When the mathematician David Hilbert tried in 1916 to build the first Theory of Everything

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Assisted by Patrick Marquet
for the translation from German

The present article is based on the two publications made by Hilbert in 1915 and 1916 ([1], [2]). Before entering into the heart of the matter, it is necessary to specify the context in which these works were realized.

We are in 1915. At that time, only two forces at work in the universe were known: the electromagnetic force and the force of gravity. The other two: the strong interaction force, binding the components of the atomic nuclei, and the weak force, responsible for beta radioactivity, would not be discovered until much later.

The upheaval introduced by Albert Einstein, with his special relativity, was finally accepted, at least by some advanced minds, since it is the only one to account for the experiment initiated by the American Abraham Michelson in 1887, which concludes that the value of the speed of light is invariant, regardless of the reference frame, fixed or immobile, in which we operate. No other credible interpretation of the speed of light was found.

However, this idea took time to become one of the pillars of modern physics, so much so that when the Nobel Prize was awarded in 1921 to its author, it was not for this idea but for his interpretation of the photoelectric phenomenon. Einstein is considered the inventor of the word "photon".

What is the discovery of relativity?

It is based on a new vision of the universe, with the appearance of a junction between two words, leading to the compound word space-time. Einstein, thus, can be considered as the inventor of space-time.

Previously, space and time were dissociated objects. Space is considered as Euclidean. That is to say that the theorem of Pythagore in three dimensions, as if we locate the position of two points A and B with the help of an orthonormal frame of reference by giving them coordinates

$$\left\{ x_A, y_A, z_A \right\} \text{ et } \left\{ x_B, y_B, z_B \right\}$$

the distance between them is:

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$$L = \sqrt{\left(x_B - x_A \right)^2 + \left(y_B - y_A \right)^2 + \left(z_B - z_A \right)^2}$$

Time is measured differently, with clocks, with mechanical systems. Before Einstein, nobody would have had the idea to join, to mix two "objects" as different as space and time, to combine meters and seconds.

Behind all this there is a geometrical vision of the cosmos. If we remove a dimension of space, the z coordinate, for example, we have the following diagram:

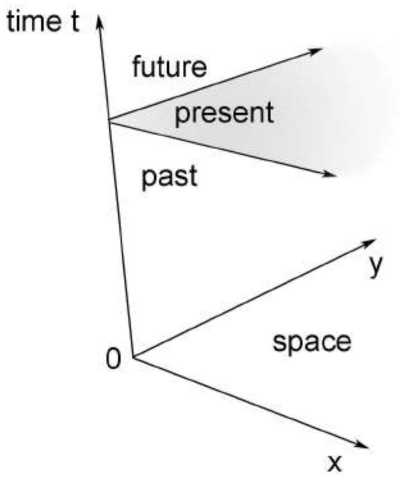


Fig.1 : Pre-relativist space

As for the time t = 0, it obviously refers, in 1915, to the instant of the creation of the universe, "by God". At that time, before the irruption of special relativity in the mode of science and physics, the question "what is then the geometry of space-time? It cannot be identified with a Riemanian mathematical space, defined by a metric. Otherwise, what sense can be given to the following formula, defining a "length » s :

(1) $$ds^2 = dx^2 + dy^2 + dz^2 + a^2 dt^2$$

where a would be a constant, in the form of a velocity, so that we can add up similar quantities. This metric is then devoid of physical meaning.

The space coordinates are obviously real : $\left\{ x, y, z \right\} \in \mathbb{R}^3$

What about time? It would never occur to anyone to imagine a negative time, nor would it occur to anyone to imagine a retrochronic time flow. This variable t therefore belongs to the set $t \in \mathbb{R}_+$.

Albert Einstein's discovery has however a very clear geometrical interpretation, through the space invented by the Russian mathematician Hermann Minkowski.

Fig. 1 : Hermann Minkowski (1864-1909)

It is him who imposes this idea of a space-time continuum, defined by the way in which expresses the length, according to a tool, qualified by the French mathematician Henri Poincaré of pseudo-metric:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

The constraint, imposed in 1905 by Einstein's special relativity, is expressed in a simple and clear way: it is enough to say that s is real. Thus it is necessary that:

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 \geq 0 \quad \text{ou :} \quad v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2} \leq c^2$$

This implies that the speed is less than c . To this metric, we associate its signature, in the form of the sequence of its signs:

$$(+ - - -)$$

When Einstein undertook to describe gravitation using a bilinear form, if we note by $\{ x_1, x_2, x_3, x_4 \}$ the coordinates of one of the points of the tangent space, $\{ x_1, x_2, x_3 \}$ locating it in space and x_4 being the time coordinate, it seems logical to write, indifferently :

$$ds^2 = dx_4{}^2 - dx_1{}^2 - dx_2{}^2 - dx_3{}^2 \text{ ou } ds^2 = - dx_1{}^2 - dx_2{}^2 - dx_3{}^2 + dx_4{}^2 \geq 0$$

This choice appears for example in the article published in 1915 by Einstein, referring to his calculation of the advance of Mercury's perihelion [3] .

Wir gehen nun in solcher Weise vor. Die $g_{\mu\nu}$ seien in »nullter Näherung« durch folgendes, der ursprünglichen Relativitätstheorie entsprechende Schema gegeben

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{matrix}\right\}, \qquad (4)$$

oder kürzere

$$\left.\begin{aligned} g_{\rho\sigma} &= \delta_{\rho\sigma} \\ g_{\rho 4} &= g_{4\rho} = 0 \\ g_{44} &= 1 \end{aligned}\right\} \qquad (4a)$$

Hierbei bedeuten ρ und σ die Indizes 1, 2, 3; $\delta_{\rho\sigma}$ ist gleich 1 oder 0, je nachdem $\rho = \sigma$ oder $\rho \neq \sigma$ ist.

Fig.2 : Einstein's signature choice

It is easy to see that this choice of signature is ubiquitous in the papers of all authors who published papers prior to Hilbert's 1916 paper. Let us quote Schwarzschild, Weyl, Droste, etc.

Some preliminaries before tackling Hilbert's conception of the geometry of space-time.

A 2D geometric object can be described by its metric. The metric of the sphere is for example:

(1) $$ds^2 = R^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

A metric that is totally regular, whatever the values of the two variables. How can we say, on the basis of this simple fact, that it is a 2-sphere? Well, we perform a change of coordinate $\theta = \arcsin(r / R)$ to shift from the set $\{ \theta , \varphi \}$ to the set $\{ r , \varphi \}$. Then we get :

(2) $$ds^2 = \frac{R^2}{R^2 - r^2} dr^2 + r^2 \mathrm{d}\varphi$$

We then notice that for $r = R$ the first term has a zero denominator. We have thus created a *coordinate singularity*. Another remark: for $r < R$ the term ds^2 is negative. The element of length ds becomes pure imaginary. This is normal: we are *outside the sphere.* For these 2D metrics, defining objects, surfaces, we see a thread emerging. The metric is a polynomial of degree two, a bilinear form, expressed with a certain set of coordinates, a priori real. If the length element is also real, it is because our definition interval has been judiciously chosen.

Otherwise, where *ds* is imaginary, we are simply outside the surface. Of course, in a formal way, we can always consider studying the behavior of this object, outside this interval of definition. But then we break our rule of the game. We are no longer in the world of the real, but in the world of the complex.

Let's go back to the question we asked. How do we know that (1) and (2) represent a sphere? To do this, we will plunge it into a three-dimensional space $\{ r , \varphi , z \}$. The physicist recognizes the "cylindrical" coordinates of a 3D Euclidean space. In this imbedding operation the length element must be expressed in the same way, especially on meridian curves with constant . We will therefore write:

(2) $$ds^2 = \frac{R^2}{R^2 - r^2} dr^2 = dr^2 + dz^2$$

It is a differential equation which immediately gives us the link between *r* and *z*. Its integration gives us:

(3) $$r^2 + z^2 = R^2$$

This surface is thus generated by the rotation of a circle centered at the origin of the coordinates, around the oz axis. It is indeed a sphere S2. We could do the same thing starting from two expressions of the metric of the torus T2:

(4) $$ds^2 = r_g^{\,2} d\theta^2 + (R_r + r_g \cos\theta)^2 d\varphi^2 \quad \text{et} \qquad ds^2 = \frac{dr^2}{-r^2 + 2\, r\, R_r + r_g^{\,2} - R_r^{\,2}} + r^2 d\varphi^2$$

In these expressions we recognize the radius r_g of the generating circle of the torus, the radius of this small circle whose center turns around an axis passing by its plane, along a circle of radius R_r. On the left we have opted for coordinates $\{ \theta , \varphi \}$ which are not a problem. On the right, switching to the representation system $\{ r , \varphi \}$ we have, as for the sphere, created coordinate singularities for the two values that cancel the denominator of the first term of the second :

(5) $$r = R_r + r_g \quad \text{et} \quad r = R_r - r_g$$

Moreover the *ds* is real only if this denominator remains positive:

(6) $$R_r - r_g < r < R_r + r_g$$

otherwise we are outside the surface. An imbedding operation in the three-dimensional Euclidean space allows to discover geometrical properties which will make appear the mode of generation of the torus. But nobody is interested in the geometrical properties of this object, for example for $r < (R_r - r_g)$. If we decided to do so, we would leave the mode of the real to enter a strange complex geometry, which then has nothing to do with tangible 2D objects.

These are objects defined by metrics whose signs are all positive. We will call them *elliptic metrics*. This is true for an unlimited number of dimensions, let's consider for example the 3D object:

(7) $$ds^2 = dr^2 + r^2(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,)$$

There, using adequate coordinate changes to make the metric of a Euclidean space reappear:

(8) $$ds^2 = dx^2 + dy^2 + dz^2$$

Thus we would find our familiar *representation space*. In (7) the points are only marked in 3D polar coordinates. But, by studying the object with the help of a family of surfaces with constant *r*, fitting together like Russian dolls, we could "read" the object with the help of this folding method. This space has geodesics which are the infinite number of lines that we can draw in this 3D space. Among these are the straight lines coming from the origin, which are perpendicular to these surfaces S2 which are spheres. The coordinates $\{\, r, \theta, \varphi \,\}$ are *Gaussian coordinates*, a concept to which Hilbert will refer in what follows.

We can leave the 3D Euclidean by imagining 3D spaces, 3D hypersurfaces defined by:

(9) $$ds^2 = f(r)\, dr^2 + r^2(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,)$$

and we would have the same folding system by spheres. We will consider the particular case:

(10) $$ds^2 = \frac{dr^2}{1 - \frac{R_s}{r}} + r^2(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,) \qquad R_s > 0$$

For the moment there is no time variable. In the perspective of an extension of what we said about 2D surfaces to 3D hypersurfaces, defined by their metric, from which we can build their geodesic curves, we will consider the existence of this hypersurface in its *ds* is real, so when $ds^2 > 0$. This gives a defining space such that $r > R_s$. By making *r* constant we can still leaf through the object with a family of spheres nested one inside the other like Russian dolls. But there is then a sphere of minimal area $4\pi R_s^{\,2}$ which corresponds to the metric:

(11) $$ds^2 = R_s^{\,2}(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,)$$

It is a *throat sphere*. But what happens in $r = R_s$? The denominator of the first term of the second member becomes zero. Is this sphere singular? No, it is still a *coordinate singularity*. We can eliminate it by changing the variable:

(12) $$r = R_s\,(\,1 + \mathrm{L_n}\,\mathrm{ch}\rho\,)$$

The metric thus becomes:

(13) $$ds^2 = R_s^2 \frac{1 + \mathrm{L_n}\,\mathrm{ch}\,\rho}{\mathrm{L_n}\,\mathrm{ch}\,\rho}\, \mathrm{th}^2\rho\, d\rho^2 + R_s^2\left(\,1 + \mathrm{L_n}\,\mathrm{ch}\,\rho\,\right)^2 \left(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,\right)$$

There is then no limit in the definition space, and can vary from minus infinity to plus infinity. The *metric potentials* are:

(14)

$$g_{\rho\rho} = R_s^2 \frac{1 + \mathrm{L_n}\,\mathrm{ch}\,\rho}{\mathrm{L_n}\,\mathrm{ch}\,\rho}\,\mathrm{th}^2\rho$$
$$g_{\theta\theta} = R_s^2\left(1 + \mathrm{L_n}\,\mathrm{ch}\,\rho\right)^2$$
$$g_{\varphi\varphi} = R_s^2\left(1 + \mathrm{L_n}\,\mathrm{ch}\,\rho\right)^2 \sin^2\theta$$

How did this operation make this metric regular? When the hyperbolic cosine is unity and its logarithm is then zero. So the denominator in the first term of the second member is always zero. Yes, but the same is true for the hyperbolic tangent. If we do a series development in the neighborhood of $\rho = 0$ you will see that $g_{\rho\rho} \to 2$. This hypersurface is therefore perfectly regular. Its throat sphere for $\rho = 0$ has a minimum area equal to $4\pi R_s^2$. To calculate this "2D volume" (a surface) you must do:

(15) $$\iint \sqrt{g}\, d\theta\, d\varphi$$

But you can convince yourself that the *non-contractibility* of the object is still present. All you have to do on this sphere is to make $\theta = \pi / 2$ and to vary φ from 0 à 2π . You get a finite perimeter $p = 2\pi R_s$. What happened? You are no longer in your comfortable three-dimensional Euclidean *representation space* (the only one you have in fact, to build a mental image).

This hypersurface is therefore a three dimensional manifold, equipped with an elliptic Riemanian metric. In this new system of axes the determinant is never zero. This means that this hypersurface is orientable. At any point one can define a vector product and the "*corkscrew rule*", which goes with it, will be the same at all points. It is obviously painful for the neurons to consider this kind of "*space bridge*" which creates a passage between two Euclidean 3D spaces (which are like "one inside the other"). We can call it a "*3D diabolo*". We will see later how the mathematician Hermann Weyl created and studied this object, in 1917.

By the way, you can take these steps with the "2D diabolo", defined by the metric:

(16) $$ds^2 = \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 d\varphi^2 \qquad R_s > 0$$

With the same change of coordinate you could check its regularity by obtaining:

(17) $$ds^2 = R_s^2 \frac{1 + \mathrm{Log\,ch}\,\rho}{\mathrm{Log\,ch}\,\rho}\,\mathrm{th}^2\rho\, d\rho^2 + R_s^2\left(1 + \mathrm{Log\,ch}\,\rho\right)^2 d\varphi^2$$

But then there is a much more "tangible" way to apprehend this surface. It is enough to imbed it in a three-dimensional space and to build its meridian ($\varphi = \text{cst}$) . You then obtain, as for the sphere, the differential equation:

(18) $$ds^2 = \frac{dr^2}{1 - \frac{R_s}{r}} = dr^2 + dz^2$$

Its solution is the "lying parabola":

(19) $$r = R_s + \frac{z^2}{4R_s}$$

This is the surface:

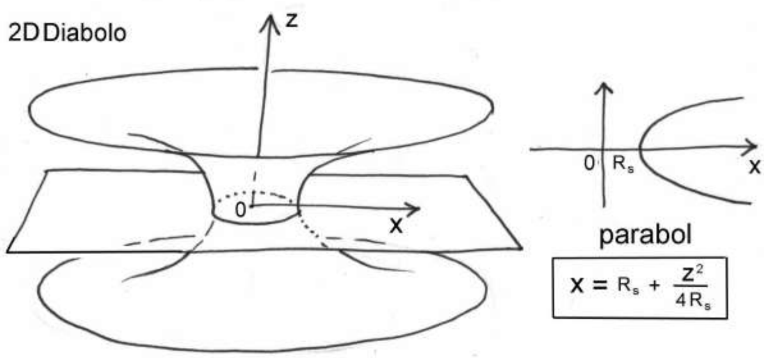
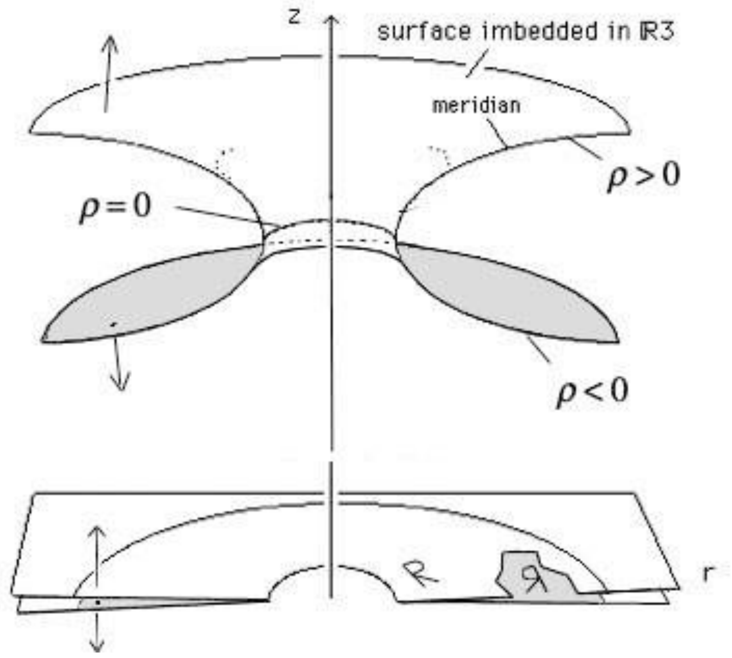


Fig.3 : The 2D diabolo.

We will find this pattern in the analysis made in 1917 by Weyl, which we will detail later.

Another system of representation: projection, and the traps of thought.

We have evoked a mode of representation of a 2-surface by imbedding it into our 3D Euclidean space. We make there a transcendent gesture, by adding an additional dimension. But how a being living in a 2D euclidean space would represent this diabolo? He could only conceive it projected in his own world. He would then imagine a strange border, represented by a circle. The objects which go on this surface, not Euclidean, cross then a circle of throat. Our inhabitant of the Euclidean space 2D can then imagine that his mode "has a place and a reverse ».



Dig.4 : Plane representation of the 2D diabolo

We can illustrate this relation of *enantiomorphy* by starting from an oriented triangle drawn on this plane, "habitat of our 2D observer". The figure below illustrates this inversion of the orientation.

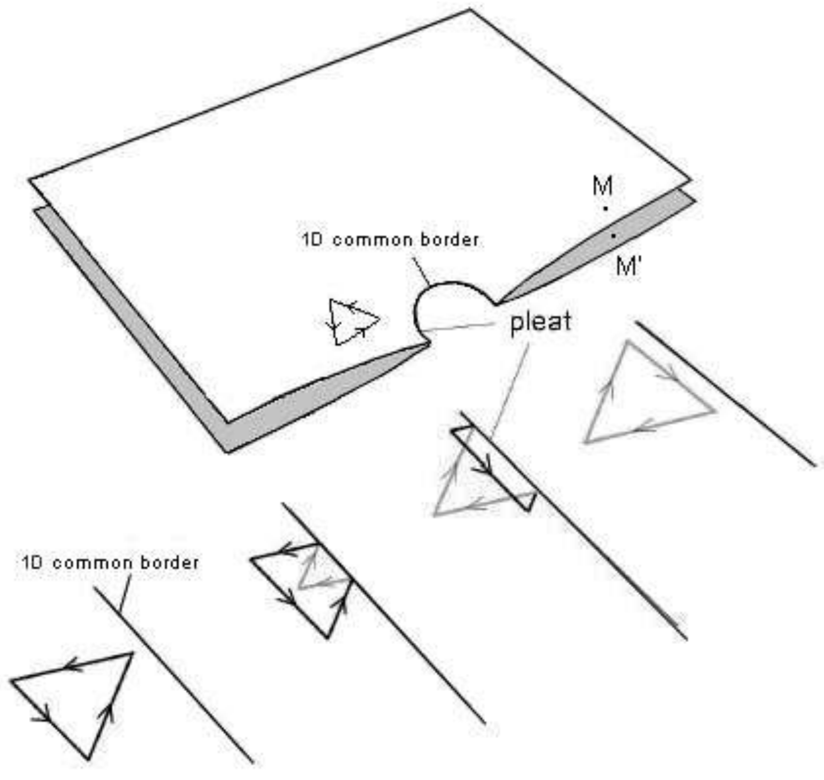


Fig.5 : Reverse orientation.

This seems obvious to us because we have the possibility of imbedding these two structures into our 3D Euclidean representation space. But for the inhabitant of this plane it would be very problematic to consider "that inside this circle there is nothing".

Let's move on to the 3D hypersurface. Here, we can no longer draw, but the idea of projection into a 3D Euclidean representation space is the same. This time, the inhabitant of this space is you, it is me. It will be very difficult to consider "that inside this *throat sphere* there is nothing", and that one cannot contract a sphere by giving it an area lower than a finite value, in a word that this 3D space is not *contractible*.

Through these 2D and 3D examples we see that the fact of using a Euclidean space of representation (the only mental tool we have) to try to read, to "understand" (etymologically "to take together") objects presenting themselves in the form of sets of points leads us to imagine objects that, in fact, do not exist. This is particularly striking for the 3D structure where we are totally unable, mentally, to get rid of this idea of "the inside of the throat sphere".

Hyperbolic surfaces and hypersurfaces.

The word hypersurface always evokes a possible representation in *a higher dimensional representation space*. We have an intuitive image of the geodesics of a surface. It is much more difficult to imagine them in 3D. In general relativity it is often said that the space-time is a hypersurface with four dimensions. Here again, the object is defined by its metric. What Einstein and Minkowski have brought is the introduction in physics of hyperbolic metrics, whose signature makes opposite signs cohabit. We can thus consider a relativistic space-time with two dimensions

(20) $$ds^2 = c^2 dt^2 - dx^2$$

The calculation of geodesics corresponds to the *variational problem*:

(21) $$\delta \int_{AB} ds = 0$$

We look for curves corresponding to paths where the distance traveled is minimal. This leads us to solve the *Lagrange equations*. These lead us to representations of x and t that are linear as a function of the parameter s. Consequently x and s are linked by the linear relation $x = v\, s$, where v is the speed. And if we impose that the length s is real, we must have:

(22) $$v = \frac{dx}{dt} < c$$

What do the theorists do then? What do we find in all the books, the courses? We find images like these, in 2D or 3D:

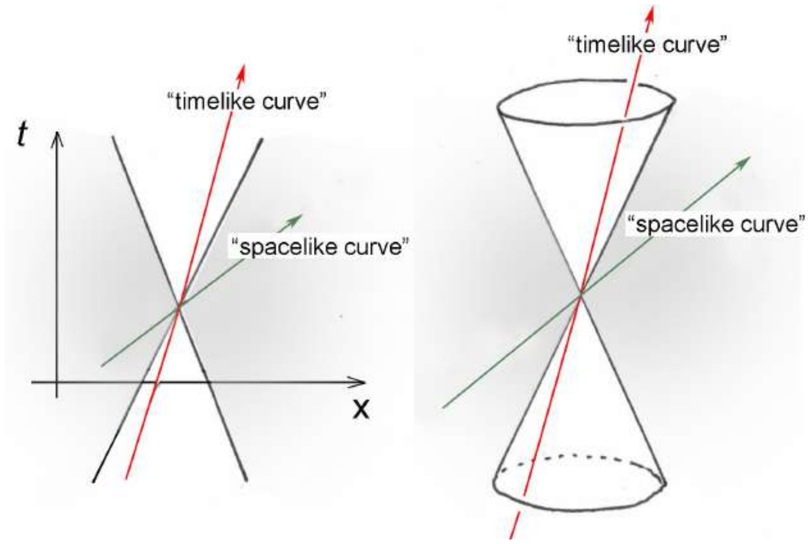


Fig.6 : The light cone.

On the left a way to represent this hyperbolic space (t,x) in two dimensions. On the right the classical "light cone". On the left figure the red curve is supposed to represent a path corresponding to $x = v\,t$ with $v < c$

The black curve represents a path with $v > c$. It is located in the "elsewhere". But in doing so, what are we doing? We are trying to build a 2D image of a hyperbolic space by projecting it into a 2D Euclidean space, of metric:

(23) $$ds^2 = c^2 dt^2 + dx^2$$

We thus create "something that does not exist", in this case this greyed-out surface or volume "outside the light cone". This space does not exist any more than this "interior of the throat sphere" that we create by trying to project the structure of the 3D diabolo into a 3D Euclidean space. This "elsewhere" exists, etymologically speaking, only in our imagination and stems from the image we have created.

The conclusion is simple:

> We simply cannot create a mental or didactic image of a structure with hyperbolic geometry. Any attempt of this kind induces an erroneous vision of things.

This preamble having been made, we will move on to the subject of the article, to the way Hilbert created his own representation of space-time and hence of the universe.

The hectic race, neck and neck, of two geniuses.

In 1915 Hilbert was 53 years old and already had an impressive record of achievement behind him, which had made him known far beyond the German borders. He loved abstraction and logic and was known, among other things, for publishing a treatise in which he defined the axioms underlying Euclidean geometry. All German and foreign mathematicians consider him a "beacon" in the discipline and know that his name will go down in the history of mathematics. He was not always interested in physics. An amusing anecdote is reported about him. When he was asked to replace the mathematician Felix Klein, who every year gave a lecture to the students of an engineering school in Göttingen, he began his lecture with these words:

> - *It is said that mathematicians and engineers have difficulty understanding each other. This is not true: they simply have nothing to do with each other.*

Fig.7 : David Hilbert (1862-1943), in 1915

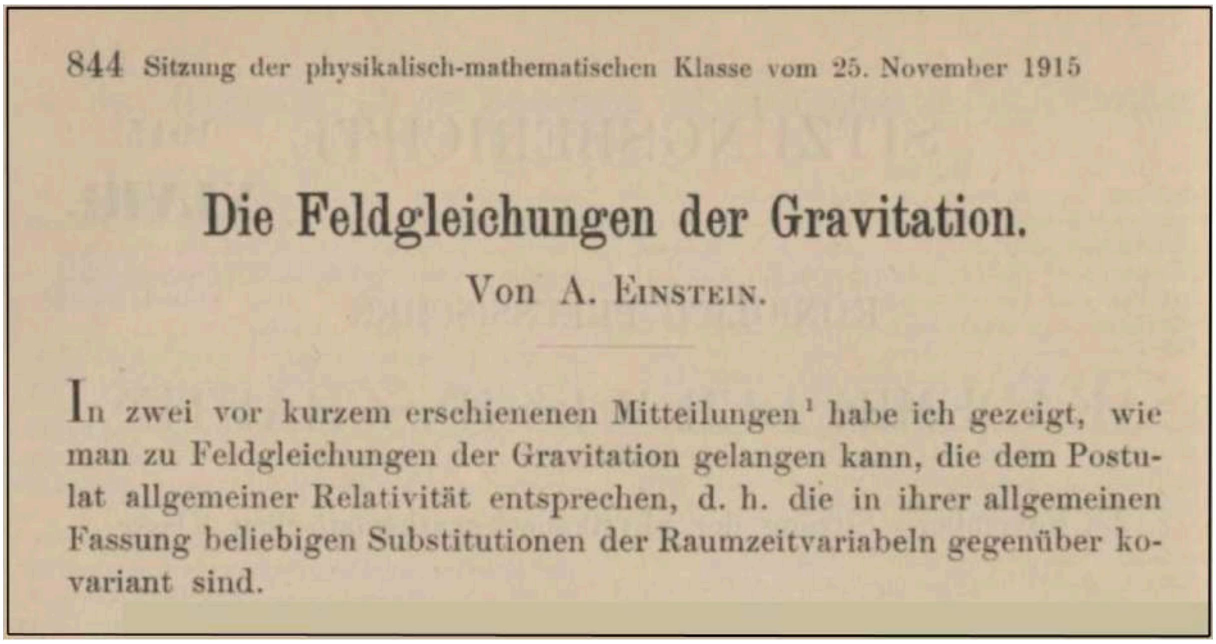
Fig.8 : Félix Klein (1849-1925) in 1915

It should be noted that at that time a similar gap existed in the field of experimental physics, at its fundamental level, where people who were physicists and chemists were working on what would later be called nuclear physics. Among them was the colorful New Zealand physicist Ernst Rutherford. Solicited during the First World War by politicians, who asked him if he could not produce, from his work, some new weapon that would allow England to overtake its adversary, Germany, he had answered them, as he was laying the foundations of the future nuclear physics:

> - *I leave it to your chemists to invent asphyxiating gases and to your engineers to invent planes, submarines and torpedoes. We, scientists, are concerned with totally different things, seeking to penetrate the secrets of matter.*

It was the meeting with the young Einstein, twenty years his junior, that was decisive for Hilbert. He then discovered a fantastic field of applications of sophisticated mathematics to physics, which from then on was no less sophisticated. He established close relations with Einstein, which could even be described as friendly and which were in any case based on a great mutual esteem. In June 1915 Einstein gave him a real lecture on relativity and Hilbert understood that there was a way to use the extremely powerful tool represented by the techniques of *calculus of variations*.

He began by applying this idea to electromagnetism, and then he became interested in gravitation. At the time, the telephone did not exist. It is thus through numerous letters that these two communicate. It so happens that a significant number of these correspondences have come down to us. They are reproduced, in whole or in part, among others by Tilman Sauer [3,], who has kindly reproduced these texts in their English translation. Hilbert communicated to Einstein his vision of things: he thought he was on the verge of unifying the only two forces known at that time, the electromagnetic force and gravitation. Einstein worked differently, by trial and error. Great mathematics is not his forte. Very intuitive, being above all a fantastic physicist, it is by trial and error that he is about to arrive, after ten years of reflection, at what will be considered as the key to a new theory, that of general relativity. But Hilbert beat him to the punch on November 20, 1915 [1]. It was only five days later that Einstein sent to the same journal: the Annals of the Prussian Academy of Sciences [4]:



844 Sitzung der physikalisch-mathematischen Klasse vom 25. November 1915

Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariabeln gegenüber kovariant sind.

Fig.9 : Albert Einstein's November 25, 2015 article entitled:
"The Field Equation of Gravitation." [4]

Hereafter is Einstein's equation where he formulates what both of them were chasing, an equation whose two members are at zero divergence.

$$\sqrt{-g} = 1. \qquad (3a)$$

Dabei ist

$$\Gamma^l_{im} = -\begin{Bmatrix} im \\ l \end{Bmatrix} \qquad (4)$$

gesetzt, welche Größen wir als die »Komponenten« des Gravitationsfeldes bezeichnen.

Ist in dem betrachteten Raume »Materie« vorhanden, so tritt deren Energietensor auf der rechten Seite von (2) bzw. (3) auf. Wir setzen

$$G_{im} = -\varkappa\left(T_{im} - \frac{1}{2} g_{im} T\right), \qquad (2a)$$

wobei

$$\sum_{\varrho\sigma} g^{\varrho\sigma} T_{\varrho\sigma} = \sum_{\sigma} T^{\sigma}_{\sigma} = T \qquad (5)$$

gesetzt ist; T ist der Skalar des Energietensors der »Materie«, die rechte Seite von (2a) ein Tensor. Spezialisieren wir wieder das Koordinatensystem in der gewohnten Weise, so erhalten wir an Stelle von (2a) die äquivalenten Gleichungen

$$R_{im} = \sum_l \frac{\partial \Gamma^l_{im}}{\partial x_l} + \sum_{\varrho l} \Gamma^l_{i\varrho} \Gamma^{\varrho}_{ml} = -\varkappa\left(T_{im} - \frac{1}{2} g_{im} T\right) \qquad (6)$$

Fig.9 : The Einstein field equation in its first form (25 nov. 1915).

Alas, four days earlier Hilbert wrote in his article this equation:

Unter Verwendung der vorhin eingeführten Bezeichnungsweise für die Variationsableitungen bezüglich der $g^{\mu\nu}$ erhalten die Gravitationsgleichungen wegen (20) die Gestalt

(21)
$$[\sqrt{g}\,K]_{\mu\nu} + \frac{\partial \sqrt{g}\,L}{\partial g^{\mu\nu}} = 0.$$

Das erste Glied linker Hand wird

$$[\sqrt{g}\,K]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \tfrac{1}{2} K g_{\mu\nu}\right),$$

Fig.10 : The Hilbert field equation (20 nov. 1015) [1].

Translation :

Using the notations of the variational derivation with respect to that we introduced above, the equation of gravitation takes the form:

$$\left[\sqrt{g}\,K\right]_{\mu\nu} + \frac{\partial\sqrt{g}\,L}{\partial g^{\mu\nu}} = 0$$

With, as first member

$$\left[\sqrt{g}\,K\right]_{\mu\nu} = \sqrt{g}\,(\,K_{\mu\nu} - \frac{1}{2} K\, g_{\mu\nu}\,)$$

For Hilbert $K_{\mu\nu}$ is the Ricci tensor, which Einstein calls $R_{\mu\nu}$. K is the scalar derived from it, the "Ricci scalar", which is designated by the letter R in Einstein. Moreover, the:

$$\frac{\partial\sqrt{g}\,L}{\partial g^{\mu\nu}} = -\sqrt{g}\,T_{\mu\nu}$$

So the Hilbert equation is written:

(24)
$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = T_{\mu\nu}$$

It is indeed in this form, which lacks the term attached to the cosmological constant:

$$\Lambda\, g_{\mu\nu}$$

that Einstein will introduce later, on Hilbert's advice, to succeed in building the first relativistic cosmological model, describing a stationary universe, that this equation will enter history.

The equation in Einstein's paper, which is 5 days later than Hilbert's, is just another equivalent form, where we can recognize, on the right, "the matter tensor T_{im}» and « the Laue scalar» T, which derives from it [2]. But Hilbert adds a construction of the equation by variational method by basing it on an action built on what he calls a "function of the universe":

(25)
$$H = K + L$$

[2] Built from the matter tensor, in the same way that the Ricci scalar R derives from the Ricci tensor R_{im}.

K is obviously the Ricci scalar. This technique will be remembered in history as the "Einstein-Hilbert action".

In the course of their intense correspondence, very warm relations were created between these two geniuses, who assumed themselves as such, perfectly conscious of their own value, and that of their partner.

Fig.11 : Albert Einstein in 1915

Indeed, Einstein and Hilbert were pursuing, at a time described as "hectic", two parallel research programs, where Einstein focused on gravitation alone, while Hilbert dreamed of uniting the two force fields, electromagnetic and gravitational. In [3] you can read about these days of agitation and doubt. But finally Einstein chose to calm down, writing to Hilbert on December 20, 1915 (see this reference, page 48) :

> *- There was a certain resentment between us, the cause of which I do not wish to analyze. I have fought against the bitterness associated with it, and with complete success, I think of you again with unmixed friendship, and I wish you would think the same way.*

In all rigor the conclusion should have been to attribute this equation to both authors. But at that time Einstein's steps were not yet completely assured. On the other hand, Hilbert's brilliant career is a recognized fact. The elder one, very sporty, leaves the premium to the younger one.

However, when we go back to these two equations there is a difference. In Einstein's equation there is a radical $\sqrt{-g}$ where the determinant of the metric is preceded by a minus sign, absent in Hilbert. In what follows we will discover why.

The cosmos in 1915.

Before presenting David Hilbert's own vision, it is important to consider what scientists knew about the universe in 1915.

Spectroscopy was born in Germany, in Heidelberg, in 1859 with the first works of Gustav Kirchhoff and Robert Bunsen, inventor of the gas burner of the same name. It allows to identify the nature of a source on the basis of its spectral signature. At the same time, with the help of a telescope installed on the roof of the Vatican, Father Angelo Secchi pursued the idea that each star is linked to a deceased person. But in the United States, in New England, at the same time the astronomer Edward Pickering undertook a vast classification of the stars, according to their spectrum, which will give birth to the Hertzprung-Russel diagram in 1900. In 1865 the Scottish genius James Clerk Maxwell published the equations that govern electromagnetism.

James Clerk Maxwell 1831-1879
(died at 48 years old)

In 1916, the Englishman Eddington followed very closely both the experimental and observational advances and the progress of the theory. He participated closely in the movement that led to the understanding of the mechanisms of energy production in the form of radiation within stars, which would exploit the understanding of radioactivity, a theory that would only become functional in 1920.

Since 1840 we know the velocities of stars (and beyond their masses), evaluated by the effect discovered by the Austrian Christian Doppler (1803-1853) and the French Armand Fizeau (1819-1896).

Thanks to Newton, celestial mechanics had taken its bearings. Confronted with the problem of the instability of the trajectories of the planets, he believed that, from time to time, it was God who, operating behind the scenes, put them back into their orbits. This idea was invalidated by the Frenchman Pierre Simon de Laplace who solved the problem using mathematics. Questioned by Bonaparte, who asked him what was the place of God in all this, he replied "that he did not need this hypothesis in his calculations". In 1902, the Englishman James Jeans formalized the mechanism of gravitational instability giving rise to stars and planets.

The first elementary particle discovered was the electron, by the Englishman J.J.Thomson in 1897, thus at the immediate dawn of the century. The idea results from the interpretation of the experiments carried out by the Englishman Crookes where a cathode placed in a vacuum tube projects its "cathodic radiation", which is deflected by a magnetic field. In 1995 the Frenchman Jean Perrin identified these "rays" as a jet of electrons. Very quickly the ratio mass-charge and determined and the latter is measured in 1911 by the American Millikan.

Fig.12 : J.J.Thomson 1856-1940

When in 1905 the New Zealander Ernst Rutherford demonstrated the corpuscular nature of matter and the existence of atoms, the idea of atoms made up of positively charged nuclei around which electrons gravitate emerged in 1913. The first model, which proposed to describe the hydrogen atom, had been put forward by the Dane Niels Bohr, who was then 28 years old.

Thus, during the first half of the 19th century, all the tools, both theoretical and observational, were developed. When Hilbert wondered about the functioning of the cosmos, the quantity of discoveries made in the few preceding years was mind-boggling and contrasted with the current stagnation of physics, astronomy and astrophysics, for half a century, where no new particle has been discovered, where the "scientists", the "finders" seem to have been replaced by an army of "researchers", five hundred times more numerous than their elders, "accumulating the data".

It remains that nobody imagines for a single second any evolution of the cosmic scene, perceived as globally homogeneous and stationary. The idea of a creation by God, at an "instant zero", is imposed in everyone's mind, whether explicitly formulated or not.

Let us add that the very beginning of the discovery of the deep nature of the force of gravity, reinterpreted in terms of geodesics of a very weakly curved space, does not change the global nature of the cosmos.

In this year 1915, which was decidedly rich in scientific events of the first magnitude, we saw that Einstein had published, on November 25, 1915, a paper presenting the equation of the gravitational field [4]. But, on the same day, he brings a second important contribution in the form of a first linear solution, which gives a precise evaluation of the advance of Mercury's perihelion [5]. The linearization is amply justified, the phenomenon, minimal, a few tens of seconds per century, can be assimilated to a disturbance. This result sounds like a thunderclap. Not only the approach of the sky phenomena with the help of a field equation represents a major paradigmatic leap, but a solution of this same equation brings the key of an enigma remained until now without solution. A work which is confirmed a few weeks later when the Austrian Karl Schwarzschild [6] , who writes to Einstein, in December 2015, that he has just constructed the linear solution.

This stimulated Hilbert who, throughout 1916, feverishly prepared the publication of an even more ambitious work, an extension of his November 1915 paper, which he published on December 23, 1916, on Christmas Eve. At the same time he announced to Einstein that he was about to synthesize electromagnetism and gravitation, in short to create what can be considered as the first Theory of Everything.

Hilbert's conception of the geometry of space-time.

If we except these tiny curvatures linked to the presence of masses, space-time remains almost flat, almost Euclidean. Einstein has, of course, brought the idea that motion alters the flow of time, but the phenomenon seems to manifest itself in a sensitive way only for "relativistic" speeds, not negligible in front of the speed of light, which do not yet belong to the world of experimental physics and in any case totally negligible in Nature. In the following figure, we can give Hilbert's representation of space-time.

Of course, the general formalism of relativity shows that the points of this space can be located by an infinite number of different coordinate systems, just as there is an infinite number of ways to locate the points on a sphere, to map it. But Hilbert keeps in mind that there must be a particular system, better than the others, that he calls "true". (*Eigentliche*[3]). The underlying idea is that of a 4D space layered by an infinity of 3D hypersurfaces stacked on top of each other.

[3] That which is most suitable, most appropriate, most in tune with reality.

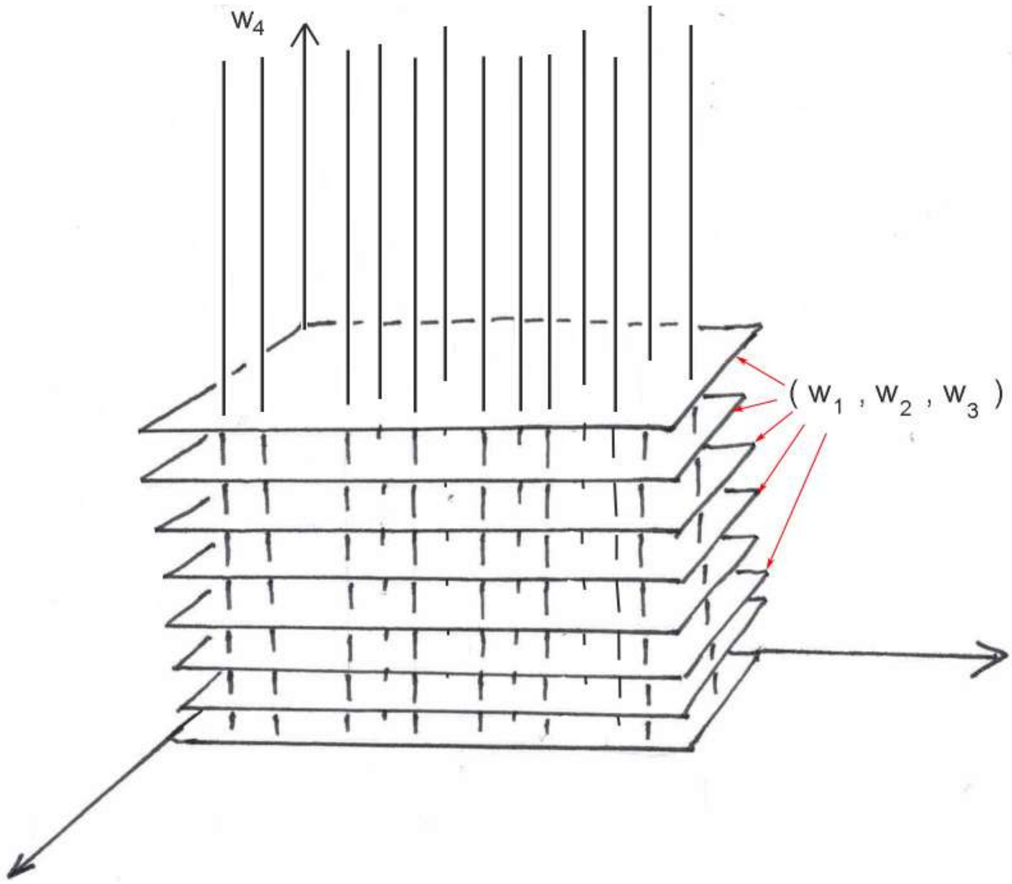


Fig.13 : The Hilbert space-time, in its primitive form.

The mass points obviously follow geodesics of this time space. They go then according to "time lines" (*Zeitlinie*). The drawing shows this family of time lines, perpendicular to the 3D hypersurfaces representing space. Of course, if there were no force, the 3D hypersurfaces of space would be 3D Euclidean spaces, and the trajectories would be parallels, perpendicular to these parallel folds. And nothing would happen. But the gravity forces result in a barely perceptible warping of the 3D hypersurfaces. Correlatively, the time lines deviate slightly from this family of parallel lines. This being the case, we remain very close to a Euclidean structure. Hilbert therefore starts by introducing four coordinates (w_1 , w_2 , w_3 , w_4) which he designates as "universal parameters" (*Weltparameter*). The first three coordinates refer to space, while w_4 refers to time. The instant w_4 = 0 represents, even if Hilbert does not formulate it explicitly, "the instant of the creation of the world by God" (Hilbert comes from a protestant family, deeply religious).

For Hilbert, time is of a different essence than space. The elegant way to account for this is to imagine that the coordinate of time is purely imaginary, which he does by writing, on the first page of his paper :

(26) $$w_1 = x_1 \qquad w_2 = x_2 \qquad w_3 = x_3 \qquad w_4 = i x_4$$

Before talking about length, as a good mathematician, Hilbert considers a bilinear form *G* constructed from the coordinates (X_1, X_2, X_3, X_4) of a vector of this four-dimensional space :

$$G(X_1, X_2, X_3, X_4) = \sum_{\mu\nu} g_{\mu\nu} X_\mu X_\nu$$

He does not specify his choice concerning this form until six pages later:

Wir konstruieren nun in einem jeden Punkte x_1, x_2, x_3 desselben die zu ihm orthogonale geodätische Linie, die eine Zeitlinie sein wird, und tragen auf derselben x_4 als Eigenzeit auf; dem so erhaltenen Punkte der vierdimensionalen Welt weisen wir die Koordinatenwerte $x_1 x_2 x_3 x_4$ zu. Für diese Koordinaten wird, wie leicht zu sehen ist,

(32) $$G(X_s) = \sum_{\mu\nu}^{1,2,3} g_{\mu\nu} X_\mu X_\nu - X_4^2$$

d. h. das Gaußische Koordinatensystem ist analytisch durch die Gleichungen

(33) $$g_{14} = 0, \quad g_{24} = 0, \quad g_{34} = 0, \quad g_{44} = -1$$

charakterisiert. Wegen der vorausgesetzten Beschaffenheit des dreidimensionalen Raumes $x_4 = 0$ fällt die rechter Hand in (32)

Fig.14 : The bilinear form preferred by Hilbert,
expressed in a Gaussian coordinate system.

This expression can then be expressed in its differential form:

(28) $$G(dx_1, dx_2, dx_3, dx_4) = \sum_{\mu\nu} g_{\mu\nu} dx_\mu \, dx_\nu$$

Hilbert then considers a curve whose points are marked with a parameter *p* :

(29) $$x_s = x_s(p) \qquad (s = 1, 2, 3, 4)$$

It specifies well that these coordinates the $x_s(p)$ are real. He then divides his curve into portions and considers the expression :

(30) $$G\left(\frac{dx_1}{dp}, \frac{dx_2}{dp}, \frac{dx_3}{dp}, \frac{dx_4}{dp}\right)$$

He then considers two cases.

Either the bilinear form, in the region where this curve is spanned, is positive:

(31) $$G(\frac{dx_s}{dp}) > 0$$

He then decided to call these portions of the curve segments (*Strecke*). He then introduced a first "*length of this segment*", according to:

(32) $$\lambda = \int \sqrt{G(\frac{dx_s}{dp})}\ dp$$

Then he considers another portion that he decides to call a timeline (*Zeitlinie*) where :

(33) $$G(\frac{dx_s}{dp}) < 0$$

and the integral calculated along this other portion of the curve will be called the proper time of this *timelike* curve:

(34) $$\tau = \int \sqrt{-G(\frac{dx_s}{dp})}\ dp$$

Finally it introduces curves of zero length (*Nullinie*), such as:

(35) $$G(\frac{dx_s}{dp}) = 0 .$$

Thanks to the expression (34) Hilbert manages to establish a link with Einstein's relativity. Indeed its bilinear form, in its differential form, is:

(36) $$G(dx_s) = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$$

It is therefore negative if we want to fit with the requirements of relativity, with a speed of light $c = 1$:

(37) $$v^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{dx_4^2} < 1$$

Thanks to the introduction of the minus sign under the root, Hilbert finds the proper time of special relativity:

(38) $$d\tau^2 = dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2$$

But now you have the origin of the *change of signature*, which has imposed itself as a standard today, in cosmology as well as in theoretical physics, without finding a trace of an article that justifies it.

In the expression (37) we find the signature introduced by Einstein which is $(+---)$. In what Hilbert writes it has become $(+++-)$ or $(-+++)$ and the signs were *reversed*.

In these first pages of his article of 1916 we find something much more serious, which will weigh on all the later development of cosmology, which is this idea of endowing its four-dimensional variety M_4, not with one length, but with two, λ and τ ! He thus speaks of two different measuring instruments. The length will be measured with a "light clock" (*Lichtuhr*). In other regions of his space-time, where his G-form is positive, he will use a tape (*Maßfaden*) to measure his length λ . But he points out that if you try to measure in one region with the instrument used in the other, it does not work. One obtains indeed then imaginary values.

He will not say more, in the rest of the article, about the nature of these mysterious regions of his space-time where the lengths are measured with this scalar. And this while he concentrates with great insistence on what can have, according to him, a physical meaning (*Physicalischer Natur*).

A little further on, Hilbert defines the light cone (*Null-kegel*: the null cone) which is in $a_s = (x_1, x_2, x_3, x_4)$ (its vertex) and whose current point has coordinates (X_1, X_2, X_3, X_4) satisfying the equation:

(39) $$G(X_1 - x_1, X_2 - x_2, X_3 - x_3, X_4 - x_4) = 0$$

And he specifies that all the time lines (timelike curves) resulting from the point a_s are located *inside* this four-dimensional part of the world whose border is the temporal separation of a_s.

All time lines (time-like curves) from the point a_s are located inside this four-dimensional part of the world whose border is the temporal separation of a_s. He then focuses on the problem of *causality* in physics, looking for a "true" coordinate system.

On this subject he states :

> *- A space-time coordinate system is called "true" (Eigentliches Raum-Zeitkoordinatensystem) if it is a system for which the following four inequalities are satisfied, with the additional condition that the determinant is negative.*

(40)

$$g_{11} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} > 0, \qquad g_{44} < 0.$$

He then adopts the definition of a system of a change of space-time coordinates also called "true", "real" (eigentliches). It is simply the system of coordinate changes which, to "true" coordinates, satisfying the inequalities (40), makes another system correspond, endowed with the same properties. The four inequalities mean that at any point event a_s the associated null cone excludes the linear space x_4 = *a4* but contains inside the line through the point $(x_1 = a_1, x_2 = a_2, x_3 = a_3, x_4 = a_4)$.

He then considers a line of universe $x_s = x_s(p)$. From (33) it follows that in a "true" space-time coordinate system we must have :

(41) $$\frac{dx_4}{dp} \neq 0$$

He deduces that along this time line, the "true" time coordinate x_4 must be systematically increasing and cannot decrease. Because a time line remains a timeline under any transformation of the coordinates, two point events located on the same time line can never correspond to the same value x_4 of the time coordinate, through a "true" space-time transformation. This means that these two events cannot be simultaneous.

We thus see that the cause and effect relations underlying the principle of causality (*Kausalitätsprinzips*), do not lead to internal contradictions in this new physics, if we take into account the inequalities (31) as part of our basic equations, which leads us to confine ourselves in "true" space-time coordinates.

Hilbert now introduces the important point of his presentation: the use of coordinates that he decides to call Gaussian, because they represent a generalization of the polar coordinate system used by Gauss in his theory of surfaces. Figure (13) illustrates the concept, which is a foliation, where these surfaces correspond to a constant value of x4. The family of orthogonal curves are geodesics along which this coordinate runs. If we opt for a coordinate system where $g_{44} = -1$ then the x_4 coordinate is identified with the *proper time* . The Gaussian coordinates then satisfy the relation (32) and correspond to:

(42) $$g_{14} = 0, \qquad g_{24} = 0, \qquad g_{34} = 0, \qquad g_{44} = -1$$ [4]

Mais, à ce stade, le choix de Hilbert reste arbitraire. Il avoue échouer à le faire émerger sur la base de considérations purement mathématiques.

[4] In the translation published by Springer an error has crept in. We find $g_{44} = 0$

How Hilbert justifies his choice.

When a Riemann space is put in its diagonalized form (absence of crossed terms) the sequence of signs attached to the different terms represents its *signature*. It is an invariant by change of coordinates (real). Thus the Einstein space-time corresponds to the bilinear form:

(43) $$G(dx_s) = g_{44}\, dx_4^{\ 2} - g_{11}\, dx_1^{\ 2} - g_{22}\, dx_2^{\ 2} - g_{44}\, dx_{33}^{\ 2}$$

and his signature is $(+---)$. In contrast, in his vision of space-time the form retained by Hilbert is :

(45) $$G(dx_s) = -g_{44}\, dx_4^{\ 2} + g_{11}\, dx_1^{\ 2} + g_{22}\, dx_2^{\ 2} + g_{44}\, dx_{33}^{\ 2}$$

and his signature is $(-+++)$. Einstein's choice is based on the identification of the fourth coordinate through $x_4 = ct$. Thus, the simple fact of imposing that the shape is positive translates this physical property which is the limitation of the speed to the speed *c*, that of light (corresponding to a zero value of the line element). The proper time is then calculated by :

(46) $$\tau = \int \sqrt{G(\frac{dx_s}{dp})}\ dp$$

Hilbert's choice is less clear and forces him to situate the real, time-like trajectories, in a representation such that $G < 0$. He must then introduce a change of sign, and opt for the relatio:

(47) $$\tau = \int \sqrt{-G(\frac{dx_s}{dp})}\ dp$$

To this we can add the idea, singular, of endowing space with a second measurement tool, giving a length, whose physical nature is not even touched upon in the article, nor in the following ones, and which will give birth to the myth of "spacelike curves".

It is then necessary to understand why Hilbert made this choice.

The explanation of the choice of an inverted signature by Hilbert.

For Einstein, this fourth coordinate is "of the same nature" as the other three. Let's take our familiar three-dimensional Euclidean space. On the surface of the Earth we will speak for example of the "length x", "width y" and "height z" of an object, knowing that these denominations are arbitrary. We can make an element of the group of rotations act on this object by totally modifying this scheme, while we do not alter the object itself. The distances between its different points remain unchanged. These rotations are part of the isometry group of this 3D Euclidean space. Instead of rotating the object we can decide to observe it from a different angle.

The objects of the space-time are movements, characterized by the energy E and the impulse p which are associated to them[5]. The isometry group of Minkowski space, the Poincaré group, has a subgroup, the Lorentz group, which happens to be the equivalent of the group of rotations and symmetries, in the Euclid group. The latter operates rotations and symmetries that preserve lengths (isometry: same length). The Lorentz group "operates rotations in four dimensions" and preserves a length, that of the impulse-energy quadrivector.

These who see things in this way, through this "group-view" are then tempted to imagine that this x_4 dimension is real and is measured in ... meters. What creates these "hyperbolic rotations" then comes from the axiomatic construction of the Lorentz group:

(48) $\quad L^T\, G\, L = G$ with the Gramm matrix $G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

This aspect is present in Einstein's thought, but not in Hilbert's, who is immediately attached to the reassuring Gaussian coordinates. Certainly, the mass points do not remain immobile. But in a world which remains very far from relativistic physics, we are very close to the Euclidean vision evoked in figure 13. So what is real for Hilbert are the space coordinates :

$$(x_1, x_2, x_3)$$

These are tangible for him. Time is another matter. Nobody can take a second between thumb and forefinger. So, Hilbert concludes, this one must be of another nature, imaginary. Finally, another point, the universe has a beginning, in $x_4 = 0$.

Hilbert is protestant. One can imagine him paraphrasing Genesis:

- *God first created a four-dimensional space whose points were marked by the coordinates*

$$(w_1, w_2, w_3, w_4)$$

Before the play began, the author had to set the stage before the curtain rose. All the objects were put in their place, the stars and their procession of planets, ready to launch themselves into their orbits. The distances that separated them being predefined and real.

[5] The mathematician J.M.Souriau added the spin, as an object of pure geometry.

- Then God decided that the fourth coordinate should be purely imaginary, according to $w_4 = i x_4$ *and that was the first day, the first moment. The Earth then began its movement around the Sun, the same as the other stars of the cosmos. Time appeared, irreversible and implacable.*

We read this interpretation of Hilbert's thought through these lines of his article. Let us quote him:

Due to the nature of the three-dimensional space in $x_4 = 0$ *, as we have posed it a priori (Vorausgesestzen: "posed first, before") the quadratic form of the variables* X_1 *,* X_2 *,* X_2 *described by the right-hand side of (see Figure 14) is necessarily positive and defined.*

Therefore the first three inequalities (40 in this paper) are satisfied, as well as the fourth, and the Gaussian coordinate system appears to be the true (eigentliches: proper) coordinate system of spacetime.

What creates events? Hilbert takes up the idea launched a century earlier by Laplace. He writes:

- If in the present time we have the data concerning the physical quantities and their first derivatives with respect to time, their future values can always be determined: the laws of physics, and this without exception, the laws of physics, have been expressed to date through a system of differential equations in which the number of unknown functions is equal to the number of these equations.

Hilbert lists these data. These are the ten potentials $g_{\mu\nu}$ $(\mu, \nu = 1,2,3)$ which emerge from the symmetrical tensor of format (4,4), representing the matter. To this we must add, at any point, the components of the quadrivector q_s $(s = 1,2,3,4)$ of electromagnetism. This makes a total of fourteen potentials. But, when counting the equations, including those found in 1840 by the genius Maxwell, Hilbert counts only ten, independent of each other.

Under such conditions, as is the case in this new physics of general relativity, he concludes that it is not possible, from the knowledge of physical quantities at the present moment, to determine future values in a unique way. Faced with this impossibility of anchoring a logic of physics on a causality based on concrete elements, Hilbert falls back on the opinion that "*to follow the essence of this new principle of relativity one must require the invariance, separately, of each postulate of physics that has a physical meaning*". And he adds "*In physics, we must consider everything that is not invariant by change of the system of coordinates, as devoid of physical meaning*". And, without any real mathematical argumentation, since he adds "*it is not mathematical problems that are important to discuss here*". And he concludes: ("*Instead I will limit myself to formulate considerations concerning this particular problem*")[6]. He concludes that his choice;

$$g_{11} = 1, \qquad g_{22} = 1, \qquad g_{33} = 1, \qquad g_{44} = -1$$

(49)

$$g_{\mu\nu} = 0 \qquad (\mu \neq \nu)$$

presents itself, for him, as the only alternative representing "the only regular solution of the basic equations of physics", and that it represents, according to him, "*a solution, and even the only regular solution of the basic equations of physics*".

Conclusion on this first part of Hilbert's 1916 article.

After the Second World War, at the turn of the seventies, a change of signature was de facto ratified, through the scientific publications that followed, without an article published in a physics journal justifying the reason. In the same way these chimeras, like "the light cone", "outside of which" is a part of space qualified as "*elsewhere*", populated with "spacelike curves", appeared, whereas this vision comes from the projection of a reality associated with a hyperbolic geometry, in a space of representation endowed with an elliptical geometry, an act of which we have shown that it generated objects exempt from reality.

We had to go back to Hilbert's 1916 paper to trace the source of these drifts. In fact, a "standard" vision of cosmology, populated by presentations that were considered as acquired, not contestable, was built on the basis of later texts, written in English, and therefore more easily assimilated in this language that had become commonplace on a planetary scale after the war. The authors, like medieval copyists, copied one another without any being able to return to the fundamental texts if they did not master German.

To this we must add that if they had simply made the effort to go back to the original version, by simply looking at equation (14) of this article they would have been able to see immediately that their interpretation was in total contradiction with the result of Schwarzschild.

It is significant that Hilbert's articles of 1915-1916 were only translated into English in 2007 ([1], [2]), i.e. ninety years after they were published. Worse still, these translations remain today, like many others, under copyright, as part of a work gathering elements of this kind made available to scientists for a price (October 2021) of 733 dollars for the printed version and 608 dollars for the

[6] In the English translation by Springer, 2007, we read : « is a mathematical problem not to be discussed here. Instead I confine myself to presenting thoughts concerning this problem in particular ». Le texte allemand est : «sind ist mathematisch hier nicht allgemein zu erörternde aufgade. Ich beschränke mich vielmehr darauf, einige besondere diese Aufgabe betreffende überlegungen anzustellen ».

digital version! We had to pay 58 dollars to buy the two pdf's corresponding to the two articles that serve as a basis for this paper, while they are working tools for researchers. If readers want to consult these translations, they will have to pay the same amount. Today, there are no scientists, presenting themselves as experts in cosmology, who have read the fundamental texts, or even know of the existence of a mass of capital texts, some of which have not yet been translated from German.

In what follows we will highlight a glaring and indisputable error of David Hilbert, the immense impact of which will be measured in all the development of cosmology that followed.

This error was first identified by the Canadian L.S. Abrams [7] in 1989, after examining the original text, published in German by Karl Schwarzschild in January 1916 [6], and comparing it with Hilbert's article of December 1916, also in German.

In 1999 a similar approach was taken by A.Loinger [8], always starting from German texts, which he reads fluently.

En 1999 l'italien S.Antoci, et l'allemand D.E Liebscher reprennent cette question [9] et installent sur arXiv les traductions faites par Liebscher, enfin disponibles en langue anglaise, 83 ans après leur publication en allemand, alors que ces textes sont considérés comme la base même de la théorie des trous noirs.

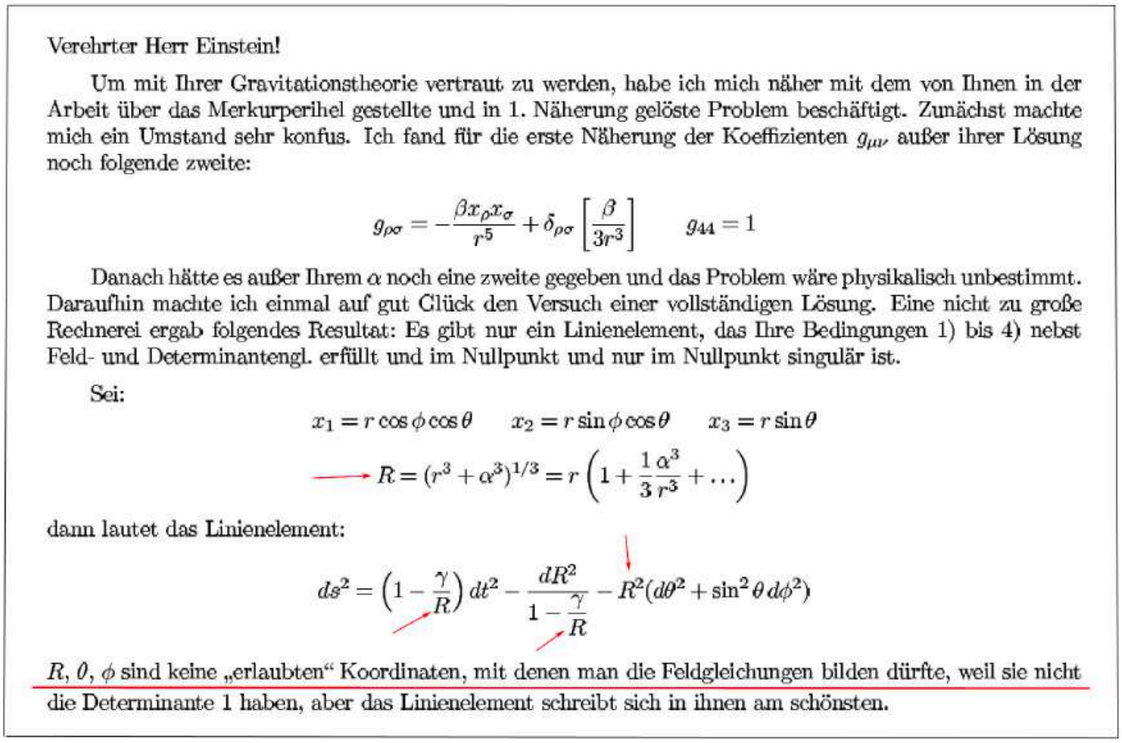
En 2001 ils montrent que l'erreur relève d'une mauvaise compréhension de la topologie de la solution trouvée par Schwarzschild [10] .

En 2003 S.Antoci réitère en publiant un article très documenté : David Hilbert and the origin of the Schwarzschild Solution [11].

Plus récemment, en 2021, le Russe Anatoli Vankov souligne ce point en concluant "Strictly talking the Black hole does not come from the general relativity theory" [12].

Hilbert's error.

Let's go back to the chronology. On November 20, 1916 Hilbert published his first paper entitled "Foundations of Physics" [1]. Five days later Einstein published his own version of the field equation [4] as well as the first solution of the linearized version of the equation, providing the first explanation of the advance of Mercury's perihelion [5]. On December 22, 1915 the mathematician Karl Schwarzschild, who avidly follows everything published in the section of the Prussian Academy's annals devoted to mathematics, writes to Einstein announcing that he has just constructed the non-linear solution of his equation, which confirms his calculation. He announces to him that he will publish an article in the same review. The text of this letter is available at the reference [3].:



Verehrter Herr Einstein!

Um mit Ihrer Gravitationstheorie vertraut zu werden, habe ich mich näher mit dem von Ihnen in der Arbeit über das Merkurperihel gestellte und in 1. Näherung gelöste Problem beschäftigt. Zunächst machte mich ein Umstand sehr konfus. Ich fand für die erste Näherung der Koeffizienten $g_{\mu\nu}$ außer ihrer Lösung noch folgende zweite:

$$g_{\rho\sigma} = -\frac{\beta x_\rho x_\sigma}{r^5} + \delta_{\rho\sigma}\left[\frac{\beta}{3r^3}\right] \qquad g_{44} = 1$$

Danach hätte es außer Ihrem α noch eine zweite gegeben und das Problem wäre physikalisch unbestimmt. Daraufhin machte ich einmal auf gut Glück den Versuch einer vollständigen Lösung. Eine nicht zu große Rechnerei ergab folgendes Resultat: Es gibt nur ein Linienelement, das Ihre Bedingungen 1) bis 4) nebst Feld- und Determinantengl. erfüllt und im Nullpunkt und nur im Nullpunkt singulär ist.

Sei:

$$x_1 = r\cos\phi\cos\theta \qquad x_2 = r\sin\phi\cos\theta \qquad x_3 = r\sin\theta$$

$$R = (r^3+\alpha^3)^{1/3} = r\left(1+\frac{1}{3}\frac{\alpha^3}{r^3}+\dots\right)$$

dann lautet das Linienelement:

$$ds^2 = \left(1-\frac{\gamma}{R}\right)dt^2 - \frac{dR^2}{1-\frac{\gamma}{R}} - R^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

R, θ, ϕ sind keine „erlaubten" Koordinaten, mit denen man die Feldgleichungen bilden dürfte, weil sie nicht die Determinante 1 haben, aber das Linienelement schreibt sich in ihnen am schönsten.

Fig.14 : Schwarzschild's letter to Einstein of December 22, 1916

Translation of the underlined passage:

> *R, θ, φ are not the coordinates "in use [7] », but it turns out that this is the best way to express the metric.*

We see that the genuine radial variable r is clearly visible. Schwarzschild introduces an "intermediate variable" (Hilfsgröβe) R, according to the relation:

$$R = \left(r^3+\alpha^3\right)^{1/3} ,$$

that Hilbert will confuse with a radial variable.

In his defense, we can say that in 1916 the mathematical tool "differential geometry" was not completely mastered. In order to point out the error with precision, we will resume Hilbert's calculation, point by point.

He begins by listing the hypotheses which are at the basis of this solution, which he chooses to describe using what is for him a touchstone, Gaussian coordinates, with:

(50) $$g_{14}=0, \qquad g_{24}=0, \qquad g_{34}=0,$$

The metric potentials are independent of time x_4 . To this he adds that they present a central symmetry (*zentrisch symmeytisch*), with respect to the origin of

---

[7] *Erlauten* : authorized, permitted, standard, in use.

the coordinates. An origin that he assimilated to the value R = 0 and not r = 0. In a more modern way one would speak of an invariance under the action of the group SO(3) and even O(3). Indeed, a geometric object which has this property does not automatically have a "center". A torus has an "axial" symmetry, but this axis only appears when we plunge it into $\mathbb{R}^3$. Formally, this axis does not exist. Similarly, returning to the metric of the "3D diabolo", this object does not have a "central symmetry", because this "center" only appears when we project it into a representation space which is the three-dimensional Euclidean space. It is invariant by action of the group O(3).

Hilbert then writes:

(51)

In agreement with Schwarzschild, if we pose:

$$w_1 = r\cos\vartheta$$
$$w_2 = r\sin\vartheta\cos\varphi$$
$$w_3 = r\sin\vartheta\sin\varphi$$
$$w_4 = l$$

We find a character *l* which translates the vision that Hilbert has of the universe. These are four coordinates which are, let us refer to the beginning of his article, universal parameters (*weltparameter*) (w_1, w_2, w_3, w_4) . What is extraordinary is that Hilbert will conduct his calculation in this coordinate system. The time t will appear only at the very end "when all the cosmic mechanics will start working".

In the Lagrange equations, which will follow, he does not manipulate the derivative $\frac{dt}{dp}$ mais la dérivée $\frac{dp}{dp}$.

This is a way for him to affirm that this metric "exists", that this geometry "pre-exists" before God decides that *l* = *it*, before starting the time race. For Hilbert the solar system exists, as it is, since the creation of the universe by God, since time zero.

Another remark, in this following you will find $\sqrt{g}$ and not $\sqrt{-g}$. So the determinant of the metric is *positive*. We have to remember that for Hilbert the curvature phenomena are exceptional accidents, almost imperceptible folds in a practically Euclidean universe. The universe associated to these coordinates (w_1, w_2, w_3, w_4) is therefore Euclidean. Does it have a length? No, not yet. It is only when this length (these lengths in the plural, if we stick to Hilbert's text, which defines two of them) that the space becomes pseudo-Euclidean and that the determinant becomes negative. Before God gives it a physical character, it is an object of pure mathematics. We could call it "metaphysical"..

-> When Einstein produced his field equation, he created a tool with which to interpret physical phenomena, accessible to astronomy. He is thus already in a four-dimensional world (x_1, x_2, x_3, t) , concret. The determinant of its metric is therefore negative and it must handle a $\sqrt{-\mathrm{g}}$.

-> When Hilbert produces his own version of the field equation he is in another universe, which he wants to be more abstract, more fundamental, the universe of (w_1, w_2, w_3, w_4). You now have an explanation for the presence of the $\sqrt{\mathrm{g}}$ in its field equation, constructed and written in the system (w_1, w_2, w_3, w_4).

It is extremely regrettable that I cannot provide a link to the English translation of Hilbert's 1915 article [1], which is still under this scandalous copyright, even though it is a key element of the world's scientific culture. So it will cost you 29 dollars if you want to check that the coordinates (x_1, x_2, x_3, x_4) are totally absent from this article where all derivatives are in $\dfrac{\partial}{\partial w_s}$. This is true for all terms, the Christoffels coefficients, the terms of the Ricci tensor.

So the Hilbert field equation refers to a space (w_1, w_2, w_3, w_4)

End of this digression. In his 1916 paper Hilbert puts the bilinear form in the form :

(52) $$F(r)\,dr^2 + G(r)(\,d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\,) + H(r)\,dl^2$$

A further step has been taken. Hilbert is then in a reference frame:

(53) $$\{\, w_1 = x_1 ,\, w_2 = x_2 ,\, w_3 = x_3 ,\, w_4 = l_1 \,\}$$

The passage in polar coordinates implies that its variable r is defined, as with Schwarzschild [10][8]; by :

(54) $$r = \sqrt{x_1^{\,2} + x_2^{\,2} + x_3^{\,2}} \;\geq 0$$

Then, thanks to the change of variable, implicit, it is passed in polar coordinates:

(54) $$\vartheta = \arcsin x_3 \qquad \varphi = \arccos x_1$$

We are thus in the coordinates (51). From (52) it is clear that Hilbert expresses his bilinear form in the system:

(55) $$(\, r \,, \vartheta, \varphi, l \,)$$

[8] What you can check in the English translation, not covered by a copyright

Other form of the system (w_1, w_2, w_3, w_4).

There is no question of time.

He then poses :

(56) $$r^* = \sqrt{G(r)}$$

It is then that he will commit a *major error*, reported since 1989 in

([7], [8], [9], [10], [11], [12]) .

He writes:

(42) $$F(r)\,dr^2 + G(r)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + H(r)\,dl^2$$

dargestellt, wo $F(r)$, $G(r)$, $H(r)$ noch willkürliche Funktionen von r sind. Setzen wir

$$r^* = \sqrt{G(r)},$$

so sind wir in gleicher Weise berechtigt r^*, ϑ, φ als räumliche Polarkoordinaten zu deuten. Führen wir in (42) r^* anstatt r ein und lassen dann wieder das Zeichen $*$ weg, so entsteht der Ausdruck

(43) $$M(r)\,dr^2 + r^2\,d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 + W(r)\,dl^2,$$

wo $M(r)$, $W(r)$ die zwei wesentlichen willkürlichen Funktionen von r bedeuten. Die Frage ist, ob und wie diese auf die allgemeinste Weise zu bestimmen sind, damit den Differentialgleichungen

Fig.15 : Hilbert's error

Translation :

- We are therefore, in the same way, entitled to interpret ($r^*, \vartheta, \varphi, l$) as polar spatial coordinates. If we introduce *r* instead of *r** in (our expression of the bilinear form) and omit the * sign again, we obtain the expression:

(57) $$M(r)\,dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + W(r)\,dl^2$$

This bilinear form is a solution of the Einstein field equation without a second member which then reduces to canceling the components of the Ricci tensor, which Hilbert denotes by $K_{\mu\nu}$. These are calculated on the basis of Christoffels symbols. Note that Hilbert, as in his 1915 paper, does all his calculations with the variables (r , ϑ, φ, l) i.e. "universal"- coordinates » (w_1, w_2, w_3, w_4) . His calculation of the geodesics is then based on the variation of the action, constructed with these same variables :

(58) $$\delta\int\left(M\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\vartheta}{dp}\right)^2 + r^2\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2 + W\left(\frac{dl}{dp}\right)^2 \right)dp = 0$$

Hs corresponding Lagrange equations are:

(59)

$$\frac{d^2r}{dp^2} + \frac{1}{2}\frac{M'}{M}\left(\frac{dr}{dp}\right)^2 - \frac{r}{M}\left(\frac{d\vartheta}{dp}\right)^2 - \frac{r}{M}\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2 - \frac{1}{2}\frac{W'}{M}\left(\frac{dl}{dp}\right)^2 = 0,$$
$$\frac{d^2\vartheta}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\vartheta}{dp} - \sin\vartheta\cos\vartheta\left(\frac{d\varphi}{dp}\right)^2 = 0,$$
$$\frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\varphi}{dp} + 2\cot\vartheta\frac{d\vartheta}{dp}\frac{d\varphi}{dp} = 0,$$
$$\frac{d^2l}{dp^2} + \frac{W'}{W}\frac{dr}{dp}\frac{dl}{dp} = 0;$$

The type ' in these equations, and in what follows, refers to a derivation with respect to *r*. The differential equations of the geodesic curves are:

(60) $$\frac{d^2w_s}{dp^2} + \sum_{\mu\nu}\begin{Bmatrix}\mu & \nu \\ & s\end{Bmatrix}\frac{dw_\mu}{dp}\frac{dw_\nu}{dp} = 0$$

We notice that we are always in these "universal" coordinates (w_1, w_2, w_3, w_4). Hilbert then computes non-zero Christofells symbols:

(61)

$$\begin{Bmatrix}11\\1\end{Bmatrix} = \frac{1}{2}\frac{M'}{M}, \quad \begin{Bmatrix}22\\1\end{Bmatrix} = -\frac{r}{M}, \quad \begin{Bmatrix}33\\1\end{Bmatrix} = -\frac{r}{M}\sin^2\vartheta,$$
$$\begin{Bmatrix}44\\1\end{Bmatrix} = -\frac{1}{2}\frac{W'}{M}, \quad \begin{Bmatrix}12\\2\end{Bmatrix} = \frac{1}{r}, \quad \begin{Bmatrix}33\\2\end{Bmatrix} = -\sin\vartheta\cos\vartheta$$
$$\begin{Bmatrix}13\\3\end{Bmatrix} = \frac{1}{r}, \quad \begin{Bmatrix}23\\3\end{Bmatrix} = \cot\vartheta, \quad \begin{Bmatrix}14\\4\end{Bmatrix} = \frac{1}{2}\frac{W'}{W}.$$

This allows him to calculate the components of the Ricci tensor [9]

(62)

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$$
\begin{aligned}
K_{22} = {} & \frac{\partial}{\partial\vartheta}\begin{Bmatrix}23\\3\end{Bmatrix} - \frac{\partial}{\partial r}\begin{Bmatrix}22\\1\end{Bmatrix} \\
& + \begin{Bmatrix}21\\2\end{Bmatrix}\begin{Bmatrix}22\\1\end{Bmatrix} + \begin{Bmatrix}22\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix} + \begin{Bmatrix}23\\3\end{Bmatrix}\begin{Bmatrix}32\\3\end{Bmatrix} \\
& - \begin{Bmatrix}22\\1\end{Bmatrix}\left(\begin{Bmatrix}11\\1\end{Bmatrix} + \begin{Bmatrix}12\\2\end{Bmatrix} + \begin{Bmatrix}13\\3\end{Bmatrix} + \begin{Bmatrix}14\\4\end{Bmatrix}\right) \\
= {} & -1 - \frac{1}{2}\frac{rM'}{M^2} + \frac{1}{M} + \frac{1}{2}\frac{rW'}{MW}
\end{aligned}
$$

$$
\begin{aligned}
K_{33} = {} & -\frac{\partial}{\partial r}\begin{Bmatrix}33\\1\end{Bmatrix} - \frac{\partial}{\partial\vartheta}\begin{Bmatrix}33\\2\end{Bmatrix} \\
& + \begin{Bmatrix}31\\3\end{Bmatrix}\begin{Bmatrix}33\\1\end{Bmatrix} + \begin{Bmatrix}32\\3\end{Bmatrix}\begin{Bmatrix}33\\2\end{Bmatrix} + \begin{Bmatrix}33\\1\end{Bmatrix}\begin{Bmatrix}13\\3\end{Bmatrix} + \begin{Bmatrix}33\\2\end{Bmatrix}\begin{Bmatrix}23\\3\end{Bmatrix} \\
& - \begin{Bmatrix}33\\1\end{Bmatrix}\left(\begin{Bmatrix}11\\1\end{Bmatrix} + \begin{Bmatrix}12\\2\end{Bmatrix} + \begin{Bmatrix}13\\3\end{Bmatrix} + \begin{Bmatrix}14\\4\end{Bmatrix}\right) - \begin{Bmatrix}33\\2\end{Bmatrix}\begin{Bmatrix}23\\3\end{Bmatrix} \\
= {} & \sin^2\vartheta\left(-1 - \frac{1}{2}\frac{rM'}{M^2} + \frac{1}{M} + \frac{1}{2}\frac{rW'}{MW}\right)
\end{aligned}
$$

$$
\begin{aligned}
K_{44} = {} & -\frac{\partial}{\partial r}\begin{Bmatrix}44\\1\end{Bmatrix} + \begin{Bmatrix}41\\4\end{Bmatrix}\begin{Bmatrix}44\\1\end{Bmatrix} + \begin{Bmatrix}44\\1\end{Bmatrix}\begin{Bmatrix}41\\4\end{Bmatrix} \\
& - \begin{Bmatrix}44\\1\end{Bmatrix}\left(\begin{Bmatrix}11\\1\end{Bmatrix} + \begin{Bmatrix}12\\2\end{Bmatrix} + \begin{Bmatrix}13\\3\end{Bmatrix} + \begin{Bmatrix}14\\4\end{Bmatrix}\right) \\
= {} & \frac{1}{2}\frac{W''}{M} - \frac{1}{4}\frac{M'W'}{M^2} - \frac{1}{4}\frac{W'^2}{MW} + \frac{W'}{rM}
\end{aligned}
$$

The calculation of the Ricci scalar follows:

[9] In the English translation, under copyright, a sign error in the equation giving K_{22} where the translator has put a plus sign instead of a minus sign in the second term of the right member.

(63)

$$K = \sum_s g^{ss} K_{ss} = \frac{W''}{MW} - \frac{1}{2}\frac{W'^2}{MW^2} - 2\frac{M'}{rM^2} - \frac{1}{2}\frac{M'W'}{M^2W}$$
$$-\frac{2}{r^2} + \frac{2}{r^2M} + 2\frac{W'}{rMW}.$$

Hilbert places himself in a coordinate system where the determinant is positive, which allows him to write:

(64) $$\sqrt{g} = \sqrt{MW}\; r^2 \sin\vartheta$$

Then :

(65)

$$K\sqrt{g} = \left\{\left(\frac{r^2W'}{\sqrt{MW}}\right)' - 2\frac{rM'\sqrt{W}}{M^{3/2}} - 2\sqrt{MW} + 2\sqrt{\frac{W}{M}}\right\}\sin\vartheta,$$

He poses

(64) $$M = \frac{r}{r - m} \qquad W = w^2 \frac{r - m}{r}$$

-> The letter *w* does not designate the modulus of the vector (w_1, w_2, w_3, w_4)

It is an unknown function. By making the change (64) Hilbert will now have two unknown functions to determine: *m* and *w* . Why such a change? It is inspired by the equation in Schwarzschild's paper and these functions *m* and *w* will turn out to be simple constants.

But, in passing, we discover the origin of lette *m* used to describe what has the dimension of a length!

It comes from:

(65)

$$K\sqrt{g} = \left\{\left(\frac{r^2W'}{\sqrt{MW}}\right)' - 2wm'\right\}\sin\vartheta,$$

Hilbert constructed his action on the basis of a function $H = K + L$, where K is the Ricci scalar. But in a portion of the universe which is empty, $L = 0$. So the variation reduces to:

(66) $$\delta \iint \iint K\sqrt{g}\, dr\, d\vartheta\, d\varphi\, dl = 0$$

Note that we are still in the coordinate system $(r, \vartheta, \varphi, l)$. These geodesics "exist", but "as God has not yet created time", the planets cannot launch themselves on these geodesic-orbits. This equation is equivalent to:

(67)
$$\delta \int w m' dr = 0$$

And the Lagrange equations then give:

(68)
$$m' = 0$$
$$w' = 0$$

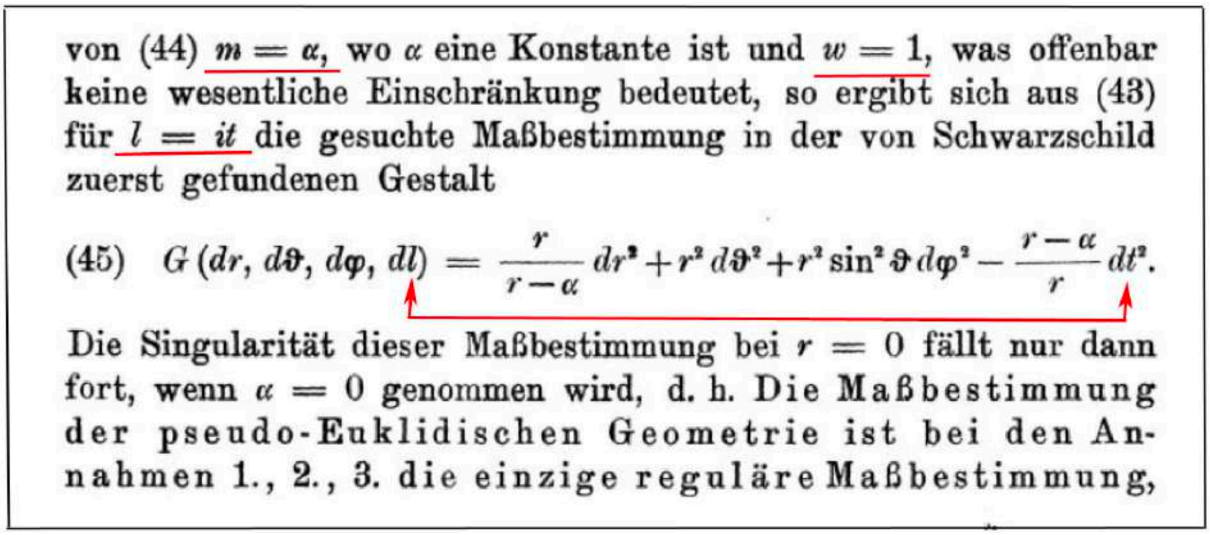
The solution constructed by David Hilbert is then written:

(69)
$$G(dr, d\vartheta, d\varphi, dl) = \frac{r}{r-\alpha} dr^2 + r^2 \sin^2\vartheta \, d\varphi^2 + \frac{r-\alpha}{r} dl^2$$

And, posing $l = i\,t$ (i.e. $w_4 = i\,x_4$, according to Hilbert's notation).

(70)
$$G(dr, d\vartheta, d\varphi, dt) = \frac{r}{r-\alpha} dr^2 + r^2 \sin^2\vartheta \, d\varphi^2 - \frac{r-\alpha}{r} dt^2$$

In the original text we find a typographical error of Hilbert who leaves his "universal variable" l in the first member [10].



von (44) $m = \alpha$, wo α eine Konstante ist und $w = 1$, was offenbar keine wesentliche Einschränkung bedeutet, so ergibt sich aus (43) für $l = it$ die gesuchte Maßbestimmung in der von Schwarzschild zuerst gefundenen Gestalt

(45) $$G(dr, d\vartheta, d\varphi, dl) = \frac{r}{r-\alpha} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2 - \frac{r-\alpha}{r} dt^2.$$

Die Singularität dieser Maßbestimmung bei $r = 0$ fällt nur dann fort, wenn $\alpha = 0$ genommen wird, d. h. Die Maßbestimmung der pseudo-Euklidischen Geometrie ist bei den Annahmen 1., 2., 3. die einzige reguläre Maßbestimmung,

Fig.16 : A typographical error by Hilbert.

At this point, Hilbert is convinced that he has found Schwarzchild's result. He writes:

[10] Error also in the translation published by Springer (still under copyright): the letter l must be replaced by the letter t in both members. Considering the number of obvious errors in this translation, available since 2007, that is to say, at the time of writing, since 14 years, it is doubtful that this translation has been read by people mastering general relativity.

- Für l = it die gesuchte Maßbestimmung in der von Schwarzschild zuerst gefunden Gestalt

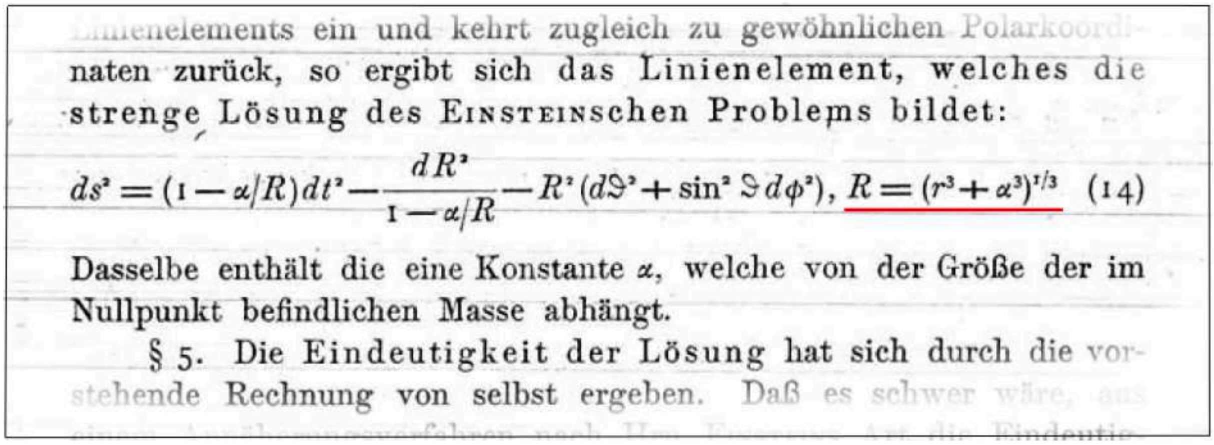
Translation :

- For l = it we find the metric first constructed by Schwarzchild.

It is this error that was very quickly propagated through successive erroneous interpretations.

The true Schwarzschild metric.

Had Schwarzschild survived that spring of 1916, when he died of an infection contracted on the Russian front, he would have immediately brought these commentators of his work back to the original form of it, where the nature of R is well specified and corresponds only to an *intermediate quantity* (Hilfsgröße) and in no case to the radial distance r.


Linienelements ein und kehrt zugleich zu gewöhnlichen Polarkoordinaten zurück, so ergibt sich das Linienelement, welches die strenge Lösung des Einsteinschen Problems bildet:

$$ds^2 = (1 - \alpha/R)dt^2 - \frac{dR^2}{1 - \alpha/R} - R^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2), \quad R = (r^3 + \alpha^3)^{1/3} \quad (14)$$

Dasselbe enthält die eine Konstante α, welche von der Größe der im Nullpunkt befindlichen Masse abhängt.

§ 5. Die Eindeutigkeit der Lösung hat sich durch die vorstehende Rechnung von selbst ergeben. Daß es schwer wäre, aus

Fig.17 : Schwarzschild's true metric, 1916

What is extraordinary is that this confusion between R and r, which is the basis of Hilbert's error, has been a standard for more than a century, even though it was obvious in Schwarzschild's result in equation (14), even to a non-German speaker. The only possible explanation was that this misinterpretation spread, like the misinterpretation of a founding text by people behaving like medieval copyists.

In fact, the generations of theorists that followed were satisfied by the fact that this solution fulfilled for them a consideration that could be considered as fundamental: to identify with the Lorentz metric at infinity. In his article of 2003 [11], the Italian mathematician Salvatore Antoci analyzes with the greatest precision the mechanism of construction, by Hilbert, of this error, putting in perspective his calculation and that of Schwarzschild. In another paper [10] , Antoci and the German D.E. Liebscher show that Hilbert's error translates an erroneous interpretation of the topology of the object.

To this we must add that for 43 years the only application that came to mind concerned the linearized form of this metric. And this is the reason why Schwarzschild himself does not provide the expression of its nonlinear solution in his coordinates. In his paper, Schwarzschild takes the case of the Sun, where the large α is then 3 km , noting:

Überhaupt geht hiernach Hrn. Einsteins Annäherung für die Bahnkurve in die strenge Lösung über, wenn man nur statt r die Größe

$$R = (r^3 + \alpha^3)^{1/3} = r\left(1 + \frac{\alpha^3}{r^3}\right)^{1/3}$$

einführt. Da $\frac{\alpha}{r}$ nahe gleich dem doppelten Quadrat der Planetengeschwindigkeit (Einheit die Lichtgeschwindigkeit) ist, so ist die Klammer selbst für Merkur nur um Größen der Ordnung 10^{-12} von 1 verschieden. Es ist also praktisch R mit r identisch und Hrn. Einsteins Annäherung für die entferntesten Bedürfnisse der Praxis ausreichend.

Fig.18 : Schwarzschild justifies his limitation to the linearized solution.

Translation:

- Nevertheless, Mr. Einstein's approach to the calculation of the geodesics is compatible with the exact solution, if we express it using r instead of:

$$R = \left(r^3 + \alpha^3\right)^{1/3} = r\left(1 + \frac{\alpha^3}{r^3}\right)^{1/3}$$

since α/r is close to twice the square of the planetary speed (taking the speed of light as the unit). For Mercury, the order of magnitude is $10^{-12.}$ *Thus R is practically equal to r and Mr. Einstein's approach is sufficient, beyond the needs of current practice.*

If Schwarzschild had finalized his work, which he did not find necessary, which he would have done by expressing the metric in the coordinates $(r, \vartheta, \varphi, t)$ this would have led him to write the true solution, which is obtained immediately by performing the change of coordinate, explicitly mentioned by Hilbert. An operation which would have made appear this true metric solution of the Einstein equation without second member.

(71)

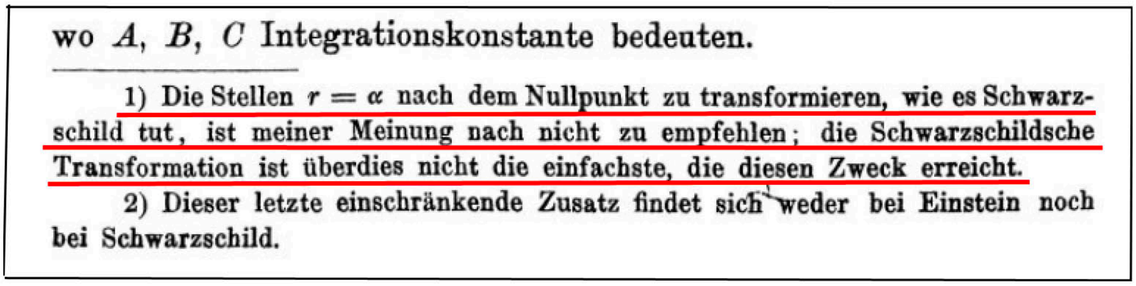
$$ds^2 = \frac{(r^3 + R_s^3)^{1/3} - R_s}{(r^3 + R_s^3)^{1/3}} c^2 dt^2 - \frac{r^4}{(r^3 + R_s^3)\left[(r^3 + R_s^3)^{1/3} - R_s \right]} dr^2 - (r^3 + R_s^3)^{2/3} (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$r \geq 0$$

→ Obviously, if we perform a series development according to the small parameter R_s/r we find the linearized solution of Einstein.

In a footnote on page 70 of his manuscript, Hilbert signs his obvious misunderstanding of the relation (number 14 without Schwarzschild's original paper; which accompanies his expression of its metric solution, according to the intermediate quantity R :

(72) $$R = \left(r^3 + \alpha^3 \right)^{1/3}$$



wo A, B, C Integrationskonstante bedeuten.

1) Die Stellen $r = \alpha$ nach dem Nullpunkt zu transformieren, wie es Schwarzschild tut, ist meiner Meinung nach nicht zu empfehlen; die Schwarzschildsche Transformation ist überdies nicht die einfachste, die diesen Zweck erreicht.

2) Dieser letzte einschränkende Zusatz findet sich weder bei Einstein noch bei Schwarzschild.

Fig.18 : Hilbert's footnote

Translation :

- *In my opinion I will not recommend, as Schwarzschild does, this transformation bringing the point r = α to the origin, especially since there are simpler ways to achieve this.*

Thus, while the expression according to the true radial coordinate r is consistent with the approach followed by Schwarzschild, the fact of presenting the result according to this intermediate quantity R is only an artifice used by Schwarzschild to stick more simply with the linearized solution of Einstein, Hilbert reverses the reasoning by considering R as the radial variable and r as an artifice of calculation to get rid of the singularity in what he believes to be the origin of the coordinates, in $R = 0$, whereas, according to (72) this point corresponds to the pure imaginary value:

(73) $$r = i \left(\left| R^3 - \alpha^3 \right| \right)^{1/3}$$

In doing so, Hilbert does not realize that he has left the domain of definition of the manifoold M_4.

An error that will lead to floods of ink and the production of theorems (Penrose, Hawking) referring to a "central singularity", existing only in the imagination of scientists who have studied this question, thus taking for a reality what belongs, mathematically speaking, to an imaginary space domain.

Coming back to this expression (71), by performing a series development we obtain the values of the metric potentials when r tends to zero:

(74) $$g_{tt} \rightarrow 0 \qquad g_{rr} \simeq \frac{3r}{R_s} \rightarrow 0 \qquad g_{\theta\theta} \rightarrow R_s^{\,2} \qquad g_{\varphi\varphi} \rightarrow R_s^{\,2}\sin^2\theta$$

At r = 0 the Kretschman scalar is nonzero, so the sphere is not a singular locus. On the other hand the determinant is zero, which indicates that the hypersurface is locally inorientable. It is doubly so, since the two potentials and are zero. If the sphere is a gorge sphere, it reflects a double inversion of space and time, a PT-symmetry, as established in . [13]]. If we consider this nullity of the determinant as a singular region" we can then conclude that the hypersurface is not a 4-manifold but a 4-orbifold.

Recall, as Hilbert notes, that these coordinates are Gaussian. We can therefore consider a layering of the hypersurface where the variable with t as parameter. This corresponds for these three-dimensional hypersurfaces to:

(75) $$d\sigma^2 = \frac{r^4}{(r^3 + R_s^{\,3})\left[(r^3 + R_s^{\,3})^{1/3} - R_s\right]}\,dr^2 + (r^3 + R_s^{\,3})^{2/3}(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

These are themselves subject to a new layering at r = Cst which gives a family of nested spheres like Russian dolls having the minimal area, corresponding to r = 0:

(75) $$4\pi R_s^{\,2}$$

The 4D hypersurface is therefore *non-contractile*. It is then :

- Either a *bordered manifold*.

- Either a geometrical object translating a space-time bridge between two Minkowski spaces, realized through a sphere of throat area $4\pi R_s^{\,2}$.

We can produce a finer description of the object by introducing a variable through the change of variable [13]. :

(77) $$r = R_s(\,1 + \mathrm{L_n}\,\mathrm{ch}\rho\,)$$

The metric then becomes:

(78)

$$ds^2 = \frac{\operatorname{Log}\operatorname{ch}\rho}{1+\operatorname{Log}\operatorname{ch}\rho}c^2dt^2 - R_s^2\frac{1+\operatorname{Log}\operatorname{ch}\rho}{\operatorname{Log}\operatorname{ch}\rho}\operatorname{th}^2\rho\, d\rho^2 - R_s^2\left(1+\operatorname{Log}\operatorname{ch}\rho\right)^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

The determinant is always zero because of the term g_{tt} but now $g_{\rho\rho} \to 2$.

This explains why theorists have massively embarked on this erroneous interpretation of Schwarzschild's solution, which is repeated on several points.

-> The fact that the non-linear aspects of the solution were taken into account only in 1939, after the publication of a key article [14] by R. Oppenheimer, without references, which we will mention in the following.

-> The fact, highly probable, that no cosmologist, starting from this "founding article" had the curiosity to take a look at the original article, in German, in which case the equation (14) of this paper should have attracted their attention.

-> The fact that this curiosity came up against the fact that these German texts were only available, until the advent of the internet and pdf files, in books that were generally expensive.

-> The fact that Schwarzschild's article was only translated into English in 1999 [6], i.e. 86 years after its publication in its original form.

-> The fact that the translations of the documents that are essential to clear up this matter were only translated into English in 2007 ([1], [2]), that is 91 years after their publication, and are still covered by a scandalous copyright.

-> In 1960 the publication by M.D.Kruskal [15] of the construction of an analytical extension allowing to "penetrate inside the Schwarzschild sphere" gives the illusion of a progress in the understanding of this geometry, with that it does not change anything to the case. This extension, which refers to an imaginary r-value, is outside the definition space of the geometric object. In the same way one could build an analytical extension allowing to study a torus inside its throat circle.

-> To this we must add, for more than half a century, the illusory conviction of a progress in the understanding of this solution, whereas the generalized contagion of this flagrant error has made thousands of publications fall into what is nothing but science fiction.

The birth act of the black hole model.

In 1939 the hypothesis of the existence of neutron stars was already circulating, although their existence was only confirmed in 1967 by the discovery of pulsars. In their article [14] J.R. Oppenheimer and H.Snyder explicitly mention them. In fact, it all started with an article published a few months earlier by R.C. Tolman, then working at Caltech [16].

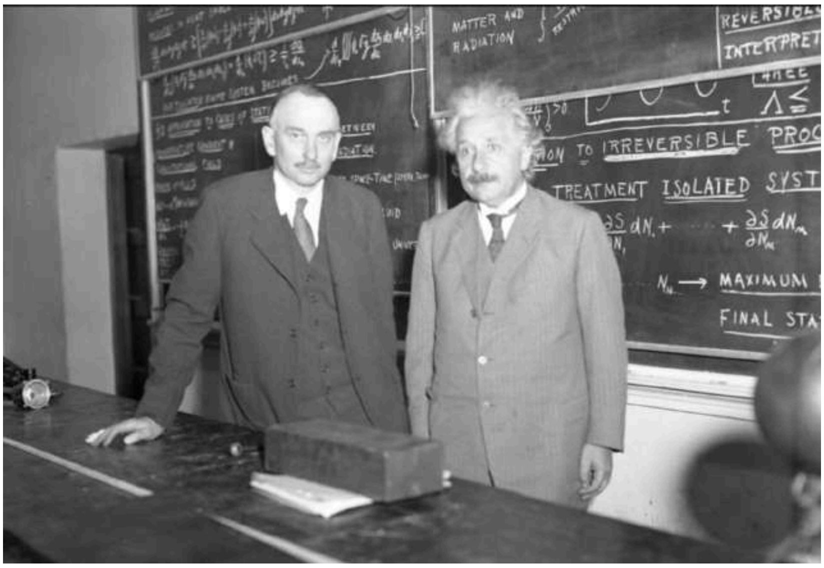


Fig.19 : Richard Tolman and Albert Einstein

We find this expression of the metric:

In the first place, since the condition of gravitational equilibrium for a fluid will on physical grounds be a static and spherically symmetrical distribution of matter, we can begin by choosing space-like coordinates r, θ and ϕ, and a time-like coordinate t such that the solution will be described by the simple form of line element

$$ds^2 = -e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + e^\nu dt^2, \quad (2.2)$$

with λ and ν functions of r alone, as is known to be possible in the case of any static and spherically symmetrical distribution of matter. With the simple expressions for the gravitational potentials appearing in (2.2), the application of the field

Fig.20 : Tolmans' line element.

This description still places us in Gaussian coordinates. We can therefore leaf through the space-time by using t as a parameter, translating a simple temporal translation and by considering the three-dimensional hypersurfaces described by the metric:

(79) $$d\sigma^2 = e^{\lambda(r)}\, dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

Hypersurfaces that can be foded through a family of area spheres $4\pi r^2$. Can this area be brought to zero? If so, this will mean that the geometric object associated with this portion of space-time is contractible.

Moreover, by introducing the two functions $e^{\nu(r)}$ and $e^{\lambda(r)}$ strictly positive, if we are in a mode governed by reals, Tolman rejects a possible modification of the hyperbolic signature which, for him, is (- - - +); or (+ - - -) according to the order of the terms.

produced by rotation, the line element outside the boundary r_b of the stellar matter must take the form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (1)$$

with $e^\nu = (1 - r_0/r)$

and $e^\lambda = (1 - r_0/r)^{-1}$.

Here r_0 is the gravitational radius, connected

Fig.21 : Tolman's expression after Oppenheimer and Snyder.

This work is in fact a resumption of the second article [17] published by Karl Schwarzschild, just before his death, where he entirely constructs the geometry inside a sphere filled with an incompressible matter, of constant density. R.Oppenheimer and H.Snyder quote him in the article they published in 1939 [14] and start again from the same form of the metric, this time to take up the question of geometry outside the mass. The question of the end of life of stars having exhausted their fusion fuel is at the center of the article. The accentuation of the gravitational redshift is evoked, as the contraction of the star increases. The free fall time of a test particle is calculated in two ways. The authors show that if we rely on the proper time, this time is finite, and very short. On the other hand, if we measure this time by using the coordinate t , which is supposed to refer to the proper time of an observer located at a great distance from the object, this time becomes infinite.

This remark signs the birth of the Black Hole model, according to the following reasoning:

-> A star, no longer able to counterbalance the force of gravity with the help of pressure, undergoes a free fall towards its center, which nothing can counteract.

-> Without undertaking a description of this phenomenon, we rely on the data emanating from the metric describing the exterior of this object, which we will call "Schwarzschild's exterior metric".
-> By calculating the free fall times of test-mass, if we find that it reaches the Schwarzschild sphere in a finite time, this time becomes infinite by an observer located at a great distance. For the latter, this implosion phenomenon seems to be like a frozen.

-> At the same time, the radiation emitted by the matter undergoes a gravitational redshift effect which becomes infinite when this signal is emitted from a point located on the Schwarzschild sphere of radius Rs. Thus, a fortiori, no radiation can cross this sphere which will be qualified as cosmological horizon.

An outside, distant observer will then perceive this object as a perfectly black disk, which will be called a black hole.

This reasoning allows us to free ourselves from the description of the collapse phenomenon by starting from the reasoning:

-> I do not feel obliged to describe a phenomenon which for me, an outside observer, lasts an infinite time.

This also allows us to reduce the description of the geometry of the object to the only geometry referring to the outside of the horizon sphere, thus to a solution of Einstein's equation that refers to a portion of the empty universe. Assuming that one can start from the solution of figure 21 to calculate the free fall time of a witness particle up to the point r = 0 , supposed to be the "center" of the object, one obtains a finite and brief value. We deduce, although we cannot carry out any observation on what happens and has happened inside the horizon sphere, that all the matter is concentrated in a *central singularity*.

This reasoning is based on the assumption that the considered expression of the solution has a physical meaning. However, as mentioned above, this is not the result found by Schwarschild in 1916 but the result of the error committed by Hilbert, by confusing the intermediate quantity $R = \left(r^3 + \alpha^3 \right)^{1/3}$ with the radial distance r. Considering to exploit a calculation referring to a value $r < r_o$ (Schwarzschild radius) one is simply outside the four-dimensional hypersurface, which can be seen immediately from the fact that the exponential functions become negative. Now :

(80) $$e^{\lambda} = -\left|n\right| \quad \rightarrow \quad \lambda = \mathrm{Ln}\left|n\right| + i\pi$$

A complex function appears, itself a function of a complex value of the variables. Thus the supposed "interior" of such an object exists only in the imagination of theorists, in the strict sense, since they decide to consider as real what is imaginary.

The emergence of surrealism in physics.

The years have passed. No theorist cares to return to the founding texts, nor to consider another model. Here are the arguments that appear in all the books and manuals intended for the training of students. As an example, we reproduce elements of section 6.8 of chapter 6 of the book of reference [18]. The choice of the form of the metric introduced by Tolman [16] is then simply presented as "reasonable".

When r becomes less than 2I (the Schwarzschild radius), the signs of the components of the metric (potential metric referring to time) g_{11} (metric potential referring to the assumed radial coordinate) change, g_{11} becomes positive ans g_{oo} negative. This forces us to reconsider the physical meaning (...) given to the variables t and r as a system of marking time and radius, inside the Schwarzschild sphere. In fact a line of universe which is described according to a t, that is to say with (r, θ, φ) constants, corresponds to $ds^2 < 0$. It's a *spacelike curve*, while a line of universe for which $ds^2 > 0$ is a timelike curve.

And there you come across the consequence of another of Hilbert's errors, that of endowing space-time with two systems of measurement, the second referring to portions of curves that he calls "segments" and for which the sign of the bilinear form is inverted.

nicht sein Vorzeichen ändert: ein Kurvenstück, für welches

$$G\left(\frac{dx_s}{dp}\right) > 0$$

ausfällt, heiße eine *Strecke* und das längs dieses Kurvenstücks genommene Integral

$$\lambda = \int \sqrt{G\left(\frac{dx_s}{dp}\right)}\, dp$$

heiße die *Länge der Strecke*

Fig.22 : The second "length" measured on Hilbert « segments ».

Translation :

> - A portion of a curve where (the form *G* is positive) will be called a segment, while (the expression giving the scalar λ) is the length of this segment.

This vision of things is in total contradiction with the one of Einstein, K. Schwarzschild, J. Droste, H.Weyl, and all the scientist-mathematicians who at that time contributed to the construction of the general relativity. In the article of Schwarzschild, for example, we read:

Ein Punkt bewege sich gemäß der Forderung

$$\left.\begin{aligned} \delta \int ds = 0, \\ \text{wobei} \qquad ds = \sqrt{\sum g_{\mu\nu} dx_\mu dx_\nu} \quad \mu,\nu = 1,2,3,4 \end{aligned}\right\} \quad (1)$$

ist, $g_{\mu\nu}$ Funktionen der Variabeln x bedeuten und bei der Variation am Anfang und Ende des Integrationswegs die Variablen x festzuhalten sind. Der Punkt bewege sich also, kurz gesagt, auf einer geodätischen Linie in der durch das Linienelement *ds* charakterisierten Mannigfaltigkeit.

Fig.23 : How Schwarzschild defines length, essentially positive.

Translation :

> *- Consider a point that moves according to (the expressions in the figure), where the variables are functions of the x variables (coordinates of points of the space-time hypersurface) and where the values of x must be considered as constant at the beginning and at the end of the path followed for the integration. Clearly, the point will have to move according to a geodesic of the variety (manifold) characterized by the element ds.*

How did Hilbert come to endow space-time with two lengths? Perhaps he had in mind to create a metaphysics, with respect to the events occurring *inside* (...) the Schwarzschild sphere.

Let's go back to the text of the reference [18].

> *- Let us take again the text of the reference It would thus appear natural (...) to treat r as a time coordinate and t as a radial*

coordinate (...). We interpret ds/c as the proper time along the lines of universe traveled by a particle. Then, as we have shown in section 4.2, this definition only has a physical meaning if . Similarly, a massive particle cannot maintain itself on a trajectory with constant r inside the Schwarzschild sphere, which would imply that $ds^2 < 0$ *along this line of universe.*

Let us quote the "standard" derivation of the so-called Schwarzschild solution, e.g., by quoting the corresponding pages of chapter 6 of the 1975 book of reference [18]. Page 186 equation (6.4) represents the most general form of the metric, in the absence of cross terms. The minus sign shows that the authors intend, with A , B , C ; D positive, to introduce the signature of the metric right away:

(6.3) $$ds^2 = Ac^2\,dt^2 - (B\,dr^2 + Cr^2\,d\theta^2 + Dr^2 \sin^2\theta\,d\varphi^2)$$

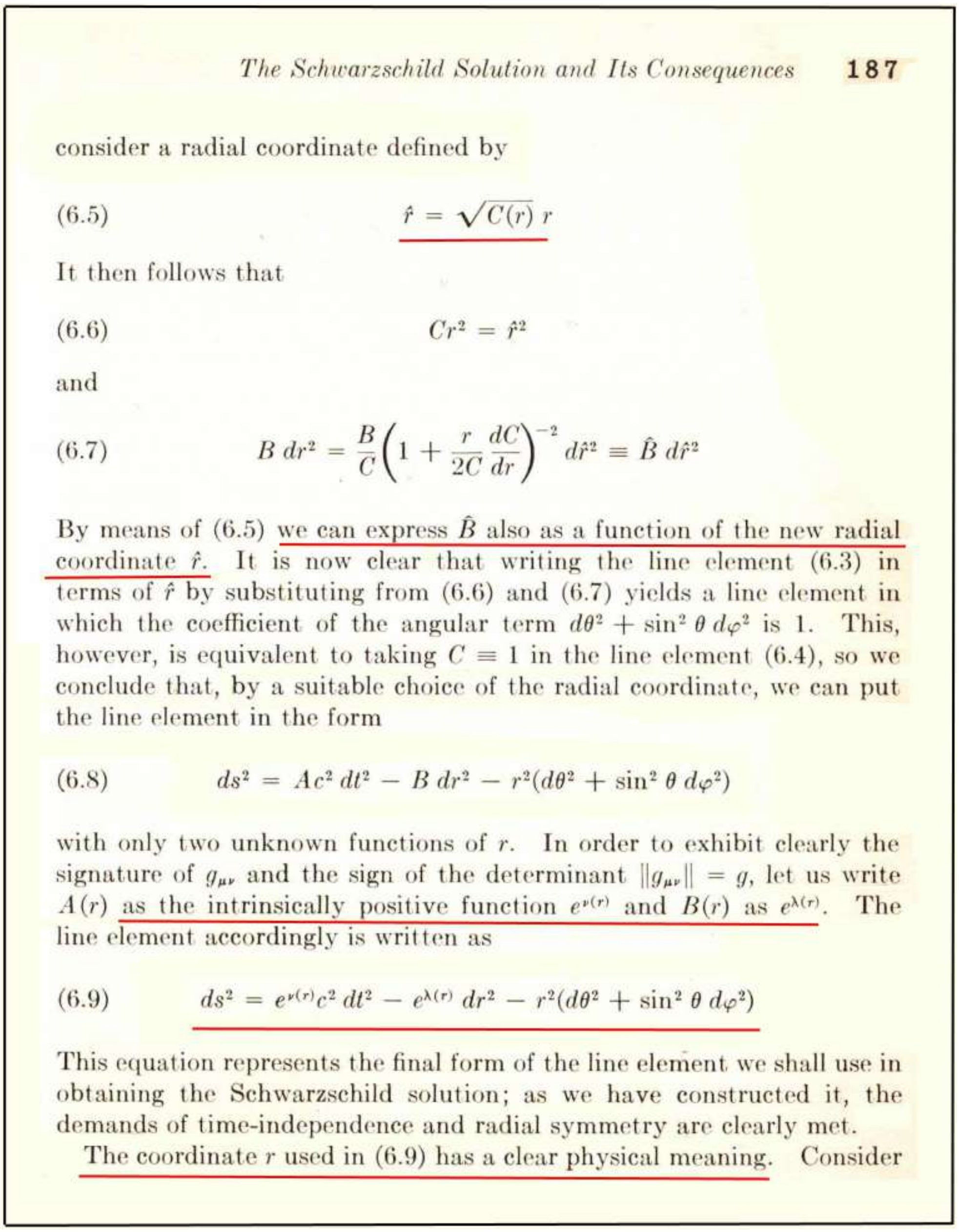
Furthermore, by our assumption of radial symmetry, the functions A, B, C, and D must be functions of r only. One more simplification of the form of the line element can be made on the basis of symmetry: we can suppose that the functions $C(r)$ and $D(r)$ which appear in (6.3) are equal. This can be seen as follows: A displacement by $\epsilon = r\,d\theta$ from the north pole ($\theta = 0$) corresponds to $ds^2 = -C\epsilon^2$, and a displacement by $\epsilon = r\,d\varphi$ along the equator ($\theta = \pi/2$) corresponds to $ds^2 = -D\epsilon^2$. If θ and φ are to represent angular coordinates, we should expect these quantities to be equal due to isotropy, which requires that $C \equiv D$. Then

(6.4) $$ds^2 = Ac^2\,dt^2 - B\,dr^2 - C(r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2)$$

The above line element represents the simplest form which is dictated by the symmetry requirements; however, it is possible to obtain a further simplification by a judicious choice of a radial coordinate. Specifically,

Fig.24 : Excerpt from page 186 of the reference [18]

But in the last lines we read "nevertheless, it is possible to obtain an additional simplification by a judicious choice of radial coordinate"



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consider a radial coordinate defined by

$$\hat{r} = \sqrt{C(r)}\, r \qquad (6.5)$$

It then follows that

$$Cr^2 = \hat{r}^2 \qquad (6.6)$$

and

$$B\, dr^2 = \frac{B}{C}\left(1 + \frac{r}{2C}\frac{dC}{dr}\right)^{-2} d\hat{r}^2 \equiv \hat{B}\, d\hat{r}^2 \qquad (6.7)$$

By means of (6.5) we can express $\hat{B}$ also as a function of the new radial coordinate $\hat{r}$. It is now clear that writing the line element (6.3) in terms of $\hat{r}$ by substituting from (6.6) and (6.7) yields a line element in which the coefficient of the angular term $d\theta^2 + \sin^2\theta\, d\varphi^2$ is 1. This, however, is equivalent to taking $C \equiv 1$ in the line element (6.4), so we conclude that, by a suitable choice of the radial coordinate, we can put the line element in the form

$$ds^2 = Ac^2\, dt^2 - B\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (6.8)$$

with only two unknown functions of r. In order to exhibit clearly the signature of $g_{\mu\nu}$ and the sign of the determinant $\|g_{\mu\nu}\| = g$, let us write $A(r)$ as the intrinsically positive function $e^{\nu(r)}$ and $B(r)$ as $e^{\lambda(r)}$. The line element accordingly is written as

$$ds^2 = e^{\nu(r)}c^2\, dt^2 - e^{\lambda(r)}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (6.9)$$

This equation represents the final form of the line element we shall use in obtaining the Schwarzschild solution; as we have constructed it, the demands of time-independence and radial symmetry are clearly met.

The coordinate r used in (6.9) has a clear physical meaning. Consider

Fig. 25 : Excerpt from page 187 of the referenc [18].

And there we see, reproduced identically, the reasoning held by Hilbert in his 1916 paper, which shows that he is at the origin of this distortion of the true Schwarzschild solution. In equation (6.9) the authors take up the introduction, in 1939, of two exponentials by Tolman [16] and Oppenheimer [14]. They even specify that these functions are "intrinsically positive" and that "these coordinates have a clear physical meaning", whereas, precisely, attributing a physical meaning to coordinates is the first source of error.

Six pages later, on page 193, we find the result of their calculation:

$$(6.47) \qquad e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r}$$
$$e^{\lambda} = \frac{1}{1 - 2m/r}$$

Fig. 25 : Extrait de la page 193 de la référence [18].

If these exponential functions are "intrinsically positive", then the variable r cannot be less than $2m$, otherwise the quantities and would correspond to:

$$(81) \qquad e^{\lambda} = -\left| 1 - \frac{2m}{r} \right| \quad \rightarrow \quad \lambda = L_n \left| 1 - \frac{2m}{r} \right| + i\pi$$

The artifact of the Kruskal coordinates.

In spite of this obvious contradiction D.Kruskal has constructed an analytical extension, so as to be able to build a description of this "interior of the Schwarzschild sphere (0 < r < 2m). The reader will find the "standard" construction of these new coordinates u and v , as well as the resulting metric, in pages 226 to 230 of the reference [18]. We will only reproduce the equations themselves. First, we have the two equations identified by (6.91) in the book:

$$(82) \qquad \xi = r + 2m\, L_n \left| \frac{r}{2m} - 1 \right|$$

$$(83) \qquad F(\xi) = \frac{1 - 2m/r}{f^2}$$

From these relations it is established by introducing the intermediate quantity η , equation (6.200) of reference [18] shows that:

$$(84) \qquad F(\xi) = \eta^2 e^{2\eta\xi}$$

We then read, in the equations (6.201) :

(85)

$$u = \left(\frac{r}{2m} - 1 \right)^{2m\eta} e^{\eta r} \operatorname{ch} \eta x^\circ$$

$$v = \left(\frac{r}{2m} - 1 \right)^{2m\eta} e^{\eta r} \operatorname{sh} \eta x^\circ$$

$$f^2 = \frac{2m}{\eta^2 r} \left(\frac{r}{2m} - 1 \right)^{1-4m\eta} e^{-2\eta r}$$

Then:

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>
> It remains only to choose the arbitrary parameter η; to do this we demand that f^2 have no zero or singularity at $r = 2m$, which requires that $\eta = 1/4m$. (Then ds^2 will vanish *only* on the light cone.) The transformation is thus, finally,

Fig. 26 : From page 229 of the reference [18].

The scalar *m* being real, η is therefore also real.

This leads to the final expression of the new variables, in two configurations.

(86) $r > 2m$:

$$u = \sqrt{\frac{r}{2m} - 1}\; e^{r/4m} \operatorname{ch} \frac{x^\circ}{4m}$$

$$v = \sqrt{\frac{r}{2m} - 1}\; e^{r/4m} \operatorname{sh} \frac{x^\circ}{4m}$$

$$f^2 = \frac{32m^3}{r} e^{r/2m}$$

(87) $$u^2 - v^2 = \left(\frac{r}{2m} - 1 \right) e^{r/2m} \qquad \frac{v}{u} = \operatorname{th} \frac{x^\circ}{4m}$$

And

(88) $r < 2m$:

$$u = \sqrt{1 - \frac{r}{2m}}\ \mathrm{e}^{r/4m}\ \mathrm{sh}\ \frac{x^\circ}{4m}$$

$$v = \sqrt{1 - \frac{r}{2m}}\ \mathrm{e}^{r/4m}\ \mathrm{ch}\ \frac{x^\circ}{4m}$$

$$f^2 = \frac{32 m^3}{r}\ \mathrm{e}^{r/2m}$$

With :

(89) $$v^2 - u^2 = \left(1 - \frac{r}{2m}\right)\mathrm{e}^{r/2m} \qquad \frac{u}{v} = \mathrm{th}\frac{x^\circ}{4m}$$

The metric then takes the form, equation (6.187) of the reference [18]:

(90) $$ds^2 = f^2(u,v)(\,dv^2 - du^2\,) - r^2(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,)$$

Although this metric is not identified with the Lorentz metric at infinity, when we form $ds = 0$ we have , see equation (6.188) of reference [18] :

(91) $$\left(\frac{du}{dv}\right)^2 = 1$$

Here is the famous Kruskal diagram:

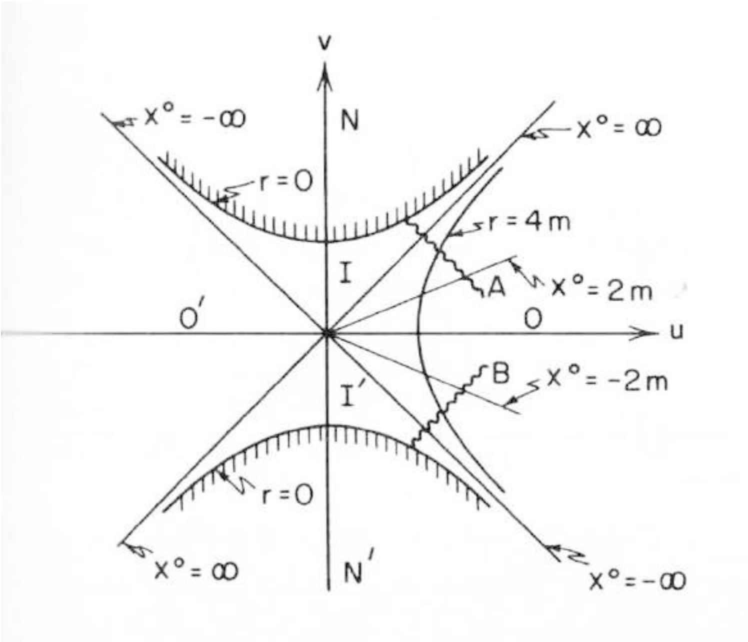

Fig.27 : Kruskal diagram.

If one follows this diagram, where all quantities become real, with a real ds one thus manages to penetrate "inside the Schwarzschild sphere". The points located at constant r are on hyperbolas. The one on the right refers to the value $r = 4m$. The path A is that of a particle with mass, which plunges towards the Schwarzschild sphere. The half line $x° = 2m$ evokes the way time evolves. In the chosen coordinate system this mass reaches this sphere in an infinite time, shown by the half line $v = u$. Then, assuming that this point-mass can penetrate inside this sphere, it continues its way and reaches the point associated with a zero value of r which is another hyperbola accompanied by hatching, which is supposed to represent "the *singularity*". A similar reasoning is associated to the trajectory B.

By a real magic wand, Kruskal seems to have transformed an unreal portion of the variety into something real, described by equations (85) to (89). But we have to remember that if with real quantities, like space-time coordinates, we can obtain a complex quantity ds, the opposite is also possible.

So we have to go back to Kruskal's approach. Let us explain the relation (84)

(92) $$F(\xi) = \eta^2 e^{2\eta\xi} = \frac{1}{16m^2} e^{\xi/2m}$$

Within the Schwarzschild sphere (83) indicates that:

(93) $$F(\xi) = \frac{1 - 2m/r}{f^2} < 0$$

Combined with the previous equation this gives us:

(94) $$e^{\xi/2m} = 16m^2 F(\xi) < 0$$

Now the exponential can only be negative if the exponent is complex:

(95) $$\xi = 2m \, \mathrm{L_n}\left[16m^2 \left|F(\chi)\right|\right] - i\pi$$

Note all the calculation has been based on the hypothesis (82), which leads to:

(96) $$\xi = r + 2m \, \mathrm{L_n}\left| \frac{r}{2m} - 1 \right| = 2m \, \mathrm{L_n}\left[16m^2 \left|F(\chi)\right|\right] - i\pi$$

There is therefore a contradiction. This relation becomes incoherent. In the two members of an equation, one cannot be real and the other complex. This analytical extension makes sense if we are in the mode of complexes, but it does not make sense in the mode of physics, which is in the world of reals.

The interpretation of Hermann Weyl (1917)

In 1915 Hermann Weyl was thirty years old. After having taught mathematics at the University of Göttingen, where he was fascinated by the revolutionary ideas introduced by Riemann and particularly by hyperbolic varieties, he found a position in Zurich, at the Federal Polytechnic, where he was offered a chair. He then met Einstein and quickly assimilated the basic concepts of relativity, first special and then general. Discovering the exact nonlinear solution found by Karl Schwarzschild, he published his interpretation in 1917 [19].

Fig.28 : Hermann Weyl in 1915

To access the German version of this article, published more than a century ago but copyrighted by Springer in a 2012 republication which also includes the English translation, it will cost you $49, regardless of the version. In this paper, unlike Hilbert's, the letter R is used to designate the Riemann scalar:

Für eine Variation des Gravitationsfeldes, die an den Grenzen des Weltgebietes 𝔊 verschwindet, gilt

$$\delta \int H\, d\omega = \int (R_{ik} - \tfrac{1}{2} g_{ik} R)\, \delta g^{ik} \cdot d\omega ;$$

darin ist R_{ik} der symmetrische Riemannsche Krümmungstensor und die Invariante

$$R = g^{ik} R_{ik} .$$

Wenden wir die eben angestellte Überlegung statt auf M auf H an (daß H auch die Differentialquotienten der g^{ik} enthält, ist dabei ganz unwesentlich), so finden wir ohne Rechnung, daß der Tensor

$$R_{ik} - \tfrac{1}{2} g_{ik} R ,$$

an Stelle von T_{ik} gesetzt, die Gleichungen (3) identisch erfüllt. Der Energie-Impulssatz ist demnach nicht nur, wie wir soeben zeigten, eine mathematische Folge der Gesetze des materiellen Vorganges, sondern auch der Gravitationsgleichungen

$$R_{ik} - \tfrac{1}{2} g_{ik} R = - T_{ik} .$$

Fig.29 : Derivation of the field equation by Weyl.

The first equation shows that Weyl immediately integrated the technique of derivation of the field equation by the variational method, with the introduction of the Ricci tensor and the scalar *R* derived from it. At the bottom, we see the field equation to which Einstein gave his name, in the form (equivalent to that of his publication of November 25, 1916 [4]) that students know today. The first thing that Weyl does is to remind us of the inequality which gives the solution a physical character:

des Elementes **e** seiner Weltlinie daselbst, das Verhältnis der dx_i bezeichnet deren Weltrichtung (Geschwindigkeit). Wir müssen voraussetzen, daß diese Richtung eine zeitartige ist, d. h. daß für sie

$$ds^2 = g_{ik}\, dx_i\, dx_k > 0$$

wird. Statt der Differentiale dx_i schreibe ich fortan, da alle unsere Betrachtungen sich auf die eine Stelle P beziehen,

Fig.30 : Length measurement, after H.Weyl [21].

Like Schwarzschild, he is perfectly clear in his choice of Gaussian coordinates (x_1, x_2, x_3, x_4). The space-time is therefore foldable by means of a sequence of three-dimensional hypersurfaces, invariant with respect to time x_4. The problem is to construct this stationary three dimensional hypersurface, described by the coordinates (x_1, x_2, x_3) and defined by its length element $d\sigma$ according to :

(97) $$ds^2 = f\, dx_4{}^2 - d\sigma^2$$

If we compare the choices made by Weyl with Schwarzschild's approach, the latter carries out his calculation with a coordinate r which is in fact his intermediate quantity R. He calculates the function f as well as the expressions of the three other metric potentials.

und bei geeigneter Verfügung über die noch willkürliche Maßeinheit der Zeit: $w = 1$. Variation von w ergibt

$$v' = 0, \quad v = \text{const.} = -2a;$$

$$f = \frac{1}{h} = 1 - \frac{2a}{r}.$$

a hängt mit der Masse m durch die Gleichung $a = \varkappa m$ zusammen; wir nennen a den *Gravitationsradius der Masse* m.

Fig.31 : Result of the calculation of the potential $g_{44} = f$ [21].

Like Schwarzschild, he makes appear what he calls the *gravitational radius* associated to a mass m. His metric can be summarized as:

(98) $$ds^2 = \left(1 - \frac{2a}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2 (\, d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\,) = \left(1 - \frac{2a}{r}\right) dt^2 - d\sigma^2$$

With the 3D hypersurface [11] defined by the metric:

(99) $$d\sigma^2 = \frac{dr^2}{1-\frac{2a}{r}} + r^2\,(\,d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\,)$$

But Weyl is not naive enough to consider that this solution could make physical sense for r < 2a, where the length element would cease to be real. This variable r (which is not the same as the one in the Schwarzschild paper) corresponds to a ray vector with coordinates (x_1, x_2, x_3). It is a 3-dimensional hypersurface which we know "exists" only for $r > 2a$. Weyl will therefore push further, to pierce the secret of its geometry, and more precisely of its topology. On these three space coordinates he can always delete one of them. So he writes:

Zur Veranschaulichung der Geometrie mit dem Linienelement $d\sigma^2$ beschränken wir uns auf die durch das Zentrum gehende Ebene $x_3 = 0$. Führen wir in ihr Polarkoordinaten ein

$$x_1 = r\cos\vartheta\,, \qquad x_2 = r\sin\vartheta\,,$$

so wird

$$d\sigma^2 = h\,dr^2 + r^2\,d\vartheta^2\,.$$

Dieses Linienelement charakterisiert die Geometrie, die auf dem folgenden Rotationsparaboloid im Euklidischen Raum mit den rechtwinkligen Koordinaten x_1, x_2, z gilt:

$$z = \sqrt{8a(r-2a)},$$

wenn dasselbe durch orthogonale Projektion auf die Ebene $z = 0$ mit den Polarkoordinaten r, ϑ bezogen wird. Die Projektion bedeckt das Äußere des Kreises $r \geqq 2a$ doppelt, das Innere überhaupt nicht. Bei natürlicher analytischer Fortsetzung wird also der wirkliche Raum in dem zur Darstellung benutzten Koordinatenraum der x_i das durch $r \geqq 2a$ gekennzeichnete Gebiet doppelt überdecken. Die beiden Überdeckungen sind durch die Kugel $r = 2a$, auf der sich die Masse befindet und die Maßbestimmung singulär wird, geschieden, und man wird jene beiden Hälften als das „Äußere" und das „Innere" des Massenpunktes zu bezeichnen haben.

Fig. 32 : The analysis of the topology of the 3D hypersurface by Weyl

[11] We recognize the metric of our "3D diabolo" from the beginning of the article.

Then he weites:

- *In order to determine the geometry which is characterized by the form of the metric giving* $d\sigma^2$ *, we will project in a corresponding plane à* $x_3 = 0$ *. If we introduce the polar coordinates:*

$$x_1 = r\cos\vartheta \qquad x_2 = r\sin\vartheta$$

$$d\sigma^2 = h\,dr^2 + r^2 d\vartheta^2$$

Ce qui lui donne :

$$d\sigma^2 = \left(1 - \frac{2a}{r}\right)dr^2 + r^2 d\vartheta^2$$

In the following, it does exactly what we did for the 3D diabolo, i.e. it sets the angle ϑ to determine the equation of the meridian, which gives it the differential equation:

(100) $$dr^2 + dz^2 = \left(1 - \frac{2a}{r}\right)dr^2$$

Whose solution is:

(101) $$z = \sqrt{8a\,(r - 2a)} \quad \text{or} \quad r = 2a + \frac{z^2}{8a} = R_s + \frac{z^2}{4R_s}$$

We find the equation(19) of the lying parabola. The hypersurface is thus a "3D diabolo" which is the projection of the 3D hypersurface in a Euclidean space (x_1, x_2, x_3). And Weyl adds :

- *Die projektion bedekt das äubere doppelt* : This projection covers twice (*doppelt*) the portion of space $r > 2a$, *das innere überhaupt nicht* : but this (3D) structure definitively does not fit the portion of space $r < 2a$ [12]. *BeI natürlicher analytischer Fortsetzung wird also der wirkliche Raum in dem zur Darstellung benuntzen Koordinaten der* x_i *das durch* $r \geq 2a$ *gekennziechnete Gebeit dopplet überdecken* : In a natural analytical extension, the real space in the coordinates used for the representation of the coordinate x_i of the point correspond two points of the 3D hypersurface. On this sphere of radius 2a, which makes the junction between these two coverings, is located the mass.

Là on voit poindre une idée tout à fait extraordinaire, assimilant les masses à des singularités topologiques. On se rappellera que la topologie, à cette époque,

[12] On dirait aujourd'hui : cette hypersurface 3D constitue le revêtement à deux feuillets de la portion d'un espace Euclidien 3 extérieur à une sphère de rayon 2a.

est en train de naître en tant que discipline mathématique. Les surfaces fermées régulières 2D sont au nombre de quatre. On a la sphère, le tore, puis la bouteille de Klein et la surface de Boy. Précisos que Félix Klein invente sa bouteille en 1882, tandis que Werner Boy, élève de Hilbert, créera sa propre surface en 1902. Comme Schwarzschild ce dernier s'engage à 35 ans, dès l'entrée en guerre de l'Allemagne, en juillet 1914 et est tué en France, où il repose, en septembre de la même année.

Weyl (who also joined the army, but was discharged for health reasons) continued his analysis of the 3D hypersurface. He is thus the first to introduce the isotropic form of the metric.

Vielleicht wird das noch deutlicher durch Einführung eines andern Koordinatensystems, auf das ich die Schwarzschildschen Formeln ohnehin um der weiteren Entwicklungen willen transformieren muß. Die Transformationsformeln sollen lauten

$$x_1' = \frac{r'}{r} x_1, \quad x_2' = \frac{r'}{r} x_2, \quad x_3' = \frac{r'}{r} x_3; \quad r = \left(r' + \frac{a}{2}\right)^2 \cdot \frac{1}{r'}.$$

Lasse ich nach Durchführung der Transformation die Akzente wieder fort, so ergibt sich

$$(12) \quad d\sigma^2 = \left(1 + \frac{a}{2r}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2), \quad f = \left(\frac{r - a/2}{r + a/2}\right)^2$$

In den neuen Koordinaten ist das Linienelement des Gravitationsraumes also dem Euklidischen *konform*; das lineare Vergrößerungsverhältnis ist

$$\left(1 + \frac{a}{2r}\right)^2.$$

Fig. 33 : Isotropic form of the Schwarzschild metric.

His variable *r* is not the previous one. Its formula (12) corresponds to equation (6.69) of reference [18]. The last expression represents the linear expansion coefficient (*lineare Vergröberungsverhältnis*).

It is clear that Weyl has perfectly integrated the fact that the various coordinates are only representations of the objects defined by their metrics and that the only object endowed with an intrinsic reality (invariant by any change of coordinates) is the element of length s . These successive choices allow us to discover their *topology*. Thus, in 1917 is the first to discover that the geometry discovered by Schwarzschild is non-contractile.

The P-symmetry that goes with the passage of the throat sphere.

Weyl thus creates this concept of a representation space and a two-sheet covering of a manifold. The homologous points of these two four-dimensional sheets can thus be identified using the same coordinates x_i. We can consider four points in the vicinity of a point of coordinates x_i which form a tetrahedron consisting of four equilateral triangles having two common vertices. We can define a positive orientation by defining a direction of travel of these triangles, considered as positive. This one defines a normal vector. The following figure shows what happens to this set of points when they cross the throat sphere. The tetrahedron with black edges is supposed to belong to one of the two tridimensional layers of the object. Let one of its faces ABC, the positive direction of travel, arbitrary, being indicated by arrows. The tetrahedron with the shaded edges A'B'C' belongs to the other layer. If we bring these two objects into coincidence we can see that the two directions are the opposite of each other

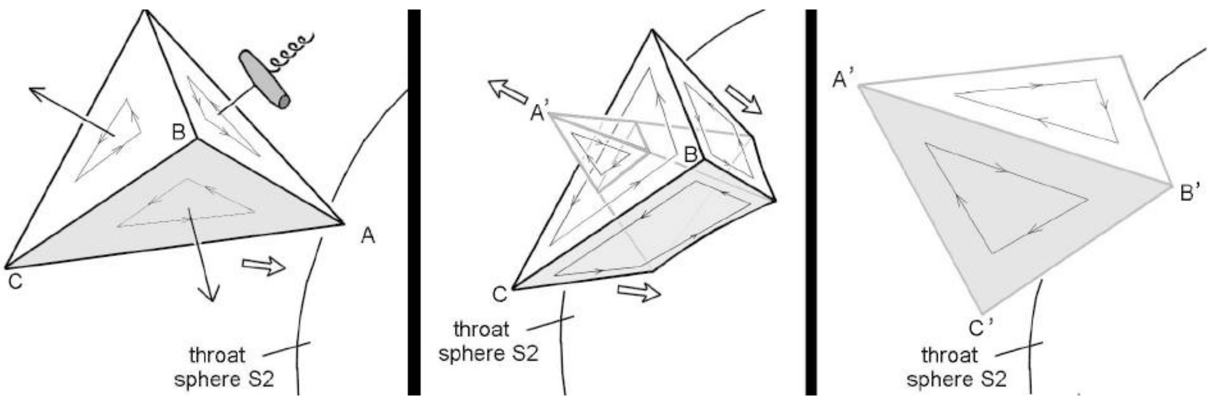

Fig.34 : Reversal of the orientation of a tetrahedron after crossing the throat sphere.

We deduce that any object crossing the throat sphere of the Schwarzschild geometry undergoes a P-symmetry.

Free fall time or escape time in Schwarzschild geometry.

The black hole model is based on the complete decoupling of the proper time of objects accompanying the implosion phenomenon that this geometry is supposed to describe and the proper time of an observer located at infinity, observing the phenomenon, which for him is supposed to last an infinite time. Let us consider radial trajectories. We have (keeping the Schwarzschild notation) :

(102) $$ds^2 = \left(1 - \frac{R_s}{R}\right) dx_4{}^2 - \frac{dR^2}{1 - \frac{R_s}{R}} \qquad s = c\tau \qquad x_4 = ct$$

The Lagrange equations give:

(103) $$\frac{d\tau}{dR} = \pm \frac{1}{c}\sqrt{\frac{R}{R_s}}$$

(104) $$\frac{dt}{dR} = \pm \frac{1}{c}\sqrt{\frac{R}{R - R_s}}$$

The integration gives the diagram:

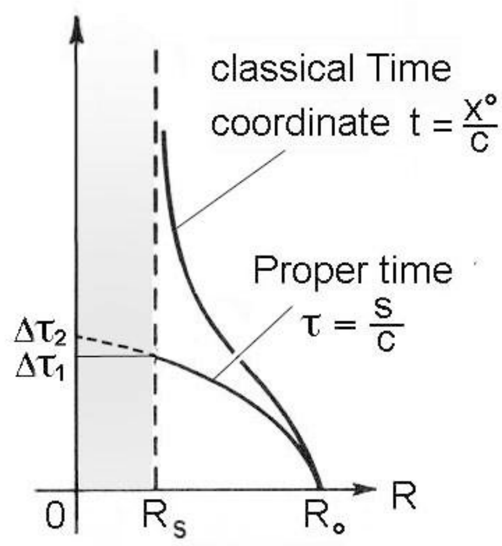

Fig. 35 : The free fall time.

The witness particle reaches the throat sphere in a finite time, in terms of its own time, but this path corresponds to an infinite time for a distant observer.

The Kerr metric.

In 1963 Roy Kerr constructed the solution of the Einstein equation without a second member [20], describing a portion of empty space, invariant by time translation and by action of the group O(2).

(105)

$$ds^2 = \left(1 - \frac{2\,m\,\rho}{\rho^2 + a^2 \cos^2\theta} \right) c^2 dt^2 - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2\,m\,\rho} d\rho^2 - \left(\rho^2 + a^2 \cos^2\theta \right) d\theta^2$$

$$- \left[\left(\rho^2 + a \right) \sin^2\theta + \frac{2m\,\rho\, a^2 \sin^4\theta}{\rho^2 + a^2\cos^2\theta} \right] d\varphi^2 - \frac{4m\,\rho\, a \sin^2\theta}{\rho^2 + a^2 \cos^2\theta}\, c\, dt\, d\varphi$$

The parameter *a* figures the importance of the rotation. If it is zero we find the Schwarzschild metric. In the plane this metric becomes:

(106)

$$ds^2 = \left(1 - \frac{2\,m}{\rho} \right) c^2 dt^2 - \frac{\rho^2}{\rho^2 + a^2 - 2\,m\,\rho} d\rho^2$$

$$- \left[\left(\rho^2 + a \right) + \frac{2m\, a^2}{\rho} \right] d\varphi^2 - \frac{4m\, a}{\rho}\, c\, dt\, d\varphi$$

Consider two rays of light emitted tangentially to a trajectory at ρ constant :

(107) $$0 = \left(1 - \frac{2\,m}{\rho} \right) c^2 dt^2 - \left[\left(\rho^2 + a \right) + \frac{2m\, a^2}{\rho} \right] d\varphi^2 - \frac{4m\, a}{\rho}\, c\, dt\, d\varphi$$

The speed of light then takes two different values, depending on the direction of emission:

(108) $$v_\varphi = c\left[2\mathrm{ma} \pm \sqrt{18\mathrm{m}^2\mathrm{a}^2 + 2\mathrm{m}\rho^3 - \rho^2 a^2 - \rho^4} \right]$$

This phenomenon is classically interpreted as a phenomenon of dragging of the coordinate system (*frame-dragging*) and is not without evoking the idea of Ernst Mach according to which matter and space would be closely linked. These two values of the photon velocity is related to the presence of a cross term in $d\varphi\, dt$ The presence in the stationary solution of such a cross term was envisaged in 1916 by the Dutchman J. Droste[13] [21], but immediately rejected by this author as unphysical.

Let us digress for a moment. Droste, a student of Lorentz, presented this work on May 27, 1916, he was thirty years old.

[13] https://fr.wikipedia.org/wiki/Johannes_Droste

Fig.36 : Johannes Droste (1886-1963)

He starts with the same assumptions as Schwarzschild, and here is his result:

This r is not the same as occurs in (4). We obtain

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) \qquad (7)$$

Fig.37 : Droste metric .

The result is absolutely identical to Schwarzchild's. In this very complete article, everything is explained. The expressions of the geodesics and the rest. Later Droste will say that when he presented his paper, he did not know that Schwarzschild had just solved this problem three months earlier.

But let us return to Roy Kerr's work:

Since we admit in his solution the presence of an azimuthal frame-dragging, what do we obtain if we consider in a solution invariant by action of O(3) a radial frame-dragging. This results in the presence of a cross term in *dRdt*.

Keeping the Schwarzschid notations, this corresponds to the Eddington- metric, which is deduced from the Schwarzschild metric by the change of variable

(109) $$t = t' + \delta \frac{R_s}{c} \, \mathrm{L_n} \left(\frac{R}{R_s} - 1 \right) \qquad \delta = \pm 1$$

Which gives:

(110)

$$ds^2 = \left(1 - \frac{R_s}{R}\right) c^2 dt'^2 - \left(1 + \frac{R_s}{R}\right) dR^2 - R^2 (d\theta^2 + \sin^2 d\varphi^2) + 2\delta \frac{R_s}{R} c \, dt' \, dR$$

This situation was recently studied in [22]. The transit time along a radial trajectory becomes:

(111) $\nu = -1$: centripetal trajectory; $\nu = 1$: centrifugal path.

$$\frac{dt'}{dR} = \frac{1}{c}\,\frac{\lambda R - \delta\,\nu\,R_s\sqrt{\lambda^2 - 1 + \frac{R_s}{R} - \frac{h^2}{R^2} + \frac{h^2 R_s}{R3}}}{\nu(R - R_s)\sqrt{\lambda^2 - 1 + \frac{R_s}{R} - \frac{h^2}{R^2} + \frac{h^2 R_s}{R3}}} \qquad \begin{matrix}\delta = \pm 1\\ \nu = \pm 1\end{matrix}$$

For radial paths ($h = 0$)

(112)

$$\frac{dt'}{dR} = \frac{1}{c}\,\frac{\lambda R - \delta\,\nu\,R_s\sqrt{\lambda^2 - 1 + \frac{R_s}{R}}}{\nu(R - R_s)\sqrt{\lambda^2 - 1 + \frac{R_s}{R}}} \qquad \begin{matrix}\delta = \pm 1\\ \nu = \pm 1\end{matrix}$$

Quand R tend vers R_s, cette contribution du temps devient infini if $\delta\,\nu < 0$

In the conditions:

(113)
$$\frac{dt'}{dR} \simeq \frac{\nu}{c}\,\frac{R - \delta\,\nu\,R_s}{R - R_s}$$

It is therefore possible to couple two metric solutions, two sheets connecting according to the sphere of groove, playing then the role of one-way membrane, surface that the masses can cross only in one direction. Let's consider the following pair of metrics. In the first one, the masses can only enter the throat sphere in a short time, but can only emerge in an infinite time, which is equivalent to an impossibility. The opposite situation with respect to the second layer, defined by the metric (91). Globally the transit, with entry into the first layer and emergence into the second, takes place in a finite time. The reverse transit is impossible.

(114) $$ds^2 = \left(1 - \frac{R_s}{R}\right)c^2 dt'^2 - \left(1 + \frac{R_s}{R}\right)dR^2 - R^2(\,d\theta^2 + \sin^2 d\varphi^2\,) - 2\,\frac{R_s}{R}c\,dt'\,dR$$

(115) $$ds^2 = \left(1 - \frac{R_s}{R}\right)c^2 dt'^2 - \left(1 + \frac{R_s}{R}\right)dR^2 - R^2(\,d\theta^2 + \sin^2 d\varphi^2\,) + 2\,\frac{R_s}{R}c\,dt'\,dR$$

A star in implosion would thus see its mass transferred into a second sheet, according to a finite time.

Epilog.

Frankfurt is the birthplace of Karl Schwarzschild. Every year a "Schwarzschild Colloquium" is held at the Advanced Studies Institute in Frankfurt, devoted to questions of cosmology and astrophysics. In 2017 the organizers of the colloquium had invited the cosmologist Juan Malcadena, member of the the Advanced Studies Institute in Princeton, USA. He began his conference, devoted to the latest advances in the field of thermodynamics of black holes by saying :

> - *In 1916, when Karl Schwarzschild published his paper, the scientific community had to spend some time before certain points were clarified. Today these problems have been well mastered.*

To show the way the community of specialists perceives these questions, since the sixties, the simplest way is to reproduce the main stream interpretation as it is presented in page 223 of the reference [18], and which translates a unanimous consensus within the community of cosmologists of today and the partisans of the Black Hole model.

The Schwarzschild Solution and Its Consequences **223**

6.8 The Schwarzschild Radius, Kruskal Coordinates, and the Black Hole

We have noted that in the Schwarzschild line element (6.53) a singularity occurs at $r = 2m$, the Schwarzschild radius; at this radius g_{11} is infinite while g_{00} is zero. Because g_{00} is zero, the spherical surface at $r = 2m$ is an infinite red shift surface, as is clear from our discussion of the red shift in Secs. 4.2 and 4.4. That is, since light emitted by a radiating atom situated on this surface would be red-shifted to zero frequency as it traveled to larger radii, the atom could not be observed.

When r becomes less than $2m$, the signs of the metric components g_{00} and g_{11} change, g_{11} becoming positive and g_{00} becoming negative. This forces us to reconsider the physical meaning of t and r as time and radial markers inside the Schwarzschild radius. Indeed a world-line along the t axis (r, θ, φ constant) has $ds^2 < 0$ and is a *spacelike* curve, while a world-line along the r axis has $ds^2 > 0$ and is a *timelike* curve. It would thus appear natural to reinterpret r as a time marker and t as a radial marker for events which occur inside the Schwarzschild radius. Since we interpret ds/c to represent the proper time along the world line of a particle, as in Sec. 4.2, we see that ds^2 must be positive along such a path. Thus a massive particle could not remain at a constant value of r inside the Schwarzschild radius since that would imply that $ds^2 < 0$ along its world-line.

Fig.44 : Standard interpretation of the Schwarzschild metric

This present article shows that this view should be reconsidered.

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