

THE TOPOLOGY OF SCHWARZSCHILD'S SOLUTION AND THE KRUSKAL METRIC

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ABSTRACT. Kruskal's extension solves the problem of the arrow of time of the "Schwarzschild solution" through combining two Hilbert manifolds by a singular coordinate transformation. We discuss the implications for the singularity problem and the definition of the mass point.

The analogy set by Rindler between the Kruskal metric and the Minkowski spacetime is investigated anew. The question is answered, whether this analogy is limited to a similarity of the chosen "Bildräume", or can be given a deeper, intrinsic meaning. The conclusion is reached by observing a usually neglected difference: the left and right quadrants of Kruskal's metric are endowed with worldlines of absolute rest, uniquely defined through each event by the manifold itself, while such worldlines obviously do not exist in the Minkowski spacetime.

1. INTRODUCTION: KRUSKAL'S EXTENSION OF THE SCHWARZSCHILD SOLUTION AND THE ARROW OF TIME

In general, a manifold cannot be covered by a single coordinate system. An atlas of coordinate systems is required, and it reflects the global properties of the manifold. An individual coordinate system can be limited by its singularities, that may be seen as singularities in the metric topology with respect to the coordinate-based topology. Coordinates, however, do not matter. Any determination of position or time is ruled by equations of motion or structure that are invariant against substitutions of new coordinates for the old. Hence, a singularity which is physically observable must have an expression in invariant quantities.

In a Riemannian manifold, all tensors that can be constructed from the metric tensor and its derivatives are the concomitants of the Riemann tensor. For a general manifold, the invariants of the Riemann tensor define singularities. When these invariants are non-singular, the singularities that limit the viability of a coordinate system are not observable – this is the folklore. It holds, however, only for a non-degenerate metric. The "Schwarzschild metric"¹ shows that in algebraically special metrics, where Killing vectors

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¹Due to compelling historical reasons, made clear [1] in an Editorial Note recently appeared in *General Relativity and Gravitation*, and accompanying an English translation of Schwarzschild's fundamental paper, we shall henceforth call Schwarzschild solution, without quotation marks, the original "Massenpunkt" solution that Karl Schwarzschild

are invariantly defined, singularities can now exist in the Killing congruences also in the case of non-singular curvature invariants. In the “Schwarzschild metric”, it is the transition of character of the Killing congruence from time-like to space-like that yields a singularity in the acceleration: with $\xi^k = [0, 0, 0, 1]$ and $\xi_{i;k} + \xi_{k;i} = 0$, we obtain

$$(1.1) \quad \xi_{i;k}\xi^k + \frac{1}{2}(\xi_k\xi^k)_{,i} = 0 ,$$

and for $u_i = \frac{\xi_i}{\sqrt{g_{lm}\xi^l\xi^m}}$,

$$(1.2) \quad u_{i;k}u^k + \frac{1}{2}(\ln \xi_k\xi^k)_{,i} = 0 .$$

The invariant square of the acceleration is found to be

$$(1.3) \quad a_i a_k g^{ik} = \frac{1}{2}(\ln \xi_m \xi^m)_{,i} \frac{1}{2}(\ln \xi_n \xi^n)_{,k} g^{ik} = \frac{1}{4} g'_{44} g'_{44} g^{11} (g^{44})^2 ,$$

where the prime denotes derivation with respect to r in, say, Hilbert’s coordinates. We find a singularity at the horizon not in curvature but in the Killing congruence. It coincides with the change in character, and it is measurable through the acceleration necessary to keep a massive body stationary at a given position.

Beyond the Schwarzschild horizon, there is no observation for an observer with a minimum distance from the horizon. Any object falling onto the horizon will not reach it in finite observer time. The proper time of the object, however, is finite, and the question of extending the manifold beyond the horizon is posed. This extension cannot be achieved when the Killing time is used as time coordinate. The Killing time is not timelike at all “inside” the horizon: its vector is there spacelike. This circumstance entails a confirmation of the argument, advanced by Marcel Brillouin [6] already in 1923, and reported in the Appendix, about a mismatch between the inner and the outer problem. As remarked by Rindler [7, 8], if the inner part of Hilbert’s solution is accepted as physical, due to the exchange of rôle that the radial coordinate and the time coordinate undergo at Schwarzschild’s two-surface, there is no way of drawing the arrows of time both inside and outside in a way exempt from contradiction. A simple glance to Fig. 1 (a), reported also by Rindler in his articles and books [7, 8], suffices for gathering that we are confronted with a contradiction. Moreover, it is not a contradiction associated with a particular choice of the coordinates. In fact, it is a flaw inherent in the geometric structure of the manifold inadvertently chosen by Hilbert, a flaw that no one-to-one coordinate transformation can remove, no matter whether it is regular or not in the sense of Hilbert and Lichnerowicz [4, 9]. This conclusion could have been sufficient for discarding Hilbert’s

published [2] in 1916, while the noun “Schwarzschild solution”, or Hilbert solution, is reserved to the metric manifold, inequivalent to Schwarzschild’s original one, later provided by Droste, Hilbert, Weyl [3, 4, 5] and attributed to Schwarzschild by these authors, as well as by nearly all the subsequent writers of relativity.

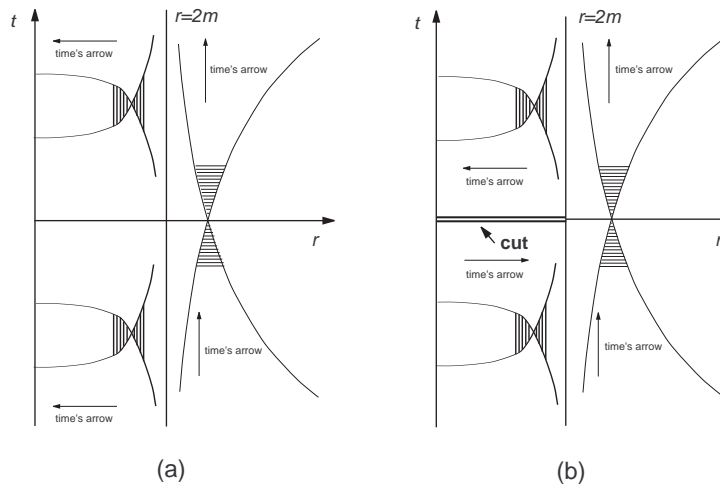


FIGURE 1. (a): r, t representation of Hilbert's manifold. Light cones and time's arrows are drawn, showing that the manifold cannot be endowed with a consistent direction of time. (b): a cut is made in Hilbert's manifold, thus allowing for a consistent choice of the time's arrows.

solution, with the inherent puzzle of the two singularities at $r = 0$ and at $r = 2m$, and for adhering to the solution that Karl Schwarzschild had originally proposed.

Coordinates do not matter – locally. The postulate that they shall not matter globally means calling manifolds of different (defined by the coordinates) topology equivalent. This is not a necessary definition of equivalence classes. When one includes the (defined by the coordinates) topology as characteristic of the manifold, only regular (with regular and non-vanishing Jacobi determinant) coordinate transformations relate equivalent manifolds. A part of the coordinate singularities become real, in particular the zeros of the determinant of the metric tensor. These have been considered in the early sixties [10, 11, 12].

The ingenuity of the relativists, as is well known, took instead a quite different route.

2. CUTTING AND PASTING TWO HILBERT MANIFOLDS

The flaw of the Hilbert manifold might be cured if it were possible to cut its inner region as is shown in Fig. 1 (b), since then the two disjoint inner parts can be endowed with opposite time arrows, thus restoring the overall coherence in the direction of time. Of course, after the cut we no longer have to do with Hilbert's manifold, but with a new manifold, whose topological properties and physical interpretation are different from those

prevailing in the previous one. Moreover, the cut introduces an artificial border without any physical *raison d'être*, imposed merely by the need to solve a contradiction. In order to heal the wound inflicted to Hilbert's manifold, by following Synge's original idea [13], one decade later Kruskal and Szekeres [14, 15] chose to effect the following audacious act of surgery: paste together two Hilbert manifolds, both affected by the same fatal illness, after suitably cutting them. One is depicted in Fig. 1 (b); the second one is its mirror image in the r, t plane, left to the imagination of the reader. Then the upper border of the cut in Fig. 1 (b) is sewn to the upper border of the cut in the mirror image manifold, and likewise for the lower borders. This way the cuts are sutured, and one gladly recognises that, after the surgery, the circulation of the arrows of time is exempt from contradiction. There is, however, a negative consequence of the surgical act: it is the impairment of the independence of the individual manifolds, with a future singularity permanently sewn to its past counterpart. One may well ask: the Kruskal extension, is it the solution for the field of a mass point? It is, of course, when we consider only one of the regions outside the horizon. It is difficult to swallow that the inner part should be considered as the field of a particle: the extension converts the problem of the "inner" time arrow into the paradox of the dissolution of the particle into an initial and a final singularity with vacuum in the time between.

3. THE ANALOGY BETWEEN THE KRUSKAL METRIC AND THE MINKOWSKI SPACETIME

The Kruskal manifold was soon noted for its considerable mathematical beauty, but looked very remote, from a physical standpoint, from the "Schwarzschild solution" and from the problem that the latter had tried to solve, after the original solution given by Schwarzschild [2] had been discarded. Since, due to the contradiction of the arrows of time, Hilbert's solution was useless, only the Kruskal solution seemed available for providing, within the theory of general relativity, a metric that could be the counterpart of the field of a point particle in Newtonian physics, but many a relativist, like Rindler, felt necessary to show that the complexity of the Kruskal manifold could be reduced to something simpler, and more familiar to the physicists, although of course not as simple and familiar as the gravitational field of a point particle in Newton's theory.

First came an attempt [16] at eliminating the strange duplication present in the Kruskal-Szekeres metric, by identifying the pair of events $(u, v, \vartheta, \varphi)$ and $(-u, -v, \vartheta, \varphi)$ in Kruskal's diagram, drawn in Fig. 2. This attempt was however unsuccessful, due to the singularity that such an identification produces on the two-sphere $u = v = 0$, and to the ambiguity that this move introduces in the direction of time. Then Rindler's attention was drawn to the analogy that he had already brought to light five years earlier, when dealing with the definition of motion with a constant acceleration in curved

spacetime [17]. Let u^i be the four-velocity of a test particle in a pseudo Riemannian manifold, whose affine connection is Γ_{kl}^i . Then the absolute derivative of the four-velocity

$$(3.1) \quad a^i \equiv \frac{du^i}{ds} + \Gamma_{kl}^i u^k u^l$$

defines the first curvature of the world-line of the particle, *i.e.* its four-acceleration [18]. Let the “Schwarzschild metric” be written in terms of Hilbert’s spherical polar coordinates r, ϑ, φ, t , like in footnote 4. It is an easy matter to verify [17], that, in keeping with Einstein’s principle of equivalence, a test particle in “Schwarzschild” exterior metric, whose spatial

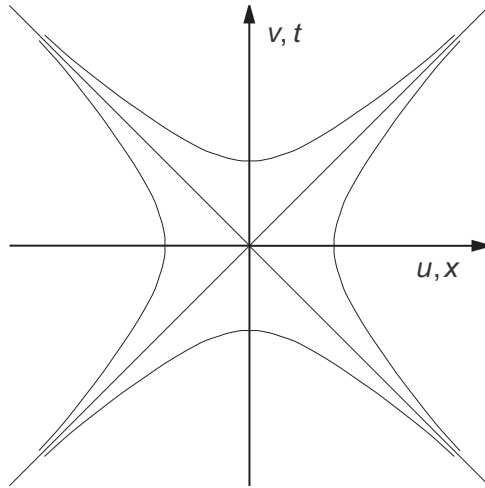


FIGURE 2. In this figure the “Bildraum” for the Kruskal manifold and a spacetime section of the Minkowski metric are juxtaposed. The coordinate axes are endowed both with Kruskal’s u, v coordinate labels, and with the x, t coordinates of the Minkowski reference system. Therefore to each point in the u, v “Bildraum” is associated a two-sphere of appropriate curvature radius, while to each point in the x, t representative space is associated a y, z plane of the Minkowski spacetime.

coordinates r, ϑ, φ are kept constant, is subject to a radially directed four-acceleration, whose norm

$$(3.2) \quad \alpha = (-a_i a^i)^{1/2} = \frac{1}{r^{3/2}(r - 2m)^{1/2}},$$

defined through equation (1.3), is constant. Let us now consider the x, t plane of Minkowski spacetime, drawn too in Fig. 2, and calculate the four-acceleration of a test particle executing a hyperbolic motion [17] in, say,

either the left or the right quadrant of that plane. Its equation of motion shall read

$$(3.3) \quad x^2 - t^2 = X^2,$$

where the constant X measures the minimum distance from the origin, attained by the test particle when $t = 0$, and the norm of the four-acceleration of the test particle is now

$$(3.4) \quad \alpha = \frac{1}{X}.$$

Rindler noticed [7, 8] that the particle with constant spatial coordinates in the “Schwarzschild solution”, when viewed in the u, v “Bildraum”, appears to execute a hyperbolic motion too, provided that the Minkowski metric be substituted for the Kruskal metric in either the left or the right quadrant of the Kruskal manifold. On this initial basis he constructed an articulated argument *ad analogiam* for gaining an understanding of the Kruskal metric through the undisputable understanding of the Rindler metric, which is merely a matter of special relativity.

4. ABOUT USING ARGUMENTS *ad analogiam* IN THE THEORY OF GENERAL RELATIVITY

A “neo Cartesian”, meant in the jokeful sense that Synge once attributed to the noun [19], may well wonder what is the rôle of arguments *ad analogiam* in the theory of general relativity. One cannot in fact forget that, when Hilbert succeeded [20] in formulating the field equations of the theory from an action principle, he ended the Communication with which he announced his achievement by expressing the hope that

damit die Möglichkeit naherückt, daß aus der Physik im Prinzip eine Wissenschaft von der Art der Geometrie werde: gewiß der herrlichste Ruhm der axiomatischen Methode, die hier wie wir sehen die mächtigen Instrumente der Analysis, nämlich Variationsrechnung und Invariantentheorie, in ihre Dienste nimmt.²

When Weyl and Levi-Civita obtained [5, 21] the reduction to quadratures for the solution of the vacuum field equations of general relativity in the case of a static, axially symmetric manifold, they did so by availing of the so called “canonical coordinates”. Let $x^0 = t$ be the time coordinate, while $x^1 = z$, $x^2 = r$ are the coordinates in a meridian half-plane, and $x^3 = \varphi$ is the azimuth of such a half-plane; then the line element of a static, axially symmetric field *in vacuo* can be written as:

$$(4.1) \quad ds^2 = e^{2\psi} dt^2 - d\sigma^2, \quad e^{2\psi} d\sigma^2 = r^2 d\varphi^2 + e^{2\gamma} (dr^2 + dz^2);$$

²An English translation: “hence the possibility gets close, that from physics originate, in principle, a science of the kind of geometry: certainly the most splendid glory of the axiomatic method, that here, as we see, takes to its service the powerful instruments of analysis, *i.e.* calculus of variations and theory of invariants.”

the two functions ψ and γ depend only on z and r . Remarkably enough, in the “Bildraum” introduced by Weyl ψ fulfils the potential equation

$$(4.2) \quad \Delta\psi = \frac{1}{r} \left\{ \frac{\partial(r\psi_z)}{\partial z} + \frac{\partial(r\psi_r)}{\partial r} \right\} = 0$$

(ψ_z, ψ_r are the derivatives with respect to z and to r respectively), while γ is obtained by solving the system

$$(4.3) \quad \gamma_z = 2r\psi_z\psi_r, \quad \gamma_r = r(\psi_r^2 - \psi_z^2);$$

due to the potential equation (4.2)

$$(4.4) \quad d\gamma = 2r\psi_z\psi_r dz + r(\psi_r^2 - \psi_z^2) dr$$

happens to be an exact differential.

The analogy of equation (4.2) with the corresponding equation in Newton’s theory was indeed impressive, and mathematically helpful. Neither Weyl nor Levi-Civita, however, thought of availing of this analogy for gaining some insight in the physical meaning of the solutions. They knew in fact that, when Schwarzschild’s solution [2] is rewritten by using Weyl’s canonical coordinates, the “source” for the “Newtonian potential” ψ , that then happens to appear at the right-hand side of equation (4.2), looks like one segment of the z -axis covered by matter with a constant linear mass density. But they also knew that this alluring analogy produced in the “Bildraum” is deceitful. As noticed by Weyl, the intrinsic form of the “source” is a completely different one: the segment covered by mass happens in fact to be Schwarzschild’s two-surface.

5. INTRINSIC VIEWPOINT. THE ABSOLUTE STATICS OF GENERAL RELATIVITY

Given the illusory character of the “Bildraum” analogy in the case of the solutions of Weyl and Levi-Civita, one shall verify whether the “Bildraum” analogy between the Kruskal metric and Minkowski spacetime has some intrinsic content too, that can be expressed in invariant way. We find in Rindler [7] the assertion that “the Kruskal diagram possesses essentially the same invariance properties as the Minkowski diagram”. We shall ask whether this claim about a possibly deceitful correspondence between “Bildraum” invariance properties is really meaningful, *i.e.* whether it can be turned into a like claim about intrinsic invariance properties shared by the manifolds.

We have already noticed that while each u, v point in Fig. 2 actually corresponds to a two-sphere, each x, t point on the same figure corresponds to a y, z plane in Minkowski spacetime. When $m \rightarrow 0$, both the original Hilbert manifold of Fig. 1 (a) and the manifold with a cut of Fig. 1 (b) tend to the Minkowski spacetime. Kruskal’s manifold is obtained by sewing together, in the way described in Sec. 2, the cuts of two manifolds like the one in Fig. 1 (b). Hence, in the limit of vanishing mass, Kruskal’s manifold

happens to be constituted by two Minkowski spacetimes, joined at the event $r = t = 0$. Therefore, already in the limit $m \rightarrow 0$, Kruskal's manifold is topologically different from the Minkowski manifold.

An analogy of intrinsic character seems promised by the following fact. A test particle executing hyperbolic motion in either the left or the right quadrant of the u, v "Bildraum", and a test particle executing for good a hyperbolic motion in Minkowski spacetime, in particular along the world lines drawn in the x, t plane of Fig. 2, both execute a motion whose four-acceleration is constant in direction and in norm, as shown by Eqs. (3.2) and (3.4) respectively. Are these particular motions an intrinsic, common feature of both the Kruskal and the Minkowski manifold?

In the Kruskal manifold, these motions occur in quadrants that are static in character, if one accepts the usual definition of "static". Let Greek indices run from 1 to 3. According to the usual definition, a region of a manifold is static if a coordinate system can be chosen for it, in which the square of the interval can be written in the form

$$(5.1) \quad ds^2 = g_{44}dt^2 + g_{\mu\nu}dx^\mu dx^\nu,$$

and the nonvanishing components of the metric do not depend on the time-like coordinate t . According to this definition, both the left and right Kruskal quadrants and the Minkowski spacetime are static; this fact appears to strengthen the analogy between the motions with constant acceleration occurring in these manifolds.

The attribution of the adjective "static" to the Minkowski manifold and to the Minkowski metric does not seem however to be an entirely convincing one. In fact, the notion of staticity is indisputably connected with the notion of rest. How is it possible to call "static" the metric of the theory of special relativity, a theory that denies intrinsic meaning to the very notion of rest? How is it possible that the Minkowski metric remain static, as it does according to our definition, after we have subjected it to an arbitrary Lorentz transformation, *i.e.* a transformation that entails relative, uniform motion? Let us start from the Minkowski metric, given by $g_{ik} = \text{diag}(-1, -1, -1, 1)$ with respect to the Galilean coordinates x, y, z, t , and perform the coordinate transformation

$$(5.2) \quad x = X \cosh T, \quad y = Y, \quad z = Z, \quad t = X \sinh T$$

to new coordinates X, Y, Z, T . We get the particular Rindler metric [22], whose interval reads

$$(5.3) \quad ds^2 = -dX^2 - dY^2 - dZ^2 + X^2dT^2.$$

How is it possible that this metric turn out to be in static form too, despite the fact that the transformation (5.2) entails uniformly accelerated motion? There is however another definition of static manifold and metric, that, although equivalent to the one given previously, is more helpful for solving our difficulty, because it is expressed in intrinsic language. It says that a metric

is static if it allows for a timelike Killing field ξ_i that is also hypersurface orthogonal, *i.e.* fulfils both the equations

$$(5.4) \quad \xi_{i;k} + \xi_{k;i} = 0,$$

and

$$(5.5) \quad \xi_{[i}\xi_{k,l]} = 0.$$

Both the Kruskal metric in the left and right quadrants and the Minkowski metric possess timelike Killing fields that obey both equations (5.4) and (5.5). There is however a fundamental difference between the two cases, whose importance has been somewhat neglected until now. In the case of the Minkowski metric, at a given event there is an infinity of timelike Killing vectors that fulfil both equations (5.4) and (5.5), and this circumstance answers our previous questions, since it says in intrinsic language that there is no privileged time axis, hence no privileged rest frame in the manifold of special relativity.

On the contrary, in the case of the left and right quadrants of the Kruskal metric, when $m > 0$ the timelike vector that obeys both equations (5.4) and (5.5) is unique: at any event in the left and right quadrants of the Kruskal manifold the metric itself provides a unique, absolute time direction; this circumstance allows one to draw through any event a unique worldline of absolute rest, that is intrinsic to the manifold. Hence, a worldline in the Hilbert manifold for which $r > 2m$, ϑ and φ have constant values is an intrinsic worldline of absolute rest.

From this result it follows that the norm of the four-acceleration a^i of a test particle on one of these worldlines, given by (3.2), is an invariant and intrinsic property of the Hilbert and of the Kruskal manifolds. In fact, this scalar is a unique outcome of equations (5.4), (5.5) and (3.2). We calculate it by availing only of the metric, without making any arbitrary choice. This scalar happens to diverge when $r \rightarrow 2m$, as shown by equation (3.2). Due to the way kept in calculating the norm (3.2), its divergence for $r \rightarrow 2m$ is as intrinsic to the Hilbert and Kruskal manifolds [23, 24] as the divergence, for $r \rightarrow 0$, of the scalars built with the Riemann tensor and with its covariant derivatives.

Also the norm (3.4) of the four-acceleration of a test particle executing a hyperbolic motion in the Minkowski manifold, *i.e.* staying on a worldline with constant values of X , Y , Z in the particular form (5.3) of Rindler's metric, diverges when $X \rightarrow 0$. This divergent behaviour is however completely different in nature from the one occurring in the Kruskal and Hilbert manifolds for $r \rightarrow 2m$. In fact, the scalar (3.4) is by no means uniquely dictated by the metric: in the Minkowski manifold there is an infinity of worldlines of hyperbolic motion through a given event; choosing one of them is completely arbitrary, and such is also, from an intrinsic viewpoint, the definition of the divergent scalar.

6. APPENDIX: HISTORICAL NOTE

When, by further elaborating the geometric interpretation of the gravitational field of a particle envisaged by Synge [13] in 1950, ten years later Kruskal [14] and Szekeres [15] succeeded in convincing an influential minority within the relativists that a major breakthrough had been achieved in the mathematical understanding of the “Schwarzschild solution”, a major problem became soon apparent too: convincing that community that a major breakthrough had been achieved also in the physical understanding. This was by no means an easy task: the possibility that the Kruskal extension might represent such an achievement was *e.g.* simply dismissed in 1962 by Dirac [25] with a polite touch of nonchalant irony. He noticed:

The mathematicians can go beyond this Schwarzschild radius, and get inside, but I would maintain that this inside region is not physical space, because to send a signal inside and get it out again would take an infinite time, so I feel that the space inside the Schwarzschild radius must belong to a different universe and should not be taken into account in any physical theory.

Dirac’s opinion had already been reached [6] four decades earlier by Marcel Brillouin, although by availing of a different argument. In 1923, after pondering for months the problem of the “catastrophe Hadamard” occurring at Schwarzschild’s singular surface, that had been raised during the discussions between Einstein and the French scientists gathered at the “Collège de France” during the Easter of 1922, Brillouin eventually wrote [6]:

On peut se demander si cette singularité ³ *limite* l’Univers, et s’il faut s’arrêter a $R = 0$; ou si, au contraire, elle *traverse* seulement l’Univers, qui continuerait au delà, pour $R < 0$. Dans les discussions, en particulier, dans celles da Pâques 1922, au Collège de France, on a généralement parlé comme si $R = 0$ caractérisait une région *catastrophique* qu’il faut dépasser pour arriver jusqu’à la véritable limite singulière, atteinte seulement pour γ infini, avec $R = -2m$. *A mon avis, c’est la première singularité, atteinte $R = 0$, $\gamma = 0$ ($m > 0$), qui limite l’Univers et qu’on ne doit pas dépasser.* [*C. R.*, t. 175 (27 nov. 1922)].

La raison en est péremptoire, quoiqu’on ait négligé de la mettre en évidence jusqu’à présent: *Pour $R < 0$, $\gamma < 0$,*

³In Brillouin’s paper, the metric

$$ds^2 = \gamma c^2 d\tau^2 - \frac{1}{\gamma} dR^2 - (R + 2m)^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad \gamma = \frac{R}{R + 2m}$$

is given. c is the velocity of light, θ and φ are the usual spherical polar angles, R is the radial coordinate. Brillouin’s pondered choice for the limit values of R ($0 < R < \infty$) puts his metric in one to one correspondence with Schwarzschild’s original metric [2].

le ds^2 ne correspond plus de tout au problème qu'on voulait traiter.

With hindsight, it must be recognised that Brillouin had given one good reason for discarding Hilbert's form of the spherically symmetric manifold of a massive particle ⁴, with its troublesome inner region, $0 < r < 2m$, and for adopting instead the manifold that Schwarzschild had deliberately decided to choose [2], when he first solved the "Massenpunkt" problem. The issue, whether Schwarzschild's two-surface is intrinsically singular or not, and whether transformations of coordinates that cancel the "catastrophe Hadamard" are allowed, despite the fact that they infringe the rules for the regularity of coordinate transformations laid down by Hilbert [4] already in 1917, and sharpened by Lichnerowicz [9] in his book of 1955, is with us since the very beginning of general relativity. It has been widely debated with opposite outcomes, if it is true that during forty years since its discovery, the overwhelming majority of relativists was convinced that the inner region of Hilbert's solution was unreachable from the outside, while during the subsequent four decades an equally overwhelming majority harbored the opposite conviction.

Brillouin's argument, however, does not rely at all on the singular character of Schwarzschild's two-surface. It simply says [6] that, because in the inner region of Hilbert's metric the radial coordinate and the time coordinate exchange their rôles as spatial and temporal coordinate respectively, in that region one is no longer considering the problem that one had set out to solve.

⁴It was Leonard Abrams, in a keen paper [26] written in 1989, who first noticed that the apparent necessity of the choice of the manifold done by Hilbert when he wrote [4] the spherically symmetric solution in the form:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad 0 < r < \infty,$$

and handed it down to the posterity as the unique "Schwarzschild solution", is just due to an arbitrary restriction inadvertently imposed by Hilbert in his calculation.

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