{\Delta}$  and diverges for orbits on the surface  $\Delta = 0$ . All Killing congruences are spacelike at  $\Delta = 0$  except for the case given by

$$\mu = \lambda J / (r_0^2 + J^2).$$

This congruence is timelike for  $r > r_0 = M + \sqrt{M^2 - J^2 - Q^2}$  and null on the surface  $r = r_0 = M + \sqrt{M^2 - J^2 - Q^2}$ .<sup>5</sup> Its acceleration diverges in the limit  $r \rightarrow r_0$ . In the case of a static metric,  $J = 0$ , the congruence turns out to be the hypersurface-orthogonal one. Hence, the surface  $r = r_0 = M + \sqrt{M^2 - J^2 - Q^2}$  is singular in the Kerr-Newman case, too. Any Killing congruence that remains timelike in the outer vicinity will show the singularity.

One can see that the singularity in the acceleration is connected with the norm of the Killing vector becoming zero or the determinant of the metric in stationary coordinates becoming zero. Let us use the notation  $\Xi = \xi_a \xi^a$ . The velocity along the orbits is given by  $u^k = \xi^k / \sqrt{\Xi}$ , the acceleration  $a^k$  by  $\Xi^2 a^k = \Xi \xi^k_{;l} \xi^l - \xi^k \xi^l \xi_b \xi^b_{;l}$ . The norm of this expression is

$$\Xi^4 a_k a^k = \frac{1}{4} \Xi^2 \Xi_{;k} \Xi_{;l} g^{kl} - \Xi (\Xi_{;l} \xi^l)^2$$

We may, of course, introduce coordinates in which  $\xi^k = \delta_0^k$ . These coordinates show that  $\Xi_{;l} \xi^l = g_{0,0} = 0$ . We obtain

$$a_k a^k = \frac{1}{4 \Xi^2} \Xi_{;k} \Xi_{;l} g^{kl}$$

When  $\Xi$  has a zero of some order  $n$  on some surface of orbits, and the space-part metric has no singularity, the norm of the acceleration has to show a second-order infinity. If the space-part metric has a singularity, but the

<sup>4</sup>To dispel any possible misunderstanding, it is stressed that no meaning is attributed to the accidental vanishing of the radial coordinate where  $\alpha \rightarrow \infty$ .

<sup>5</sup>The physical interpretation of the congruence is a swarm of particles kept at rest in the field. This rest, in the static case ( $J = 0$ ), is just no change in the spatial coordinates. In the general case, this "rest frame" undergoes a drag by the rotation of the source. The drag is given by  $\mu \neq 0$ .

determinant of the metric remains finite, as in the Schwarzschild case, the infinity of  $a_k a^k$  is still of first order. If  $a_k a^k$  becomes infinite while  $\Xi$  has no zero, it must be due to a zero of the determinant of the metric tensor in the coordinates defined.<sup>6</sup>

#### 4. The analytic extensions of the “Schwarzschild” solution

As previously shown, Hilbert's solution was born out of the accidental choice of the radial coordinate  $r$  produced by setting  $\mu = 0$  in equation (4); hence there is no reason to accept as unavoidable consequences of the very field equations of general relativity all the features stemming from this choice, in particular the existence of the region for  $r < 2m$ . However Hilbert's solution was soon perceived as the unique “Schwarzschild” solution and as such it became the obligatory starting point of all the theoretical exertions. The curious circumstance that what was initially meant to model the field of a “Massenpunkt” displayed two singularities, one at the “Schwarzschild radius”, and a second one for  $r = 0$ , instead of the single one appearing in Newtonian physics, suggested the idea that one of them had to be spurious<sup>7</sup>. Since the Kretschmann scalar happened to be finite at the “Schwarzschild radius”, while it was infinite at  $r = 0$ , the conviction arose that the “true” singularity was the one at  $r = 0$ . Therefore the singularity displayed by the components of the metric at  $r = 2m$  had to be a mere mathematical mishap, devoid both of geometrical and of physical meaning. A reason had to be given for the wrongdoing, and it was found in a presumed inadequacy of the coordinate system at  $r = 2m$ .

The search thus started for different coordinate systems that allowed to erase the singular behaviour displayed by  $g_{ik}$  at  $r = 2m$ . Already in 1924 Eddington had unintentionally succeeded in the task (Eddington 1924) by rewriting the static “Schwarzschild” solution in stationary form through the introduction of what would have been called the Eddington-Finkelstein coordinates<sup>8</sup>. In 1933 Lemaître (1933) achieved the same result by rewriting the “Schwarzschild” metric with cosmological term in time-dependent form. Another solution to the problem was given in 1950 by Synge with a geometrically inspired paper (Synge 1950) that represents the now forgotten forerunner of the maximal extensions of the “Schwarzschild” metric obtained by Kruskal (1960) and Szekeres (1960).

All these exertions entail coordinate transformations  $x'^i = f^i(x^k)$  whose derivatives happen to be singular at the “Schwarzschild” radius in just the appropriate way for providing a transformed metric that is regular there. One cannot help noticing that the restriction to admissible coordinate transformations, which looked mandatory in the old papers, with the lapse of the decades has become optional and dependent on taste. In the time span that goes from Hilbert's paper (1917) to, say, the publication of Lichnerowicz' book (Lichnerowicz 1955) with his axioms inscribed in the first chapter, transformations like the ones needed to efface the “Schwarzschild” singularity were simply disallowed.

Nevertheless, the rule was violated here and there, and already in Synge's paper one finds an explicit program of transgression, since for the latter author “it is precisely the non-regular transformations which are interesting” (Synge 1950). But the value of scalars cannot be altered by any transformation, however “interesting”. Therefore in all the alternative forms of the “Schwarzschild” metric mentioned in this section the singularity in the metric components at the “Schwarzschild” radius is canceled, but the norm  $\alpha$  of the acceleration of the hyperbolic motion on the invariantly specified Killing orbit remains infinite at the position of the erased singularity. Since  $\alpha$  has a well defined physical meaning, an infinite value of  $\alpha$  in the middle of a manifold should not be so light-heartedly overlooked: either this singularity should be removed from that position, or a physical argument for its existence there should be given.

Already in 1916 Karl Schwarzschild had deliberately sent the singularity in the “Nullpunkt”, thus stipulating that there his idealised vacuum model ceased to be physically meaningful, because the source of the field had been attained.

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<sup>6</sup>When singular coordinate transformations are forbidden, the zeros of the determinant of the metric tensor acquire a physical interpretation (Kreisel 1967). In addition, the Schwarzschild mass can be interpreted as result of the gravitational field outside the Schwarzschild sphere (Treder 1975, Treder 1978).

<sup>7</sup>Hilbert did not share this opinion; however, he did not consider the singularities displayed by his solution really worth of much study. He believed in fact that the vacuum solutions with singularities were just “an important mathematical tool for approximating characteristic regular solutions”. To him, only the latter were capable to represent reality in an immediate way. See Hilbert (1917), page 70.

<sup>8</sup>Eddington's coordinate transformation was purportedly retrieved by Finkelstein in 1958.

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