Mass inversion in a critical neutron star: An alternative to the black hole model.

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FIRST PART

Keywords: Black hole, Einstein field equations, Schwarzschild metric.

Abstract: In this first part, we present a documented historical background about the Schwarzschild solution to the field equations of general relativity, as originally considered by Karl Schwarzschild, as well as Johannes Droste, Hermann Weyl and Albert Einstein, in its linearized form. We also detail how David Hilbert assumed this question, with a pure imaginary time. Finally, we discuss extensions of the solution through analytic continuation and their geometry implications.

Introduction

The black hole model remains a belief within the scientific community.

It is true that in the recent history of science, the existence of many objects and phenomena has been conjectured long before they have been effectively observed. Yet many of them had a lot to stir up skepticism. An example was antimatter, imagined by Sir Arthur Schuster in 1898, theorized by Paul Dirac in 1928, before the first observation of positrons in 1932. Later, the phenomenon described in 1935 by Einstein, Podolsky and Rosen, named the EPR paradox, was demonstrated decades after its authors presented it as a denunciation of the infallibility of quantum mechanics.

Swiss-American astrophysicist Fritz Zwicky described how massive stars could violently end as supernovæ, in a memorable Caltech lecture course in 1931. According to him, such a star running out of fusion fuel would end in a violent gravitational collapse at an incredible high speed, converting its potential energy into kinetic energy. This outburst of a very large quantity of mass would compress the internal iron core into a totally new object: a neutron star.

This model is at first greeted with general skepticism, and very few scientists agree with such a singular idea. But after tremendous efforts, Zwicky managed to gradually demonstrate the phenomenon.

Nowadays, no one would venture to question the existence of antimatter, or quantum entanglement involved in the EPR paradox, or supernovae. As for astronomical phenomena, the situation is very simple. The cosmos is vast. If the supernova phenomenon statistically occurs in our galaxy at a rate of only one per century, the extension to all observable galaxies brings their number to tens of thousands.

Related to supernovæ was the question of the existence of neutron stars. The first ones were discovered in 1967 as pulsars (highly magnetized rotating neutron stars acting as pulsed radiation sources). The relation with supernovæ was soon established, one of these pulsars being located in the middle of the Crab Nebula, remnant of a supernova whose visible explosion in 1054 is testified by Chinese records. Today, their number has reached hundreds of samples, in every corner of the galaxy, including near distances. No reasonable scientist would now doubt of their very existence.

But this is not the same at all about stellar black holes, whose existence is practically inferred by just one observation: the Cygnus X-1 binary system, whose distance from us is evaluated at 6,000 light-years. This rarity is very abnormal. At such a large distance, its estimated mass (about eight solar masses) could result from cumulative observational errors.

This very anomalous situation gives this hypothetical object the nature of a belief, in conflict with the scientific method.

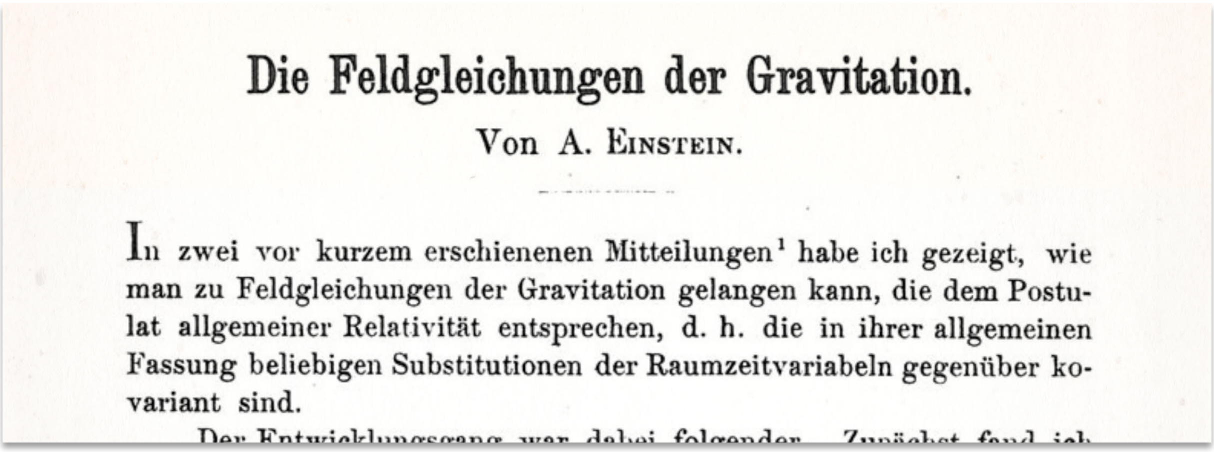
As for supermassive black holes, all that can be said for the moment is that these are very important concentrations of matter (hundreds of thousands to billions of solar masses) located in the center of galaxies. They have a very large mass, but not a very high density; the one located in the center of Milky Way can be represented by a sphere half the solar system in diameter, filled with matter having the same density as water. In any event they are new objects, which could be the remainder of quasar phenomena, but their signature does not allow, in its current state, to name them black holes.

It is therefore perfectly licit today to question the black hole model, while proposing an alternative scenario to the fate of a critical neutron star. Indeed, in X-ray binaries for example, a stable neutron star (the *accretor*) can capture material from a companion star (the *donor*) and thus becomes at some point destabilized. To refuse such a study would be unscientific and would be tantamount to the defense of a dogma.

Prelude to the birth of the black hole model

Before literally getting to the heart of the matter, it is necessary to situate the context in which Schwarzschild first produced a nonlinear, exact solution of Einstein's field equation in 1916. In public's imagination, Einstein got a superhero origin story by the media. Actually, thanks to his remarkable ability to assimilate both the mathematics of his time and the palette of physical phenomena, he was able to produce an impressive succession of results, including in 1905 the discovery of the law of the photoelectric effect. But at that time a number of German and Dutch scientists followed this path carefully, and were immediately able to understand and interpret any progress made. Schwarzschild was one of them.

In 1915, we find first important texts resulting from the collaboration-competition between Einstein and the great mathematician Hilbert. In November 1915, Einstein presented a finalized version of his theory of general relativity in a series of four papers at the Royal Prussian Academy of Sciences in Berlin [1] [2] [3] [4]. At first, his field equation is not divergence-free, but he has the intuition that for distances small enough and for moderate velocities (which corresponds to the Newtonian approximation) his equation should give the corresponding equations of fluid mechanics, namely the Euler conservation equations. He modifies his field equation to make it divergence-free in [4], whose original title is:



Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariabeln gegenüber kovariant sind.

Der Entwicklungsgang war dabei folgender. Zunächst fand ich

Fig. 1 – Einstein's original paper, November 25, 1915
"The Field Equations of Gravitation."

And here is the corresponding field equation:

Ist in dem betrachteten Raume »Materie« vorhanden, so tritt deren Energietensor auf der rechten Seite von (2) bzw. (3) auf. Wir setzen

$$G_{im} = -\varkappa\left(T_{im} - \frac{1}{2} g_{im} T\right), \tag{2a}$$

wobei

$$\sum_{\varrho\sigma} g^{\varrho\sigma} T_{\varrho\sigma} = \sum_{\sigma} T_{\sigma}^{\sigma} = T \tag{5}$$

gesetzt ist; T ist der Skalar des Energietensors der »Materie«, die rechte Seite von (2a) ein Tensor. Spezialisieren wir wieder das Koordinatensystem in der gewohnten Weise, so erhalten wir an Stelle von (2a) die äquivalenten Gleichungen

$$R_{im} = \sum_{l} \frac{\partial \Gamma_{im}^{l}}{\partial x_l} + \sum_{\varrho l} \Gamma_{i\varrho}^{l} \Gamma_{ml}^{\varrho} = -\varkappa\left(T_{im} - \frac{1}{2} g_{im} T\right) \tag{6}$$

$$\sqrt{-g} = 1. \tag{3a}$$

Wie stets nehmen wir an, daß die Divergenz des Energietensors

Fig. 2 – The field equation finalized by Einstein, 25 November 1915.

In the left hand-side of his equation (6): the Ricci tensor.
The following is the English translation [4] from 1997:

When there is "matter" in the space under consideration, its energy tensor occurs on the right-hand sides of (2) and (3), respectively. We set

$$\underbrace{G_{im} = -\kappa\left(T_{im} - \frac{1}{2} g_{im} T\right)}_{\text{divergenceless form}}, \tag{2a}$$

where

$$\sum_{\rho\sigma} g^{\rho\sigma} T_{\rho\sigma} = \sum_{\sigma} T_{\sigma}^{\sigma} = T. \tag{5}$$

T is the scalar of the energy tensor of "matter," and the right-hand side of (2a) is a tensor. If we specialize the coordinate system again in the familiar manner, we get in place of (2a) the equivalent equations

$$R_{im} = \sum_{l} \frac{\partial \Gamma_{im}^{l}}{\partial x_l} + \sum_{\rho l} \Gamma_{i\rho}^{l} \Gamma_{ml}^{\rho} = -\kappa\left(T_{im} - \frac{1}{2} g_{im} T\right) \tag{6}$$

Einstein's constant

$$\sqrt{-g} = 1. \tag{3a}$$

We assume, as usual, that the divergence of the energy tensor of matter vanishes

Fig. 3 – Translation of Einstein's founding paper.
The field equation with zero-divergence after the modification of November 25, 1915.

Einstein is obviously interested in phenomena Newtonian mechanics cannot explain. On November 18, 1915, he publishes a linearized solution of the field equation without a second member, resulting of a null Ricci tensor. [3]

Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie.

Von A. Einstein.

In einer jüngst in diesen Berichten erschienenen Arbeit, habe ich Feld-
gleichungen der Gravitation aufgestellt, welche bezüglich beliebiger

Fig. 4 – Einstein explains the perihelion motion of Mercury on November 18, 1915.

In this solution, Einstein chooses to give the value -1 to the determinant of his metric solution, a hypothesis Schwarzschild will take again.

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§ 1. Das Gravitationsfeld.

Aus meinen letzten beiden Mitteilungen geht hervor, daß das Gravitationsfeld im Vakuum bei geeignet gewähltem Bezugssystem folgenden Gleichungen zu genügen hat

$$\boxed{\sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\nu\alpha} = 0,} \qquad (1)$$

wobei die $\Gamma^{\alpha}_{\mu\nu}$ durch die Gleichung definiert sind

$$\Gamma^{\alpha}_{\mu\nu} = -\begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} = -\sum_{\beta} g^{\alpha\beta} \begin{bmatrix} \mu\nu \\ \beta \end{bmatrix} = -\frac{1}{2} \sum_{\beta} g^{\alpha\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x_{\nu}} + \frac{\partial g_{\nu\beta}}{\partial x_{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x_{\alpha}} \right).$$

Machen wir außerdem die in der letzten Mitteilung begründete Hypothese, daß der Skalar des Energietensors der »Materie« stets verschwinde, so tritt hierzu die Determinantengleichung

$$\boxed{|g_{\mu\nu}| = -1.} \qquad (3)$$

Es befinde sich im Anfangspunkt des Koordinatensystems ein Massenpunkt (die Sonne). Das Gravitationsfeld, welches dieser Massen-

Fig. 5 –Field equation with no right hand-side, and determinant -1.

It is worth mentioning, and this will be of great importance later on, that Einstein opts for a metric signature with a time variable (as well as a proper time) having real values:

ohne mich auf die Frage einzulassen, ob es die einzig mögliche sei.

Wir gehen nun in solcher Weise vor. Die $g_{\mu\nu}$ seien in »nullter Näherung« durch folgendes, der ursprünglichen Relativitätstheorie entsprechende Schema gegeben

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{matrix}\right\}, \qquad (4)$$

oder kürzere

$$\left.\begin{aligned} g_{\rho\sigma} &= \delta_{\rho\sigma} \\ g_{\rho 4} &= g_{4\rho} = 0 \\ g_{44} &= 1 \end{aligned}\right\}. \qquad (4a)$$

Hierbei bedeuten ρ und σ die Indizes 1, 2, 3; $\delta_{\rho\sigma}$ ist gleich 1 oder 0, je nachdem $\rho = \sigma$ oder $\rho \neq \sigma$ ist.

Fig. 6 – Einstein's choice of a metric signature (+ – – –).

The solution of his field equation is a four-dimensional hypersurface. It is clear that his choice of variable is within $\mathbb{R}^4$ and the element of length *ds* is real.

In the following excerpt, he gives an approximate expression of the metric potential g_{44}, introducing a quantity α, a Greek letter that Schwarzschild also choses to designate the constant of integration of his nonlinear solution in January 1916.

Erste Approximation.

Es ist leicht zu verifizieren, daß in Größen erster Ordnung den Gleichungen (1) und (3) sowie den eben genannten 4 Bedingungen genügt wird durch den Ansatz

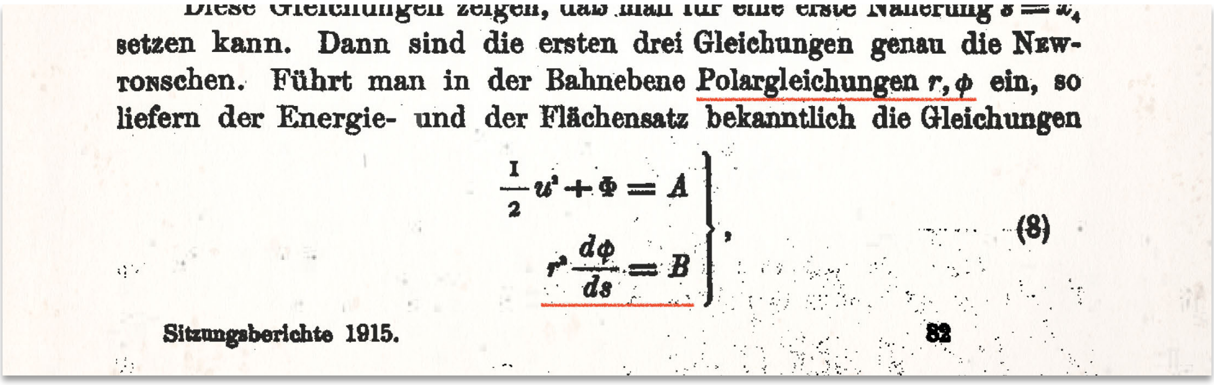
$$\left.\begin{aligned} g_{\rho\sigma} &= -\delta_{\rho\sigma} + \alpha\left(\frac{\partial^2 r}{\partial x_\rho \partial x_\sigma} - \frac{\delta_{\rho\sigma}}{r}\right) = -\delta_{\rho\sigma} - \alpha\frac{x_\rho x_\sigma}{r^3} \\ \boxed{g_{44} = 1 - \frac{\alpha}{r}} & \end{aligned}\right\}. \qquad (4\,b)$$

Die $g_{4\rho}$ bzw. $g_{\rho 4}$ sind dabei durch Bedingung 3 festgelegt. r bedeutet die Größe $+\sqrt{x_1^2 + x_2^2 + x_3^2}$, α eine durch die Sonnenmasse bestimmte Konstante.

Fig. 7 – Einstein's approximation, linearized solution of his field equation.

Letters x_1, x_2, x_3 denote real space coordinates from which he designates as polar coordinate $\sqrt{x_1^2 + x_2^2 + x_3^2}$

Using polar coordinates, he finds the Newtonian area law:



Diese Gleichungen zeigen, daß man für eine erste Näherung $s = x_4$ setzen kann. Dann sind die ersten drei Gleichungen genau die Newtonschen. Führt man in der Bahnebene Polargleichungen r, ϕ ein, so liefern der Energie- und der Flächensatz bekanntlich die Gleichungen

$$\left.\begin{aligned} \frac{1}{2}u^2 + \Phi &= A \\ r^2\frac{d\phi}{ds} &= B \end{aligned}\right\}, \qquad (8)$$

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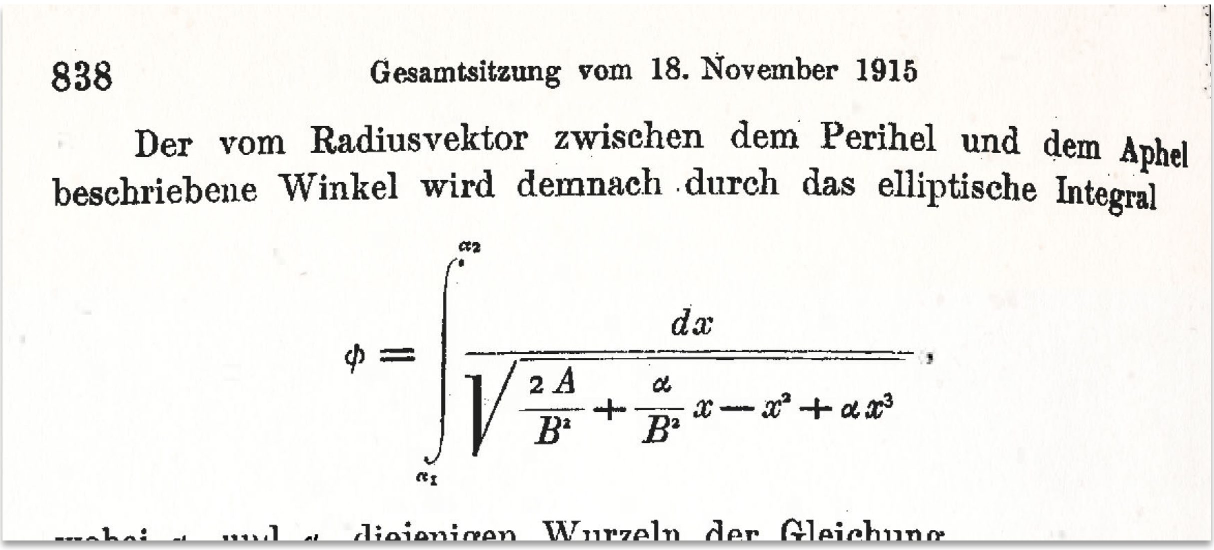
Fig. 8 – Einstein: area law (Kepler's second law of planetary motion).

Going from the variable *r* to its inverse 1/*r*, he expresses the form of the geodesic solution:



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Der vom Radiusvektor zwischen dem Perihel und dem Aphel beschriebene Winkel wird demnach durch das elliptische Integral

$$\phi = \int_{\alpha_1}^{\alpha_2} \frac{dx}{\sqrt{\frac{2A}{B^2} + \frac{\alpha}{B^2}x - x^2 + \alpha x^3}},$$

wobei α_1 und α_2 diejenigen Wurzeln der Gleichung

Fig. 9 – Quasi Newtonian solution.

To stick with the chronology, we have to mention the communication presented by the great mathematician Hilbert at the Göttingen Academy of Sciences, in the session of November 20, 1915. [5]

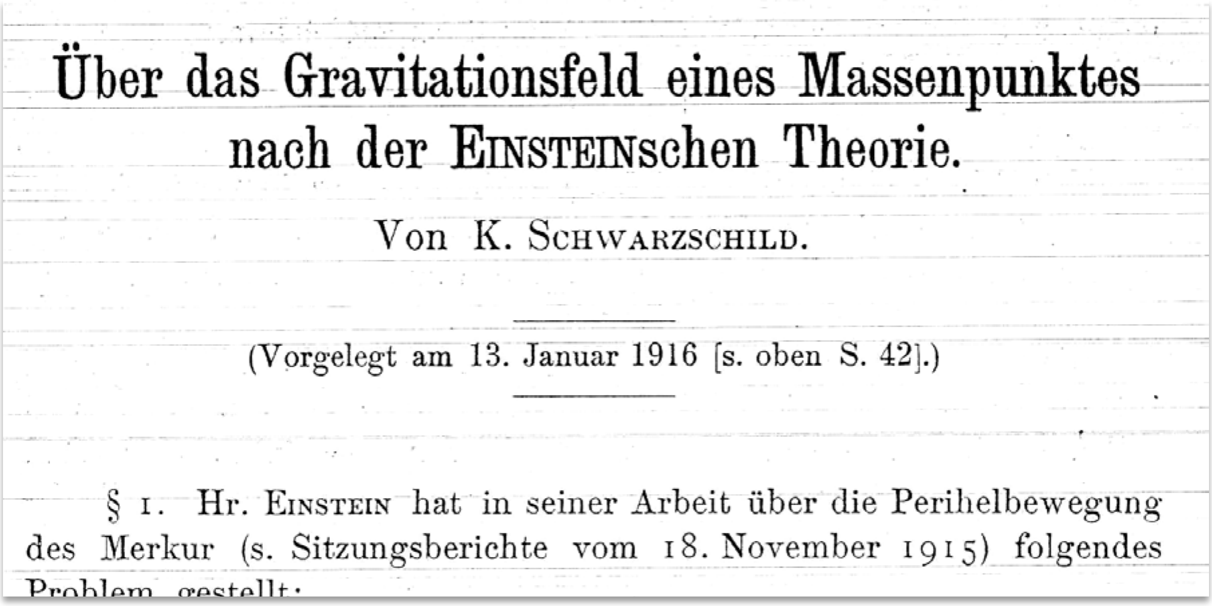
He gradually got involved about a possible mathematization of physics through a variational approach. The same month as Einstein's presentation of his finalized version of general relativity, Hilbert presented a first paper entitled "The Foundations of Physics", extremely ambitious. At that time,

physics boils down to two sets: gravitation and electromagnetism. Everyone thinks that the one who would succeed uniting these two worlds (in what will later be called a "unified field theory") will master all the physics of his time. This is the meaning of Hilbert's paper.

Such a work will be pursued by Einstein, who will also fail in this venture. We know today that in order to combine gravitation and electromagnetism together, four dimensions are not enough. To begin with, a fifth dimension would be mandatory: Kaluza's fifth dimension.

But very quickly, Hilbert did not feel satisfied with his article and decided to withdraw it to make changes.[1] As will be seen later, he will present a second communication in December 1916 [6] and it is this version, very similar to the first one, that we will comment on.

At that time of war, Karl Schwarzschild, 43 years old and already the father of three children, joined with the rank of lieutenant, by patriotism, to fight on the Russian front. He is already an astronomer and a confirmed mathematician. Taking note of Einstein's paper, he publishes in January and February 1916, not one article, but two, in which he presents his nonlinear exact solution of the field equation. [7] [8] Here is the title of the first paper:



Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie.

Von K. Schwarzschild.

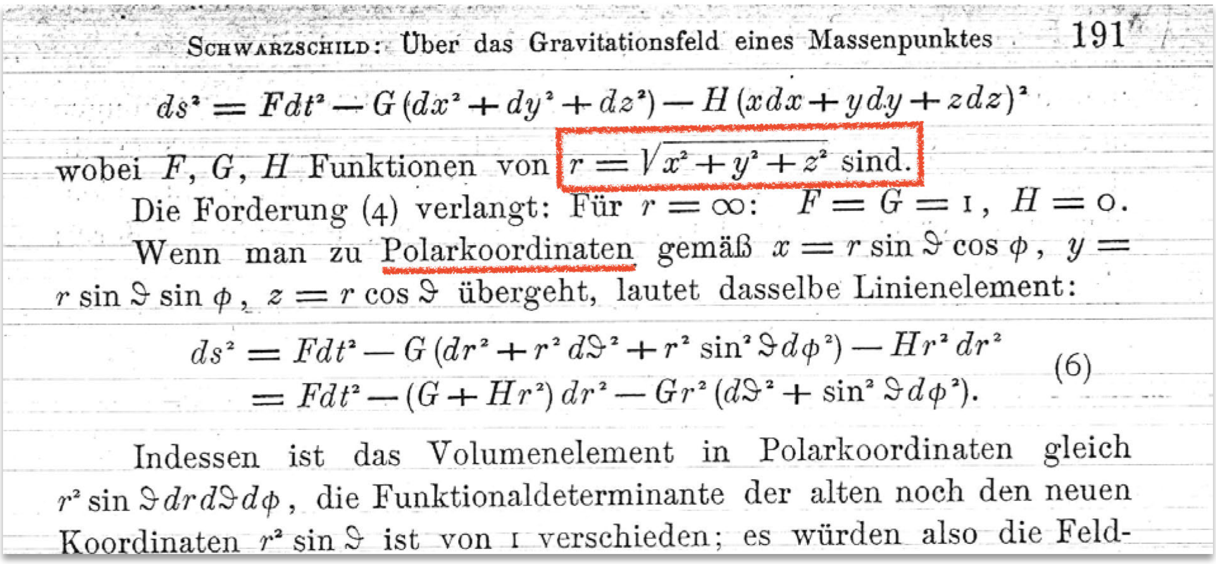
(Vorgelegt am 13. Januar 1916 [s. oben S. 42].)

§ 1. Hr. Einstein hat in seiner Arbeit über die Perihelbewegung des Merkur (s. Sitzungsberichte vom 18. November 1915) folgendes Problem gestellt:

Fig. 10 – First Schwarzschild's paper on January 13, 1916
"On the gravitational field of a mass point according to Einstein's theory."

[1] Although Hilbert's first communication was presented during the session of November 20, 1915, Hilbert will prevent its publication for months in order to make modifications. In particular, he will change the field equation *a posteriori* to use the divergenceless version presented by Einstein on November 25. The final version of his paper will be published in 1916, keeping the date of November 20, 1915. Hilbert did not "invent the field equation" five days before Einstein as one can sometimes naively hear it.

He quickly situates his solution by setting coordinates: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$$ds^2 = Fdt^2 - G(dx^2+dy^2+dz^2) - H(xdx+ydy+zdz)^2$$

wobei F, G, H Funktionen von $\boxed{r=\sqrt{x^2+y^2+z^2} \text{ sind.}}$

Die Forderung (4) verlangt: Für $r=\infty$: $F=G=1$, $H=0$.

Wenn man zu Polarkoordinaten gemäß $x=r\sin\vartheta\cos\phi$, $y=r\sin\vartheta\sin\phi$, $z=r\cos\vartheta$ übergeht, lautet dasselbe Linienelement:

$$\begin{aligned} ds^2 &= Fdt^2 - G(dr^2+r^2d\vartheta^2+r^2\sin^2\vartheta d\phi^2) - Hr^2dr^2 \\ &= Fdt^2 - (G+Hr^2)dr^2 - Gr^2(d\vartheta^2+\sin^2\vartheta d\phi^2). \end{aligned} \quad (6)$$

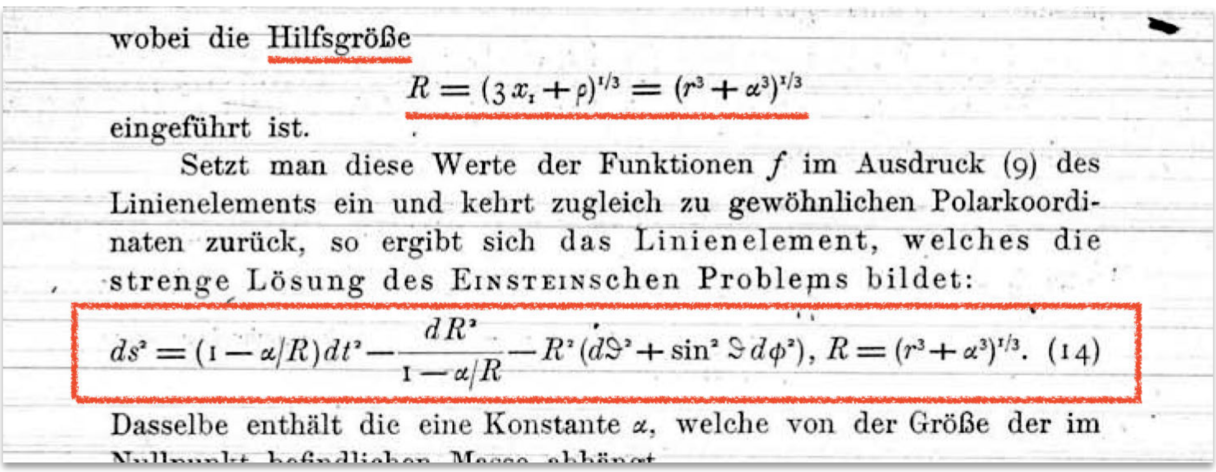
Indessen ist das Volumenelement in Polarkoordinaten gleich $r^2\sin\vartheta dr d\vartheta d\phi$, die Funktionaldeterminante der alten noch den neuen Koordinaten $r^2\sin\vartheta$ ist von 1 verschieden; es würden also die Feld-

Fig. 11 – Schwarzschild defines his coordinates.

"*One calls* ***t*** *the time and* ***x****,* ***y****,* ***z*** *the rectangular coordinates*". These coordinates are real. If he had opted for coordinates that could take on imaginary values, he would have mentioned it. So he chooses a representation in $\mathbb{R}^3$. Then he goes to a polar coordinate system, writing $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, $z=r\cos\theta$ with

$$r=\sqrt{x^2+y^2+z^2}$$

which implies $r\geq 0$, at least for the choice of a representation where variables (x , y , z) belong to $\mathbb{R}$. At this point, it should be noted that *r* is not a radial distance, but a simple *space marker*, a simple number. Schwarzschild then introduces what he calls an "auxiliary quantity" (*Hilfsgröße*):


wobei die Hilfsgröße

$$R=(3x_1+\rho)^{1/3}=(r^3+\alpha^3)^{1/3}$$

eingeführt ist.

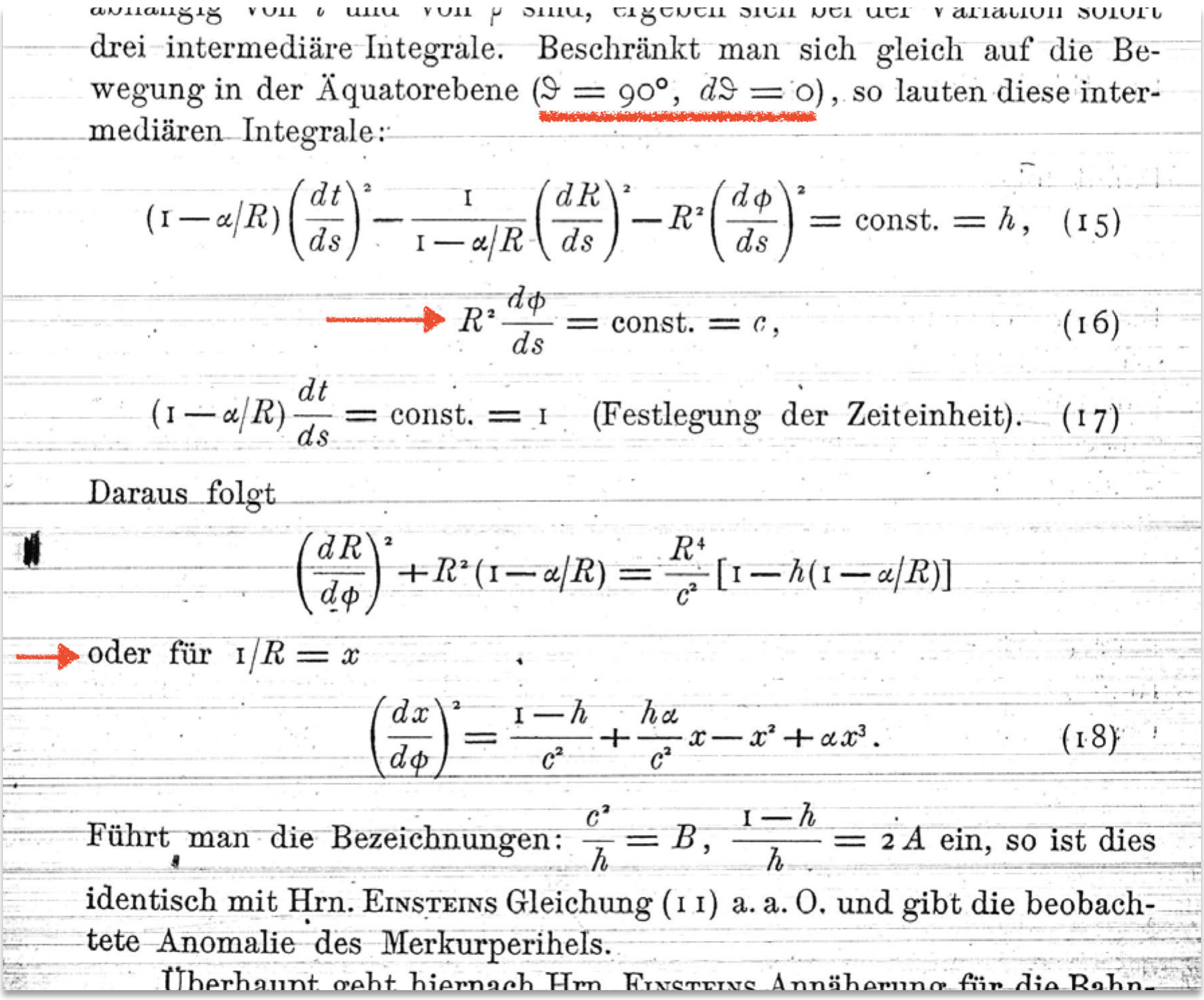
Setzt man diese Werte der Funktionen f im Ausdruck (9) des Linienelements ein und kehrt zugleich zu gewöhnlichen Polarkoordinaten zurück, so ergibt sich das Linienelement, welches die strenge Lösung des Einsteinschen Problems bildet:

$$\boxed{ds^2=(1-\alpha/R)dt^2-\frac{dR^2}{1-\alpha/R}-R^2(d\vartheta^2+\sin^2\vartheta d\phi^2),\quad R=(r^3+\alpha^3)^{1/3}. \quad (14)}$$

Dasselbe enthält die eine Konstante α, welche von der Größe der im Nullpunkt befindlichen Masse abhängt.

Fig. 12 – The solution expressed with auxiliary quantity *R*.

In so doing, he calls his constant of integration α in order to tally with Einstein's 1915 paper [3]. Using this set of variables $\{t, R, \theta, \varphi\}$ he calculates the geodesics. Like Einstein, he notes that these geodesics are part of planes and he chooses, like him, the plane $\theta = \pi/2$. It is again with this auxiliary quantity R that he expresses the area law:



abhängig von t und von ϕ sind, ergeben sich bei der Variation sofort drei intermediäre Integrale. Beschränkt man sich gleich auf die Bewegung in der Äquatorebene ($\vartheta = 90°$, $d\vartheta = 0$), so lauten diese intermediären Integrale:

$$(1-\alpha/R)\left(\frac{dt}{ds}\right)^2 - \frac{1}{1-\alpha/R}\left(\frac{dR}{ds}\right)^2 - R^2\left(\frac{d\phi}{ds}\right)^2 = \text{const.} = h, \quad (15)$$

$$R^2\frac{d\phi}{ds} = \text{const.} = c, \quad (16)$$

$$(1-\alpha/R)\frac{dt}{ds} = \text{const.} = 1 \quad \text{(Festlegung der Zeiteinheit).} \quad (17)$$

Daraus folgt

$$\left(\frac{dR}{d\phi}\right)^2 + R^2(1-\alpha/R) = \frac{R^4}{c^2}[1-h(1-\alpha/R)]$$

oder für $1/R = x$

$$\left(\frac{dx}{d\phi}\right)^2 = \frac{1-h}{c^2} + \frac{h\alpha}{c^2}x - x^2 + \alpha x^3. \quad (18)$$

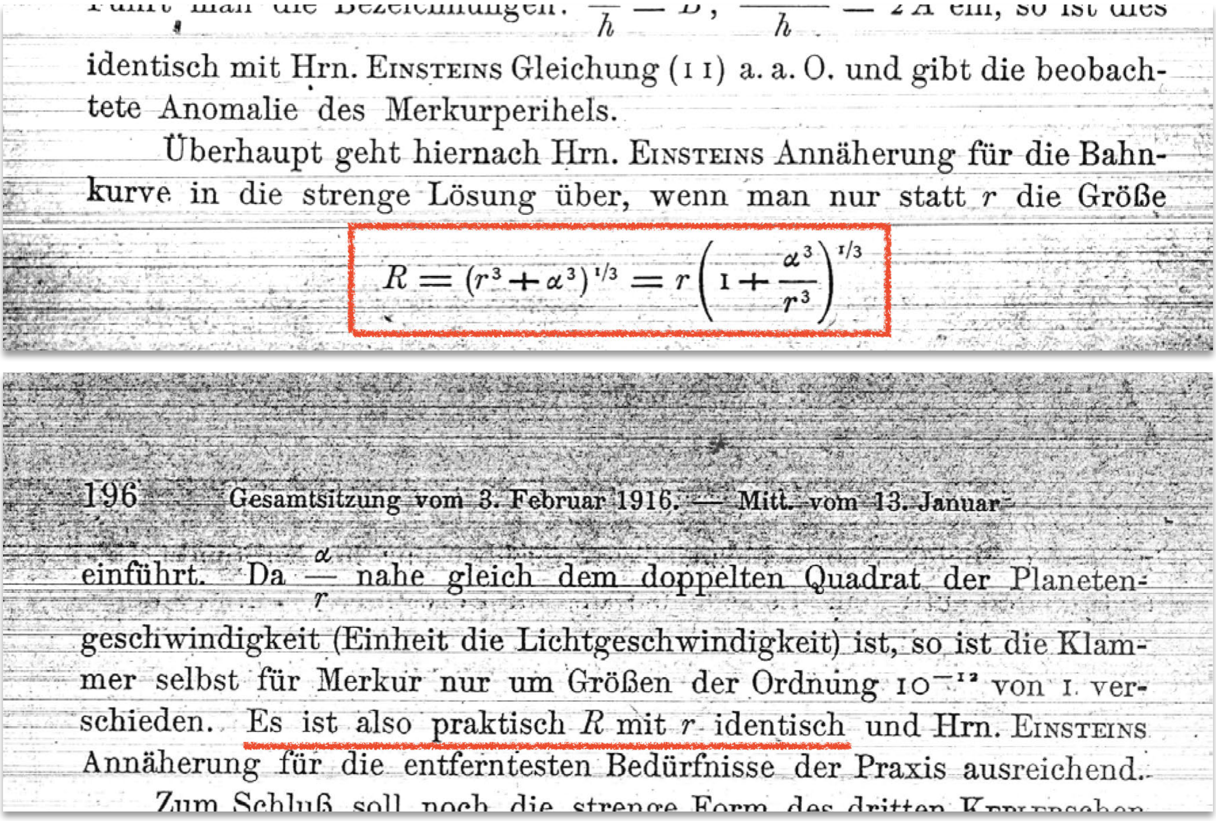
Führt man die Bezeichnungen: $\frac{c^2}{h} = B$, $\frac{1-h}{h} = 2A$ ein, so ist dies identisch mit Hrn. Einsteins Gleichung (11) a. a. O. und gibt die beobachtete Anomalie des Merkurperihels.

Überhaupt geht hiernach Hrn. Einsteins Annäherung für die Bahn-

Fig. 13 – Schwarzschild calculates the geodesics $\Phi_{(R)}$: eq. (18)

The same applies to equation (18) for the expression of the solution in the form of an integral based on the variable $x = 1/r$ (and not on the variable r) as in Einstein's paper. It is clear that he orientates the expression of his solution to stick closely to the Einstein's result.

But this detail is quite secondary to him, inasmuch as the conditions of planetary astronomy makes the two quantities practically equal, which he notes a little further on (*Es ist also praktisch R mit r identisch*):



identisch mit Hrn. Einsteins Gleichung (11) a. a. O. und gibt die beobachtete Anomalie des Merkurperihels.

Überhaupt geht hiernach Hrn. Einsteins Annäherung für die Bahnkurve in die strenge Lösung über, wenn man nur statt r die Größe

$$\boxed{R = (r^3 + \alpha^3)^{1/3} = r\left(1 + \frac{\alpha^3}{r^3}\right)^{1/3}}$$

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einführt. Da $\frac{\alpha}{r}$ nahe gleich dem doppelten Quadrat der Planetengeschwindigkeit (Einheit die Lichtgeschwindigkeit) ist, so ist die Klammer selbst für Merkur nur um Größen der Ordnung 10^{-12} von 1 verschieden. Es ist also praktisch R mit r identisch und Hrn. Einsteins Annäherung für die entferntesten Bedürfnisse der Praxis ausreichend.

Fig. 14 – Schwarzschild's solution meets Einstein's.

Translation of the underlined passage:

"*Therefore r is virtually identical to R*"

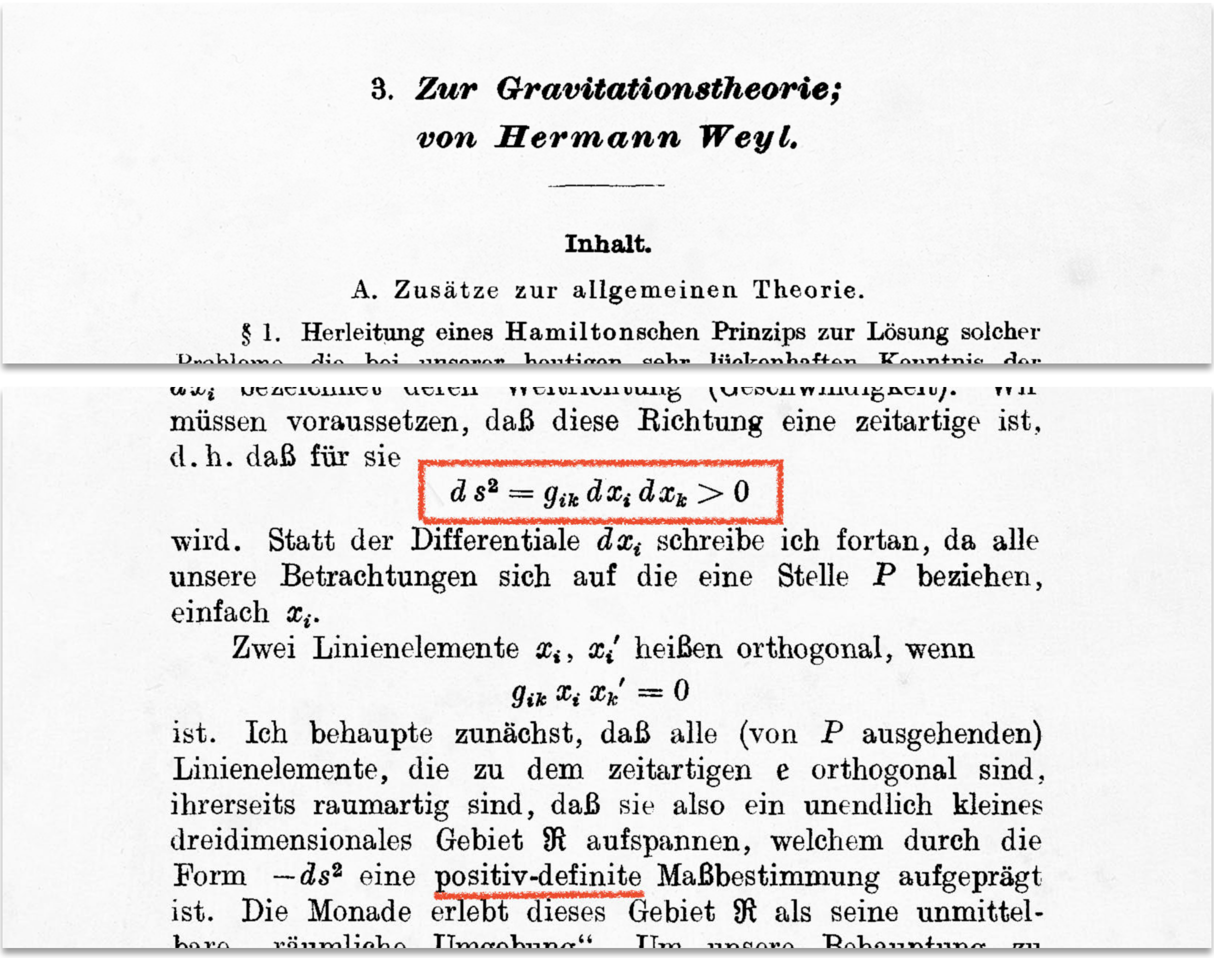
After Schwarzschild's death, his work was first presented to the community of mathematicians by Frank [9] then quickly taken up by Droste [10], Weyl [11] and of course, Hilbert [6].

It is the media, mostly American, that will fashion Einstein's image as an isolated genius, author of a theory that a tiny number of people would have been able to understand. This does not in any way reduce Einstein's merits, but he was in fact at the forefront of physics, in which a relatively large number of researchers, mainly German-speaking, were already involved. Among them, Schwarzschild.

Scientific articles were then published in German, and at that time accessing such work is done by reading "offprints", sample copies on paper, which are transmitted by postal services. Subsequently, the same papers are made available, grouped in books, but still in German. It was only in the 1970s, more than half a century later, that this documentation was translated in English, and later on made available as PDF files. The modern distribution of these files through the Internet is something very recent. Only a few decades ago, in order to distribute simple copies of articles, researchers had to type their works on tracing papers, then use them to expose a UV-sensitive yellow sheet. And finally, embedding the copies within ammonia vapors to reveal the content. I personally experienced this technique in the 1960s. The scanner, an essential part of photocopiers, will only be put on the market in the late 1960s.

All this to tell the works of Einstein and Schwarzschild have spread to the scientific community in the form of commentaries, not in their original form.

However, one should note that Droste [10] and Weyl [11] chose the element of length as always positive, associated with a metric signature (+ - - -):



3. ***Zur Gravitationstheorie;***
von Hermann Weyl.

Inhalt.

A. Zusätze zur allgemeinen Theorie.

§ 1. Herleitung eines Hamiltonschen Prinzips zur Lösung solcher

müssen voraussetzen, daß diese Richtung eine zeitartige ist, d. h. daß für sie

$$\boxed{d s^2 = g_{ik}\, dx_i\, dx_k > 0}$$

wird. Statt der Differentiale dx_i schreibe ich fortan, da alle unsere Betrachtungen sich auf die eine Stelle P beziehen, einfach x_i.

Zwei Linienelemente x_i, x_i' heißen orthogonal, wenn

$$g_{ik}\, x_i\, x_k' = 0$$

ist. Ich behaupte zunächst, daß alle (von P ausgehenden) Linienelemente, die zu dem zeitartigen e orthogonal sind, ihrerseits raumartig sind, daß sie also ein unendlich kleines dreidimensionales Gebiet $\mathfrak{R}$ aufspannen, welchem durch die Form $-ds^2$ eine positiv-definite Maßbestimmung aufgeprägt ist. Die Monade erlebt dieses Gebiet $\mathfrak{R}$ als seine unmittel-

Fig. 15 – Excerpt of Weyl's 1917 paper.

Still in Weyl's paper, we find such a signature and the mention of a real and essentially positive polar coordinate *r*.

B. Theorie des statischen rotationssymmetrischen Gravitationsfeldes.

§ 4. Massenpunkt ohne und mit elektrischer Ladung.

Für das Folgende ist es nötig, zu der Schwarzschildschen Bestimmung des Gravitationsfeldes eines ruhenden Massenpunktes²) einige Bemerkungen zu machen. Ein dreidimensionales kugelsymmetrisches Linienelement hat bei Benutzung geeigneter Koordinaten notwendig die Gestalt

$$d\sigma^2 = \mu(dx_1^2 + dx_2^2 + dx_3^2) + l(x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3)^2,$$

wo μ und l nur von der Entfernung

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

abhängen. Über die Skala, in der diese Entfernung gemessen wird, kann noch so verfügt werden, daß $\mu = 1$ ausfällt; das möge geschehen. Für das vierdimensionale Linienelement haben wir den Ansatz zu machen

$$ds^2 = f\,dx_4^2 - d\sigma^2,$$

wo auch f nur eine Funktion von r ist. Setzen wir noch

$$1 + l r^2 = h$$

Fig. 16 – In Weyl (1917): the polar coordinate and the signature.

Similar mention with Droste, in his paper communicated in English by Hendrik Lorentz sixteen days after the death of Schwarzschild, but published only the following year. [10]

Physics. — "*The field of a single centre in* EINSTEIN'S *theory of gravitation, and the motion of a particle in that field.*". By J. DROSTE. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 27, 1916).

For a centre at rest and symmetrical in all directions it is easily seen that

$$ds^2 = w^2\,dt^2 - u^2\,dr^2 - v^2\,(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2), \quad . \quad . \quad (2)$$

w, u, v only depending on r, and (ϑ, φ) representing polar coordinates. Now, if g_{ij} and therefore also g^{ij} are all zero, if $i \neq j$, G

Fig. 17 – Droste specifies on May 27, 1916 the signature he chooses.

It does not seem to come to anyone's mind to extend this solution for $r < \alpha$ which would imply a modification of the signature of the metric.

It is then that Hilbert presents the second communication of his long article "The Foundations of Physics":

Die Grundlagen der Physik.

(Zweite Mitteilung.)

Von

David Hilbert.

Vorgelegt in der Sitzung vom 23. Dezember 1916.

In meiner ersten Mitteilung [1]) habe ich ein System von Grundgleichungen der Physik aufgestellt. Ehe ich mich zur Theorie der Integration dieser Gleichungen wende, erscheint es nötig, einige allgemeinere Fragen sowohl logischer wie physikalischer Natur zu erörtern.

Zunächst führen wir an Stelle der Weltparameter w_s (s —

Fig. 18 – Hilbert's second communication, December 23, 1916.

In this sequel to his first communication presented the year before, he takes his work over and includes the Schwarzschild solution published in January. Hilbert knows that he died. Einstein delivered a memorial lecture on Karl Schwarzschild [12] on June 29, 1916 at the opening meeting of the Royal Prussian Academy of Sciences, Berlin. But Hilbert does not seem to focus much on this result, which appears to him as a detail in the vast scientific saga he intends to carry out on both gravitation and electromagnetism. Let's see his mention of Schwarzschild's solution.

In his paper, Hilbert mentions a central symmetry (*zentrischsymmetrisch*):

die Grundlagen der Physik. 71

Gravitation einer in der Umgebung des Nullpunktes zentrischsymmetrisch verteilten Masse anzusehen [1]). Im gleichen Sinne ist auch der Massenpunkt als der Grenzfall einer gewissen Verteilung der Elektrizität um einen Punkt herum aufzufassen, doch sehe ich an dieser Stelle davon ab, die Bewegungsgleichungen desselben aus meinen physikalischen Grundgleichungen abzuleiten. Ähnlich verhält es sich mit der Frage nach den Differentialgleichungen für die Lichtbewegung.

Fig. 19 – Hilbert's hypothesis of a central symmetry.

He then introduces polar coordinates and designates a fourth variable by the letter *l*:

2. Die $g_{\mu\nu}$ sind von der Zeitkoordinate x_4 unabhängig.

3. Die Gravitation $g_{\mu\nu}$ ist zentrisch symmetrisch in Bezug auf den Koordinatenanfangspunkt.

Nach Schwarzschild ist die allgemeinste diesen Annahmen entsprechende Maßbestimmung in räumlichen Polarkoordinaten, wenn

$$\begin{aligned} w_1 &= r\cos\vartheta \\ w_2 &= r\sin\vartheta\cos\varphi \\ w_3 &= r\sin\vartheta\sin\varphi \\ w_4 &= l \end{aligned}$$

gesetzt wird, durch den Ausdruck

$$F(r)\,dr^2 + G(r)\,(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2) + H(r)\,dl^2 \tag{42}$$

dargestellt, wo $F(r)$, $G(r)$, $H(r)$ noch willkürliche Funktionen von r sind. Setzen wir

$$r^* = \sqrt{G(r)},$$

so sind wir in gleicher Weise berechtigt r^*, ϑ, φ als räumliche Polarkoordinaten zu deuten. Führen wir in (42) r^* anstatt r ein und lassen dann wieder das Zeichen * weg, so entsteht der Ausdruck

$$M(r)\,dr^2 + r^2\,d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 + W(r)\,dl^2, \tag{43}$$

wo $M(r)$, $W(r)$ die zwei wesentlichen willkürlichen Funktionen

Fig. 20 – Hilbert opts for polar coordinates.

There are several things to note in this excerpt of fig. 20. We can read in the underlined passage of hypothesis #3:

> "*The gravitation is centrally symmetric with respect to the origin of coordinates*."

We find again the theme of a solution with a central symmetry, and not a spherical symmetry.

In equation (42) he presents the bilinear form of his metric, and then introduces the hypothesis $r^* = \sqrt{G(r)}$ where this variable *r** quickly transforms into *r*. The translation is:

> "*If we introduce r* in (42) instead of r and then eliminate the sign *, the result is the expression (43).*"

This hypothesis will then totally guide his solution. Incidentally, Hilbert indicates the nature of his temporal coordinate *l*:

den gemachten Annahmen 1., 2., 3., dar. Nehmen wir als Integrale von (44) $m = \alpha$, wo α eine Konstante ist und $w = 1$, was offenbar keine wesentliche Einschränkung bedeutet, so ergibt sich aus (43) für $l = it$ die gesuchte Maßbestimmung in der von Schwarzschild zuerst gefundenen Gestalt

$$(45)\quad \boxed{G(dr, d\vartheta, d\varphi, dl) = \frac{r}{r-\alpha}\,dr^2 + r^2\,d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 - \frac{r-\alpha}{r}\,dt^2.}$$

Die Singularität dieser Maßbestimmung bei $r = 0$ fällt nur dann

Fig. 21 – Hilbert's pure imaginary time and his formulation of the solution.

For Hilbert, spacetime is a *fiber bundle*, built from a real space (x , y , z) but the fiber, i.e. the fourth coordinate *l*, is purely imaginary.

Regarding Schwarzschild's solution, this passage is absolutely crucial and will guide a century of scientific work. Hilbert makes a confusion when he writes:

> "*then for* $l = i\,t$ (43) *results in the desired metric in the form first found by Schwarzschild*."

If his solution is completed (45) it should be written:

$$ds^2 = -(1-\frac{\alpha}{r})\,dt^2 + \frac{dr^2}{1-\frac{\alpha}{r}} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2$$

whereas the real Schwarzschild solution is:

$$ds^2 = (1-\frac{\alpha}{R})\,dt^2 - \frac{dR^2}{1-\frac{\alpha}{R}} - R^2 d\vartheta^2 - R^2 \sin^2\vartheta\, d\varphi^2 \qquad R^3 = (r^3 + \alpha^3)^{1/3}$$

The difference is obvious. Hilbert confused his coordinate *r* with the auxiliary variable *R* introduced by Schwarzschild. He also chooses a reverse signature.

In the rest of the paper, Hilbert comes back to this choice of variable in a footnote, considering it merely pointless. He sees it only as a wish to reject the singularity back to the origin.

$$r^2 \sin^2 \vartheta \frac{d\varphi}{dp} = B, \tag{48}$$

$$\frac{r-\alpha}{r} \frac{dt}{dp} = C, \tag{49}$$

wo A, B, C Integrationskonstante bedeuten.

1) Die Stellen $r = \alpha$ nach dem Nullpunkt zu transformieren, wie es Schwarzschild tut, ist meiner Meinung nach nicht zu empfehlen; die Schwarzschildsche Transformation ist überdies nicht die einfachste, die diesen Zweck erreicht.

2) Dieser letzte einschränkende Zusatz findet sich weder bei Einstein noch bei Schwarzschild.

Fig. 22 – The footnote in Hilbert's paper mentioning Schwarzschild's work.

Translation:

> "*To transform the locations to the origin, as Schwarzschild does, is not to be recommended in my opinion; Schwarzschild's transformation is moreover not the simplest that achieves this goal*."

Hilbert considers that what happens at this Schwarzschild sphere is a *true singularity*, whereas it will be classically shown afterward that it is only a *coordinate singularity*, that can be cancelled out with an appropriate change of variable.

Geometry signification

The solution to Einstein's equation is a four-dimensional hypersurface. Einstein, Schwarzschild, Frank, Droste, and Weyl relied on the idea that the element of length *s*, hence the proper time, the only thing independent of the choice of coordinates on this hypersurface, is a real quantity.

Let's go back to Schwarzschild's 1916 solution. Schwarzschild does not make it explicit, simply because he does not see the point in it. He merely wants to show the same result as Einstein's explanation of the precession of the perihelion of Mercury. No one at this time thinks astronomical objects requiring a nonlinear treatment of the problem could really exist. His quantity α is nothing but what will later be called the Schwarzschild radius R_s. He immediately notices that for the Sun, it is equivalent to one hundred thousandth of its diameter. As it happens, in his second paper publish in February 1916 [8] which we will come back to, he built a geometry within a sphere filled with a fluid of constant density, a non-singular solution, so he

does not find relevant to deal in his first "exterior solution" paper with a region of space that is related to the "interior solution".

In 2011, Christian Corda [13] makes this solution explicit:

$$ds^2 = \frac{(r^3+\alpha^3)^{1/3}-\alpha}{(r^3+\alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4}{(r^3+\alpha^3)\left[(r^3+\alpha^3)^{1/3}-\alpha\right]} dr^2 - (r^3+\alpha^3)^{2/3}(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Keep in mind that such a metric can be expressed in any arbitrary coordinate system. These coordinates are then merely numbers, *space markers*. The only size having an intrinsic property is the length s. Then, all boils down to the choice for this length.

If we decide that this length s is real, then such an expression designates a 4D hypersurface. The choice of angles θ and φ implies a solution with a spherical symmetry (instead of a central one). Helical geodesics exist on this hypersurface, that can be expressed giving a fixed value to θ, and fixing the coordinate r. The variables t and φ linearly depend on the parameter s. Thus, they are both related by a linear function. Spatial projection of such geodesic can be calculated: they simply become circles.

It is therefore possible to travel along this hypersurface, drawing parallel circles. If their perimeter can reach $+\infty$ it is however limited to less than $2\pi\alpha$. If the value of the angle θ is varied then this family of parallel circles generates a family of parallel spheres. Let us note that we deliberately use the word "parallel" and not "concentric" because the Schwarzschild solution has a spherical symmetry and not, as written by Hilbert, a central one.

As we skim through this series of parallel spheres, by reducing their area, at some point a minimal value is reached (if we decide that the hypersurface is real). This situation is met:

- Either for the value $R = \alpha$ in the case of what we call "Hilbert's representation" (t, R, θ, φ) (introducing the auxiliary quantity R according to Schwarzschild).
- Or for the value $r = 0$ in Schwarzschild's representation (t, R, θ, φ).

This hypersurface is thus uncontractible. How to interpret it?

One can of course decide it is a manifold with boundary. A boundary, incidentally, along which the determinant of the metric is zero in Schwarzschild's representation. This will be discussed in the second part of this article. But in the 1960s, other paths were considered.

Extensions of the solution and their meaning

Decades after the emergence of the Schwarzschild solution, it is known only through subsequent comments published afterward. I was very surprised at a recent international conference on black holes [2] held in Frankfurt (Schwarzschild's birthplace), specifically entitled "The Karl Schwarzschild Meeting", that no attendee, including German natives, had never read the founding papers I have just mentioned, as if the origin of the model were lost in the mists of the past. In his masterly lecture in front of all the delegates, Juan Maldacena said:[3]

— *The Schwarzschild solution has confused us over a hundred years and it has forced us to sharpen our views on space and time. It has led to a sharper understanding of Einstein's theory. Experimentally, it is explaining several astrophysical observations. Its quantum aspects have been a source of theoretical paradoxes that are forcing us to understand better the relation between spacetime, geometry and quantum mechanics.*

This text suggests that the immediate understanding of the Schwarzschild solution remained limited for a while, but thanks to further studies, a century of work made it possible to improve this reading by introducing an extended view of spacetime.

Why do astrophysicists refer to the Schwarzschild solution?

As will be discussed in the second part of this article, Schwarzschild had completed in February 1916 his first "exterior solution" [7] with a second "interior solution" [8] related to the geometry within a sphere filled by a fluid of constant density. At this point, the geometry describing a star immersed in a vacuum was completely achieved.

But progress in observations would reveal a new problem. In our galaxy, half of the stellar systems are multiple star systems. So there should exist a quite large number of binaries where one of the two stars became a subcritical neutron star, that could still be described with the two exterior and interior Schwarzschild metrics. Its companion star is then called the *donor*, as it

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[2] 3rd Karl Schwarzschild Meeting on Gravitational Physics and the Gauge/Gravity Correspondence (KSM 2017), 24–28 July 2017, FIAS, Frankfurt am Main, Germany.

[3] https://indico.fias.uni-frankfurt.de/event/4/session/17/contribution/39

continues to emit stellar wind that is gravitationally captured by its small companion, named the *accretor*.

In his first paper, evaluating the critical radius $(\alpha = R_s = 2m)$ and relating it to the mass responsible for this geometry, he had concluded that beyond a certain value, this "Schwarzschild radius" could exceed the star radius.

This is the question asked by observers to theoreticians:

— *Assuming that a supernova only left a subcritical neutron star after its gravitational collapse, what happens when the additional mass supplied by the stellar wind of a donor companion star triggers the geometric criticality in the neutron star?*

A question impossible to avoid. Theoreticians therefore imagined that the exterior Schwarzschild solution, yet referring to a portion of an empty space, could describe a new state of matter, popularized by the name "black hole".

The proposed analytic extensions

What is the current presentation of this Schwarzschild solution? Take for example the book of Adler, Bazin and Schiffer. [14] Page 187, equations (6.4) and (6.5) the authors reiterate Hilbert's hypothesis, writing:

$$ds^2 = Ac^2dt^2 - B\,dr^2 - C(d\theta^2 + \sin^2\theta\,d\varphi^2) \qquad \hat{r} = \sqrt{C(r)}\ r$$

A few lines later, the solution was again particularized, see equation (6.9):

Quelques lignes plus loin, la solution a de nouveau été particularisée, voir l'équation (6.9):

$$ds^2 = e^{\nu(r)}c^2dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

Unless the functions $\nu(r)$ and $\lambda(r)$ have the possibility to be imaginary, the impression is this writing militates for a signature $(+---)$. In this realistic view, the metric refers to a hypersurface where the only relevant variable, independent of the choice of coordinates, is the quantity *s*. This is the proper time, lived by any test particle. For a distant observer however, this proper time is identified to the variable *t*. Thus, we conclude that the coordinate time *t* is the one experienced by the distant observer. The following figure is found in every book about general relativity. It compares the free fall time measured either by the clock tied to a test particle falling towards the Schwarzschild sphere, or the one of a distant observer.

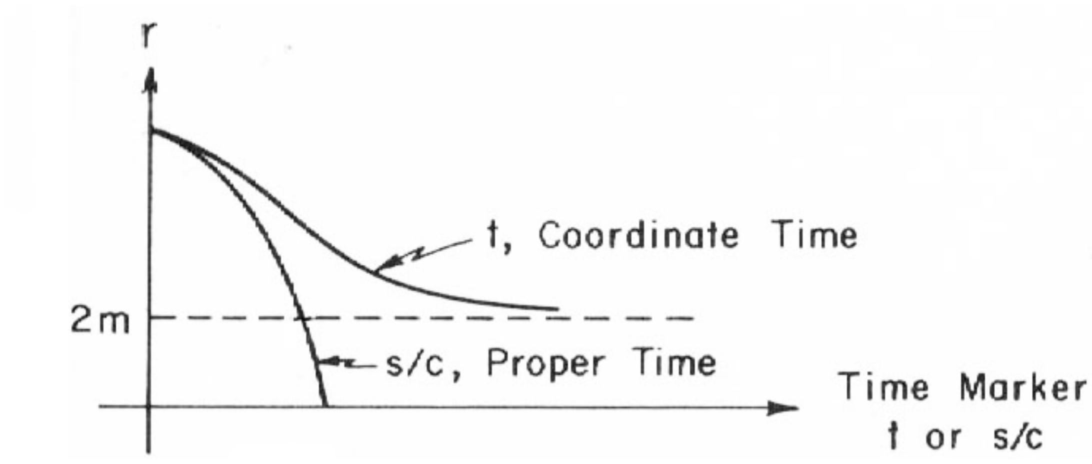

Fig. 23 – Fall toward the origin of a Schwarzschild geometry
in terms of coordinate time t and proper time on the test particle s/c.

Thus, whatever the phenomenon witnessed by the observer, it is supposed to unfold before his eyes during an infinite amount of time, which allows the theoretician to add:

— *I do not feel bound to describe the result of a process which, to my eyes, takes place in an infinite time.*

In passing, the New Zealand mathematician Roy Kerr [15] extended this type of solution to a "rotating black hole" thanks to a metric with a less constraining symmetry. The surface that has been called the "event horizon" takes the topology of a torus. But the frozen time, this freeze-frame, is also present in this model.

In 1960, Joseph Kruskal [16] and George Szekeres [17] constructed a first analytic extension, which Maldacena described as "extending the solution to cover the full spacetime".

We shall also cite Corda, who implements another analytic extension of the solution. [13]

In 1989, Canadian physicist Leonard S. Abrams published a paper [18] on the Schwarzschild solution entitled "Black Holes: The Legacy of Hilbert's Error" pointing out the same problems as those mentioned above. Questions also taken up by Italian physicist Salvatore Antoci. [19] [20] Those articles had practically no echo within the scientific community, except for a text

positioned in a blog [4] by a mathematician and computer scientist, W. D. Clinger, presenting himself as a topology specialist. He resumes the arguments of Abrams, writing:

— *The paper is well-written, and its math is almost (but not quite) correct.*

A little further on, his criticism gets stronger: *"Where Abrams went badly wrong."* He then provides the correct way to proceed:

— *In reality, the "quasiregular singularity" at the central point mass of the original Schwarzschild spacetime can be removed by allowing the radial coordinate r to go negative.*

In other words, it is enough to consider the extension of the solution for the values of $r = \sqrt{x^2 + y^2 + z^2} \;< 0$ i.e. in an imaginary portion of spacetime…

For that matter he quotes Corda, who does the same and constructs a description of the gravitational collapse of the neutron star which ends at a negative value $r = -\alpha = -R_s < 0$.

To the extent that these authors grant themselves the freedom to extend spacetime to an imaginary portion, this can no longer be criticized. On the other hand, these people call "crackpots" anyone who does not feel satisfied with such a formula.

Conclusion

In 1916, Einstein then Schwarzschild developed solutions to the field equation, the first through linearization and the second nonlinearly, which refer to real values of the coordinates and to a real value of the length *s*, measured on the four-dimensional hypersurface using the metric. This work is then taken up by different authors, like Frank, Droste, Weyl, always with the same perspective. This approach implies the constancy of an explicit metric signature $(+---)$.

On this panorama comes the interpretation given by the mathematician David Hilbert, who starts from a different vision where time, and proper time, are presented as pure imaginary quantities, as opposed to real space coordinates, which goes hand in hand with a choice of metric signature $(-+++)$ and explains the hyperbolic property of this solution.

[4] http://www.internationalskeptics.com/forums/showthread.php?t=231833

When half a century later, astrophysicists are asked to provide a model able to describe a destabilized neutron star, they opt for the solution found by Hilbert, deciding to extend it, via analytic continuation, to an imaginary portion of spacetime, where the element of length becomes purely imaginary and where the signature of the metric is inverted. The goal is not to stop at the surface of the Schwarzschild sphere, and to be able to carry their investigations towards the "interior of the object".

It all depends on what is considered to fall within the scope of physics or not.

In the second part of this paper, we will present a different scenario, in which all quantities remain real and, as a corollary, the metric keeps a signature $(+---)$.

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