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About the « Janus Cosmological Model of J.P.Petit

(translated by J.P.Petit)

Before all let us give our conclusion :

The « Janus Cosmoloogical Model » is physically (and mathematically) unconsistent

The Janus equations are the following :

(1a) $$G^{(+)}_{\mu\nu} = \chi \left[T^{(+)}_{\mu\nu} + \sqrt{\frac{g^{(-)}}{g^{(+)}}}\, T^{(-)}_{\mu\nu} \right]$$

(1b) $$G^{(-)}_{\mu\nu} = -\chi \left[-\sqrt{\frac{g^{(+)}}{g^{(-)}}}\, T^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu} \right]$$

With $G^{(+)}_{\mu\nu} = R^{(+)}_{\mu\nu} - \frac{1}{2} R^{(+)}\, g^{(+)}_{\mu\nu}$ $\qquad G^{(-)}_{\mu\nu} = R^{(-)}_{\mu\nu} - \frac{1}{2} R^{(-)}\, g^{(-)}_{\mu\nu}$

The classical definition of $T^{(+)}_{\mu\nu}$ which ensures its tensorial conservation with respect to $g^{(+)}_{\mu\nu}$ is :

$$\sqrt{-g^{(+)}}\; T^{(+)}_{\mu\nu} \equiv -\frac{2\,\delta S_{\text{matter}(+)}}{\delta g^{(+)}}$$

Where $S_{\text{matter}(+)}$ refers to the action of the ordinary matter. There is no need to give the definition of $T^{(-)}_{\mu\nu}$, which was not precised in the works of Petit and d'Agostini.

The « Janus Model » does not fit the Bianchi identities. In effect the system (1a) + (1b) goes with :

(2a) $$\nabla^{\nu}_{(+)} G^{(+)}_{\mu\nu} = 0$$

(2b) $$\nabla^{\nu}_{(-)} G^{(-)}_{\mu\nu} = 0$$

$\overline{T}_{\mu\nu} = -\frac{w}{\overline{w}} T_{\mu\nu}$ Consider the case $T^{(-)}_{\mu\nu} = 0$ so that the Janus system becomes :

(3a) $$G^{(+)}_{\mu\nu} = \chi\, T^{(+)}_{\mu\nu}$$

(3b) $$G^{(-)}_{\mu\nu} = -\chi\, T^{(+)}_{\mu\nu}$$

Let us write :

$$g^{(+)}_{\mu\nu} = g_{\mu\nu} \qquad g^{(-)}_{\mu\nu} = \overline{g}_{\mu\nu}$$

$$\sqrt{-g^{(+)}} = w \qquad \sqrt{-g^{(-)}} = \overline{w}$$

$$G^{(+)}_{\mu\nu} = G_{\mu\nu} \qquad G^{(-)}_{\mu\nu} = \overline{G}_{\mu\nu}$$

$$T^{(+)}_{\mu\nu} = T_{\mu\nu} \qquad \overline{T}_{\mu\nu} = -\frac{w}{\overline{w}} T_{\mu\nu}$$

The the Janus system becomes :

(4a) $$G_{\mu\nu} = \chi T_{\mu\nu}$$

(4b) $$\overline{G}_{\mu\nu} = \chi \overline{T}_{\mu\nu}$$

with (4c) :

$$\overline{T}_{\mu\nu} = -\frac{w}{\overline{w}} T_{\mu\nu}$$

The authors have introduced the factor $\frac{\overline{w}}{w}$ is order to cure a difficulty to some unconsistency linked to a simplified model but as will be shown further this does not prevent the severe unconsistency in the case of the hydrostatic equilibrium when we consider the cas of a self-gravitating star, in the Newtonian limit $c \to \infty$

The central point is based on the constainsts

(5a) $$\nabla^{\nu} T_{\mu\nu} = 0$$

(5b) $$\overline{\nabla}^{\nu} \overline{T}_{\mu\nu} = 0$$

where $\overline{\nabla}$ is the connection linked to $\overline{g}_{\mu\nu}$.

To illustrate such point let us consider the simple case where the « positive » matter comes both from a background source $T^{o}_{\mu\nu}$ (for example a star, or the sun in our solar system), considered as a sphere filled by a uniform distribution of « dust », i.e $T^{1}_{\mu\nu} = \rho_1 \, u_\mu \, u_\nu$, then :

(6a) $$T_{\mu\nu} = T^{o}_{\mu\nu} + \rho_1 \, u_\mu \, u_\nu$$

(6b) $$\overline{T}_{\mu\nu} = \overline{T}^{o}_{\mu\nu} + \overline{\rho}_1 \, \overline{u}_\mu \, \overline{u}_\nu$$

where

(7) $$\bar{u}_\mu = \frac{u_\mu}{N} \quad \text{with} \quad N^2 \equiv -\bar{g}^{\mu\nu} u_\mu u_\nu$$

(8) $$\bar{\rho}_1 = -N^2 \frac{w}{\bar{w}} \rho_1$$

(9) $$\bar{T}^o_{\mu\nu} = -\frac{w}{\bar{w}} T^o_{\mu\nu}$$

Here the covariant 4-velocity field u_μ is, defined with respect to the metric $g_{\mu\nu}$, so that $g^{\mu\nu} u_\mu u_\nu = -1$. Considered with respect to the second metric $\bar{g}_{\mu\nu}$ the co-vectorial field defines in a unique way the equivalent 4-velocity field $\bar{g}-$unitary $\bar{u}_\mu$ (with $\bar{g}^{\mu\nu} \bar{u}_\mu \bar{u}_\nu = -1$) as defined above.

Now consider the two conservation laws (5a) and (5b).

Let us first concentrate on the movement of the test dust matter. The laws (5a) and (5b) the following constrainst :

(10) $$\nabla_\mu u^\mu = 0$$

(11) $$\nabla_\mu (\rho_1 u^\mu) = 0$$

(12) $$\bar{\nabla}_\mu \bar{u}^\mu = 0$$

(12) $$\bar{\nabla}_\mu (\bar{\rho}_1 \bar{u}^\mu) = 0$$

The physical meaning of the equation (10) is the following. It shows that the lines of the universe of the matter (defined by $u^\mu = g^{\mu\nu} u_\nu$) are geodesics of $g_{\mu\nu} \equiv g^{(+)}_{\mu\nu}$, while the third equation (12) says that the same positive matter is also ruled (by the equations "$-$") to obey another equations of the movement $\bar{\nabla}_\mu \bar{u}^\mu = 0$ which shows that the line of the universe defined by $\bar{u}^\mu = \bar{g}^{\mu\nu} \bar{u}_\nu$ must be geodesics derived from the $\bar{g}_{\mu\nu} \equiv \bar{g}^{(+)}_{\mu\nu}$ metric. But the 4-velocity field $\bar{u}^\mu$ is not independent of u^μ. Considered as a covariant field it is basically the same through a renormalization factor $\bar{u}^\mu = u^\mu / N$, equation, so that $\bar{u}^\mu = \bar{g}^{\mu\nu} u_\nu / N = \bar{g}^{\mu\sigma} g_{\sigma\nu} u^\nu / N$. As the two metrics $g_{\mu\nu} \equiv g^{(+)}_{\mu\nu}$ and $\bar{g}_{\mu\nu} \equiv \bar{g}^{(+)}_{\mu\nu}$ are a priori different I don't see how it could be possible (considering a complex general time dependent solution, defined by arbitrary Cauchy data for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$) to have the same matter following different motion equations. If we consider for example some initial velocity data for a a test dust, such velocity would be supposed to folllow at the same time two distinct rules of evolution, which is mathematically absurd for a classical theory !

Another physico-mathematical contradiction may arise from equations (4a) and (4b) applying such system to the structure of a self-gravitating star, in Newtonian limit. Consider a background source corresponding to a perfect fluid :

(13) $$T_{\mu\nu} = T^{o(+)}_{\mu\nu} = (\rho c^2 + p) u_\mu u_\nu + p g_{\mu\nu}$$

I will limit the analysis to the almost Newtonian conditions. I will show that this theory is self contradictory and does not lead to any physical solution.

I recall that the linearized solution of the Einstein equations may be written :

(14) $$g_{oo} = -(1 - 2\frac{U}{c^2}) \quad ; \quad g_{ij} = +(1 + 2\frac{U}{c^2})\,\delta_{ij}$$

where U is the newtonian potential from Poisson equation :

(15) $$\Delta U = -4\pi G\,\frac{T_{oo}}{c^2}\left(1 + 0(\frac{1}{c^2})\right) = -4\pi G\,\rho\left(1 + 0(\frac{1}{c^2})\right)$$

Due to the formal symmetry of the system (4a) + (4b) we get the corresponding linearized solution :

(16) $$\overline{g}_{oo} = -(1 - 2\frac{\overline{U}}{c^2}) \quad ; \quad \overline{g}_{ij} = +(1 + 2\frac{}{c^2})\,\delta_{ij}$$

where the quasi Newtonian potential $\overline{U}$ obeys :

(17) $$\Delta \overline{U} = -4\pi G\,\frac{\overline{T}_{oo}}{c^2}\left(1 + 0\left(\frac{1}{c^2}\right)\right) = -4\pi G\,\overline{\rho}\left(1 + 0\left(\frac{1}{c^2}\right)\right)$$

from (9) with $w / \overline{w} = 1 + 0(\frac{1}{c^2})$ $\overline{\rho}$ is simply - ρ . So that :

(18) $$\overline{U} = -U\left(1 \quad + 0\left(\frac{1}{c^2}\right)\right)$$

Now I shift to another thing that shows the unconsistency of the « Janus Model ». After equation (4c)

(19) $$\overline{T}_{ij} = -\frac{w}{\overline{w}}\,T_{ij} = -\left(1 + 4\frac{U}{c^2} + 0(\frac{1}{c^4})\right)T_{ij}$$

It is now very important to take in charge the consequences of the equations (5a) and (5b) which act on the same energy-impulsion tensor.

I recall :

(20) $$\nabla_\nu T^\nu_\mu = \frac{1}{w}\partial_\nu (w T^\nu_\mu) - \frac{1}{2}\partial_\mu g_{\alpha\beta} T^{\alpha\beta}$$

If i refers to space :

(21) $$\nabla_\nu T^\nu_i = \frac{1}{w}\partial_\nu (w T^\nu_i) - \frac{1}{2}\partial_i g_{\alpha\beta} T^{\alpha\beta}$$

In the Newtonian approximation, in the last term the contribution from $\alpha = \beta = 0$ is dominant because $T^{oo} = 0(c^2)$ while $T^{oi} = 0(c^1)$ and $T^{ij} = 0(c^o)$. Then

(22) $$0 = \nabla_\nu T^\nu_i = \partial_j (T^j_i) - \frac{T^{oo}}{c^2}\partial_i U + 0\left(\frac{1}{c^2}\right) = \partial_j (T^j_i) - \rho\partial_i U + 0\left(\frac{1}{c^2}\right)$$

I recall that in the Newtonian approximation the order of magnitude of T_{ij} is unity, i.e. is when $c \to \infty$.

For example, for a perfect moving fluid we have $T_{ij} = \rho v^i v^j + p\delta_{ij} + 0(1/c^2)$. Then the above equation (when fullfilled by $\frac{1}{w}\partial_o (w T^o_i) = \partial_t (\rho v^i) + 0(1/c^2)$) is nothing (when $c \to \infty$) but the classical hydrodynamical Euler equation. I have considered a static case, with the equilibrium of a self-gravitating star.

Now, consider the second conservation law (5b). We shall have :

(23) $$\bar{\nabla}_\nu \bar{T}^\nu_i = \frac{1}{\bar{w}}\partial_j (\bar{w}\bar{T}^j_i) - \frac{1}{2}\partial_i \bar{g}_{\alpha\beta} \bar{T}^{\alpha\beta}$$

Thus, finally :

(24) $$0 = \bar{\nabla}_\nu \bar{T}^\nu_i = \partial_j (\bar{T}^j_i) - \bar{\rho}\partial_i \bar{U} + 0(1/c^2)$$

In this second Euler equation : $\bar{T}^j_i \to - T^j_i$ $\qquad \bar{\rho} \to -\rho$ $\qquad \bar{U} \to -U$ then

(25) $$0 = \bar{\nabla}_\nu \bar{T}^\nu_i = -\partial_j (\bar{T}^j_i) - \rho\partial_i U + 0(1/c^2)$$

which contradicts the classical Euler equation (22).

If the star is filled by a perfect fluid this static equilibrium implies both

(26) $$\partial_i p = +\rho\partial_i U \qquad \text{and} \qquad \partial_i p = -\rho\partial_i U$$

CONCLUSION : The system of coupled equations of the « Janus Model » are mathematically and physically contradictory.