Practical Calculations in General Relativity in the Case of Diagonal Metrics

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The present text focuses exclusively on the exploration of a space-time manifold (M, g), where the metric g is diagonal.

In the first chapter, a comparative study is undertaken to evaluate two distinct methods for accurately calculating the components of classical curvature-related tensors, such as the curvature tensor, the Riemann tensor, the Ricci tensor, and the Einstein tensor.

The first approach adopted is the standard method, which relies on the use of the vector field basis $(\partial_0, \partial_1, \partial_2, \partial_3)$.

The second method, known as the "tetrad method," distinguishes itself by using an alternative basis $(\theta_0, \theta_1, \theta_2, \theta_3)$, specially chosen to orthonormalize the metric g and thus facilitate calculations and geometric interpretations.

These two methods are applied in the context of a specific metric h that exhibits diagonal properties and spherical symmetry. This metric encompasses well-known metrics such as Schwarzschild, Reissner-Nordström, and Friedmann-Lemaître-Robertson-Walker metrics.

The second chapter is dedicated to solving the Einstein equation for the metric h in the presence of a general energy-momentum tensor. Efforts are focused on the analysis and resolution of the Einstein equation to deduce the initial properties of the functions appearing in the metric.

Examples are presented to illustrate the application of the obtained results. Special attention is given to the study of the Friedmann-Lemaître-Robertson-Walker metric, which is used to describe the expansion of the Universe in the field of cosmology. Subsequently, a more detailed analysis is devoted to the Schwarzschild and Reissner-Nordström metrics, representing solutions for a non-charged and charged massive object in a curved space-time, respectively.

Finally, general results are demonstrated regarding the general form of the metric in these specific cases.

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Chapter 1

Calculations of Objects Associated with Curvature for Diagonal Metrics

1.1 Generalities and Notations

Throughout this chapter, we consider a differentiable manifold M of dimension 4. We have a maximal atlas on M, and at each point p in M, we have a chart of the form:

$$(U, \varphi := (x^0, x^1, x^2, x^3))$$

where $p \in U$ and the calculations will be performed on this chart. Thus, we have a diffeomorphism $\varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^4$, and for each $i \in \{0, 1, 2, 3\}$, we have an infinitely differentiable function:

$$x^i : U \longrightarrow \mathbb{R}$$

called the **i-th coordinate function**.

We denote $\mathscr{C}^\infty(M)$ as the set of infinitely differentiable functions f from M to $\mathbb{R}$ (on each chart U, the function $f_{|U} : U \longrightarrow \mathbb{R}$ is such that $f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^4 \longrightarrow \mathbb{R}$ is infinitely differentiable in the usual sense). Throughout this chapter, all considered objects will be infinitely differentiable.

To simplify notation, we will use the symbol M instead of U from now on. It will be understood that we are working in a chart with coordinates.

References for the concepts discussed in this chapter are:

[2], [6], [8], [9], [11], [17], [20].

1.1.1 Tangent and Cotangent Bundles

We begin by defining the notion of vectors and tangent spaces.

Definition 1.1.1.1: Tangent Space at a Point

Let $p \in M$.

(i) A **vector at p in M** is a function:

$$v : \mathscr{C}^\infty(M) \longrightarrow \mathbb{R}$$

such that for all $f_1, f_2 \in \mathscr{C}^\infty(M)$ and all $a, b \in \mathbb{R}$:

(a) $v(af_1 + bf_2) = av(f_1) + bv(f_2)$;

(b) $v(f_1 f_2) = f_1(p)v(f_2) + f_2(p)v(f_1)$.

(ii) The **tangent space T_pM at p in M** is defined as the set of vectors at p in M.

We notice that if $f := a$ is constant on M, then $v(f) = 0$ because of point (i.b):

$$v(1) = 2v(1)$$

i.e., $v(1) = 0$. Therefore, using point (i.a), we have:

$$\begin{aligned} v(f) &= v(a) \\ &= av(1) \\ &= 0 \end{aligned}$$

Notation 1.1.1.2: Notation for Tangent Space at a Point

Let $p \in M$.

(1) We define two operations on T_pM. For all $v_1, v_2 \in T_pM$, $a \in \mathbb{R}$, and $f \in \mathscr{C}^\infty(M)$, we have:

(a) $(v_1 + v_2)(f) := v_1(f) + v_2(f)$;

(b) $v_1(af) := av_1(f)$.

(2) The element $\partial_{i,p} : \mathscr{C}^\infty(M) \longrightarrow \mathbb{R}$ is defined for all $f \in \mathscr{C}^\infty(M)$ by:

$$\partial_{i,p}(f) := \frac{\partial}{\partial x^i}\left(f \circ \varphi^{-1}\right)_{|\varphi(p)}.$$

We have the following classical result.

Proposition 1.1.1.3: Vector Space Structure of T_pM

(i) We have:

$$\partial_{i,p} \in T_pM.$$

(ii) The $\partial_{i,p}$ are linearly independent.

(iii) Let $p \in U$. The triplet $(T_pM, +, .)$ is an $\mathbb{R}$-vector space of dimension 4 generated by the $\partial_{i,p}$, *i.e.*, we have:

$$T_pM = \operatorname{span}_{\mathbb{R}}\left(\partial_{i,p}\right).$$

Proof. (i) Let $f_1, f_2 \in \mathscr{C}^\infty(M)$ and $a, b \in \mathbb{R}$. Using linearity of differentiation, composition on the left, and evaluation, we have:

$$\begin{aligned} \partial_{i,p}(af_1 + bf_2) &= \frac{\partial}{\partial x^i}\left((af_1 + bf_2) \circ \varphi^{-1}\right)_{|\varphi(p)} \\ &= \frac{\partial}{\partial x^i}\left(af_1 \circ \varphi^{-1} + bf_2 \circ \varphi^{-1}\right)_{|\varphi(p)} \\ &= a\frac{\partial}{\partial x^i}\left(f_1 \circ \varphi^{-1}\right)_{|\varphi(p)} + b\frac{\partial}{\partial x^i}\left(f_2 \circ \varphi^{-1}\right)_{|\varphi(p)} \\ &= a\partial_{i,p}(f_1) + b\partial_{i,p}(f_2) \end{aligned}$$

Using the product rule for differentiation, we have:

$$\begin{aligned} \partial_{i,p}(f_1 f_2) &= \frac{\partial}{\partial x^i}\left((f_1 f_2) \circ \varphi^{-1}\right)_{|\varphi(p)} \\ &= \frac{\partial}{\partial x^i}\left((f_1 \circ \varphi^{-1})(f_2 \circ \varphi^{-1})\right)_{|\varphi(p)} \\ &= f_1\left(\varphi^{-1}(\varphi(p)\right)\frac{\partial}{\partial x^i}\left(f_2 \circ \varphi^{-1}\right)_{|\varphi(p)} + f_2\left(\varphi^{-1}(\varphi(p)\right)\frac{\partial}{\partial x^i}\left(f_1 \circ \varphi^{-1}\right)_{|\varphi(p)} \\ &= f_1(p)\partial_{i,p}(f_2) + f_2(p)\partial_{i,p}(f_1) \end{aligned}$$

Therefore, we have:

$$\partial_{i,p} \in T_pM.$$

(ii) Let $a^0, a^1, a^2, a^3 \in \mathbb{R}$ such that:

$$\sum_{i=0}^{3} a^i \partial_{i,p} = 0$$

i.e., for all $f \in \mathscr{C}^\infty(M)$, we have:

$$\sum_{i=0}^{3} a^i \partial_{i,p}(f) = 0.$$

Since $x^j : M \longrightarrow \mathbb{R} \in \mathscr{C}^\infty(M)$, we have:

$$\begin{aligned}
0 &= \sum_{i=0}^{3} a^i \partial_{i,p}(x^j) \\
&= \sum_{i=0}^{3} a^i \frac{\partial}{\partial x^i} \left(x^j \circ \varphi^{-1}\right)_{|\varphi(p)} \\
&= \sum_{i=0}^{3} a^i \delta_j^i (\varphi^{-1}(\varphi(p))) \\
&= \sum_{i=0}^{3} a^i \delta_j^i \\
&= a^j
\end{aligned}$$

Thus, the $\partial_{i,p}$ are linearly independent.

(iii) Let $v \in T_pM$, $f \in \mathscr{C}^\infty(M)$, and $q \in M$. Let's define:

$$F := f \circ \varphi^{-1} : \mathbb{R}^4 \longrightarrow \mathbb{R}.$$

$$z := (z^0, z^1, z^2, z^3) := \varphi^{-1}(q) \quad y := (y^0, y^1, y^2, y^3) := \varphi^{-1}(p).$$

Let's denote:

$$\lambda := (\lambda^0, \lambda^1, \lambda^2, \lambda^3) : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R}^4 \\ s & \longmapsto & (1-s)y + sz \end{array}$$

as the line segment connecting y and z in $\mathbb{R}^4$. Since F is differentiable on $\mathbb{R}^4$ and $\lambda' = z - y$, by the fundamental theorem of calculus and the chain rule, we have:

$$\begin{aligned}
f(q) - f(p) &= F(z) - F(y) \\
&= F(\lambda(1)) - F(\lambda(0)) \\
&= [F(\lambda(s))]_0^1 \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} (F \circ \lambda)(s) \mathrm{d}s \\
&= \int_0^1 \sum_{i=0}^{3} \left(\lambda^i\right)'(s) \partial_i F(\lambda(s)) \mathrm{d}s \\
&= \int_0^1 \sum_{i=0}^{3} (z^i - y^i) \partial_i F((1-s)y + sz) \mathrm{d}s \\
&= \sum_{i=0}^{3} (z^i - y^i) \int_0^1 \partial_i F((1-s)y + sz) \mathrm{d}s \\
&= \sum_{i=0}^{3} (z^i - y^i) G_i(z) \\
&= \sum_{i=0}^{3} (x^i \circ \varphi(q) - x^i \circ \varphi(q)) G_i(\varphi(q))
\end{aligned}$$

With:

$$G_i(z) := \int_0^1 \partial_i F((1-s)y + sz)\mathrm{d}s.$$

Thus, the function f is given by:

$$f : q \longmapsto f(p) + \sum_{i=0}^{3}(x^i \circ \varphi(q) - x^i \circ \varphi(p))G_i(\varphi(q)).$$

Since v is linear and zero on constants, and using the properties satisfied by v, we apply v to obtain:

$$\begin{aligned}
v(f) &= v(f(p)) + \sum_{i=0}^{3} v\left((x^i \circ \varphi(q) - x^i \circ \varphi(p))G_i(\varphi(q))\right) \\
&= v(f(p)) + \sum_{i=0}^{3}(x^i \circ \varphi(q) - x^i \circ \varphi(p))_{q=p} v\left(G_i(\varphi(p))\right) + \sum_{i=0}^{3} G_i(\varphi(p))v(x^i \circ \varphi - x^i \circ \varphi(p)) \\
&= \sum_{i=0}^{3} v(x^i \circ \varphi)\partial_{i,p}(f)
\end{aligned}$$

By setting:

$$v^i := v(x^i \circ \varphi)$$

we have:

$$v = v^i \partial_{i,p} \in \mathrm{Vect}_{\mathbb{R}}\left(\partial_{i,p}\right).$$

□

Definition 1.1.1.4: Tangent and Cotangent Bundles

(i) (a) The **tangent bundle of** M is defined as:

$$\begin{aligned}
TM &:= \bigcup_{p \in M} \{(p, v),\ v \in T_pM\} \\
&= \{(p, v),\ p \in M \ \wedge\ v \in T_pM\}
\end{aligned}$$

It is naturally equipped with a projection map:

$$\begin{array}{rccc}
\pi : & TM & \longrightarrow & M \\
& (p, v) & \longmapsto & p
\end{array}$$

(b) A **vector field** is a section of TM, *i.e.*, a map $X : M \longrightarrow TM$ such that $\pi \circ X = \mathrm{Id}_M$.

(ii) (a) Let $p \in M$. The **cotangent space** T_p^*M **at** p **of** M is defined as the dual of T_pM, *i.e.*, we have:

$$T_p^*M := (T_pM)^*.$$

The elements of T_p^*M are called **covectors**.

(b) The **cotangent bundle** of M is defined as:

$$\begin{aligned}
T^*M &:= \bigcup_{p \in M} \left\{(p, \phi),\ \phi \in T_p^*M\right\} \\
&= \left\{(p, \phi),\ p \in M \ \wedge\ \phi \in T_p^*M\right\}
\end{aligned}$$

It is naturally equipped with a projection map:

$$\begin{array}{rccc}
\pi^* : & T^*M & \longrightarrow & M \\
& (p, \phi) & \longmapsto & p
\end{array}$$

(c) A **covector field** or **1-form (differential)** is a section of T^*M, *i.e.*, a map $\alpha : M \longrightarrow T^*M$ such that $\pi^* \circ \alpha = \mathrm{Id}_M$.

Example 1.1.1.5: Tangent and Cotangent Bundles

(1) For every $i \in \{0,1,2,3\}$, we denote the section:

$$\begin{array}{rccc} \partial_i : & M & \longrightarrow & TM \\ & p & \longmapsto & (p, \partial_{i,p}) \end{array}$$

The family $\mathscr{C} := (\partial_0, \partial_1, \partial_2, \partial_3)$ forms a basis of vector fields, *i.e.*, for every $p \in M$, $(\partial_{0,p}, \partial_{1,p}, \partial_{2,p}, \partial_{3,p})$ is a basis of T_pM.

(2) We denote $\left(\mathrm{d}x^{0,p}, \mathrm{d}x^{1,p}, \mathrm{d}x^{2,p}, \mathrm{d}x^{3,p}\right)$ as the dual basis of $(\partial_{0,p}, \partial_{1,p}, \partial_{2,p}, \partial_{3,p})$, *i.e.*, for every $j \in \{0,1,2,3\}$, we have a linear form:

$$\mathrm{d}x^{j,p} : T_pM \longrightarrow \mathbb{R}$$

such that:

$$\mathrm{d}x^{j,p}\left(\partial x_{i,p}\right) = \delta_i^j.$$

We then define the section:

$$\begin{array}{rccc} \mathrm{d}x^j : & M & \longrightarrow & T^*M \\ & p & \longmapsto & \left(p, \mathrm{d}x^{j,p}\right) \end{array}$$

The family $\mathscr{C}^* := \left(\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\right)$ forms a basis of covector fields, *i.e.*, for every $p \in M$, $\left(\mathrm{d}x^{0,p}, \mathrm{d}x^{1,p}, \mathrm{d}x^{2,p}, \mathrm{d}x^{3,p}\right)$ is a basis of T_p^*M.

Throughout, we will omit the subscript "p". Thus, we identify vectors with vector fields and covectors with covector fields. For example, we have:

$$\mathrm{d}x^j\left(\partial_i\right) = \delta_i^j.$$

1.1.2 The Language of Tensors

Let $E, F, E_1, \ldots, E_k$ be $\mathbb{R}$-vector spaces.

- We denote:

$$\mathscr{L}_k(E_1 \times \cdots \times E_k, F)$$

the set of **k-linear maps from $E_1 \times \cdots \times E_k$ to F**.

- We denote:

$$E^* := \mathscr{L}(E, \mathbb{R})$$

the **dual space of** E and $E^{**} := (E^*)^*$ the **bidual space of** E. There exists a canonical isomorphism between E and its bidual E^{**} given by:

$$\begin{array}{ccc} E & \longrightarrow & E^{**} \\ u & \longmapsto & (\phi \longmapsto \phi(u)) \end{array}$$

From now on, we identify them.

In this subsection, we consider the bases $\mathscr{B} := (e_1, \ldots, e_n)$ and $\mathscr{B}' := (f_1, \ldots, f_m)$ on E and F, respectively. We denote $\mathscr{B}^* := (e^1, \ldots, e^n)$ and $\mathscr{B}'^* := (f^1, \ldots, f^m)$ as the bases of E^* and F^*, respectively, derived from $\mathscr{B}$ and $\mathscr{B}'$ (where $e^i(e_j) = \delta_j^i$ and $f^i(f_j) = \delta_j^i$).

Definition 1.1.2.1: Tensor Product

The **tensor product of vector spaces** $E_1, \dots, E_k$ is defined as:

$$E_1 \otimes \cdots \otimes E_k := \mathscr{L}_k(E_1^* \times \cdots \times E_k^*, \mathbb{R}).$$

The elements of $E_1 \otimes \cdots \otimes E_k$ are called **tensors**.

Example 1.1.2.2: Usual Examples

(1) For any $T \in E \otimes F$, we have a decomposition:

$$T = \sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij} e_i \otimes f_j = T_{ij} e_i \otimes f_j.$$

The second equality is the **Einstein convention** that we adopt from now on. Thus, we have:

$$T(e^k, f^l) = T_{ij} e_i \otimes f_j (e^k, f^l) = T_{ij} e_i(e^k) f_j(f^l) = T_{ij} \delta_i^k \delta_j^l = T_{kl}.$$

(2) We have:

$$E \otimes E := \mathscr{L}_2(E^* \times E^*, \mathbb{R}) \ , \ E \otimes E^* := \mathscr{L}_2(E^* \times E, \mathbb{R})$$

For all $k, l \in \mathbb{N}$, we have for example:

$$E^{\otimes(k+1)} := E^{\otimes k} \otimes E \qquad E^{\otimes k} \otimes E^{\otimes l} := E^{\otimes(k+l)}$$

$$E^{\otimes 1} := \mathscr{L}(E^*, \mathbb{R}) = E^{**} = E \qquad \left(E^{\otimes k}\right)^{\otimes l} := E^{\otimes(kl)}$$

$$E^{\otimes 0} := \mathbb{R}$$

There exists a canonical isomorphism $(E^*)^{\otimes k} = \left(E^{\otimes k}\right)^*$ which allows us to identify these two vector spaces.

We define:

$$\mathscr{T}^{k,l} E := E^{\otimes k} \otimes (E^*)^{\otimes l}$$

Thus, we construct the set:

$$\mathscr{T} E := \bigoplus_{k,l \geq 0} \mathscr{T}^{k,l} E.$$

- The elements of $\mathscr{T}^k E := \mathscr{T}^{k,0} E$ (with $k > 0$) are called **contravariant tensors**.
- The elements of $\mathscr{T}^{0,l} E$ (with $l > 0$) are called **covariant tensors**.
- The elements of $\mathscr{T}^{k,l} E$ (with $k, l > 0$) are called **mixed tensors**.

Naturally, we define addition and scalar multiplication between tensors of the same type, making the triple $(\mathscr{T} E, +, \cdot)$ an $\mathbb{R}$-vector space. A tensor T of type (k, l) admits a unique decomposition of the form:

$$T = T^{i_1 \cdots i_k}_{j_1 \cdots j_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l}$$

We say that this is the **decomposition of T in the bases $\mathscr{B}$ and $\mathscr{B}^*$**.

Below, we define a multiplication operation.

Definition 1.1.2.3: Tensor Products

Let $x_1, \dots, x_k \in E$ and $\psi^1, \dots, \psi^k \in E^*$. We have an element:

$$x_1 \otimes \cdots \otimes x_k \otimes \psi^1 \otimes \cdots \otimes \psi^l \ \in \ \mathscr{T}^{k,l} E := E^{\otimes k} \otimes (E^*)^{\otimes l}$$

defined for all $\phi^1, \dots, \phi^k \in E^*$ and $y_1, \dots, y_l \in E$ by:

$$\left(x_1 \otimes \cdots \otimes x_k \otimes \psi^1 \otimes \cdots \otimes \psi^l\right)(\phi^1, \dots, \phi^k, y_1, \dots, y_l) := \prod_{i=1}^{k} \phi^i(x_i) \times \prod_{j=1}^{l} \psi^j(y_j).$$

This product generalizes by linearity to arbitrary multilinear forms. Therefore, the product $S \otimes T$ of tensors of order (k_1, l_1) and (k_2, l_2) is of order $(k_1 + k_2, l_1 + l_2)$.

Example 1.1.2.4: Simple Examples of Tensor Products

(1) The **product of** $x, y \in E$ is an element $x \otimes y \in E \otimes E = \mathscr{L}_2(E^* \times E^*, \mathbb{R})$ defined by:

$$\forall \phi, \psi \in E^*,\ (x \otimes y)(\phi, \psi) := \phi(x)\psi(y).$$

(2) The **product of** $\phi, \psi \in E^*$ is an element $\phi \otimes \psi \in E^* \otimes E^* = \mathscr{L}_2(E \times E, \mathbb{R})$ defined by:

$$\forall x, y \in E,\ (\phi \otimes \psi)(x, y) := \phi(x)\psi(y).$$

(3) The **product of** $x \in E$ **and** $\psi \in E^*$ is an element $x \otimes \psi \in E \otimes E^* = \mathscr{L}_2(E^* \times E, \mathbb{R})$ defined by:

$$\forall \phi \in E^*, \forall y \in E,\ (x \otimes \psi)(\phi, y) := \phi(x)\psi(y).$$

These products are associative, non-commutative, and distributive with respect to +.

The quadruple $(\mathscr{T}E, +, \cdot, \otimes)$ is thus a graded $\mathbb{R}$-algebra.

We conclude this subsection with the contraction of tensors.

Definition 1.1.2.5: Contractions of Tensors

Let $T := x_1 \otimes \cdots \otimes x_k \otimes y^1 \otimes \cdots \otimes y^l \in E^{\otimes k} \otimes (E^*)^{\otimes l}$ be a tensor of type (k, l) with $k, l \geq 1$, $i \in \{1, \dots, k\}$, and $j \in \{1, \dots, l\}$. The **contraction of** T **along indices** i **and** j is the tensor:

$$[T]^i_j = y^j(x_i) x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_k \otimes y^1 \otimes \cdots \otimes y^{j-1} \otimes y^{j+1} \otimes \cdots \otimes y^l$$

It is a tensor of type $(k-1, l-1)$.

These contractions are associative and distributive with respect to +. It can be shown that they are independent of the chosen basis.

Example 1.1.2.6: Trace of a $(1,1)$ Tensor

Let $T \in \mathscr{T}^1_1 E$ be a tensor of type $(1, 1)$. Since

$$\mathscr{T}^1_1 E = E \otimes E^* = \mathscr{L}(E)$$

T can also be viewed as a linear map on E. Let's denote its decomposition in the bases $\mathscr{B}$ and $\mathscr{B}^*$ as:

$$T := \mathrm{T}^i_j e_i \otimes e^j.$$

The **trace of** T is defined as:

$$\mathrm{tr}(T) := [T]^1_1.$$

Therefore, we have:

$$\begin{aligned}\operatorname{tr}(T) &= [T]_1^1 \\ &= \mathrm{T}_j^i e^j(e_i) \\ &= \mathrm{T}_j^i \delta_i^j \\ &= \mathrm{T}_j^j\end{aligned}$$

This definition is the same as the standard definition of trace for an endomorphism. The trace is independent of the chosen basis.

1.1.3 Tensor Fields

We denote $\mathscr{T}^{k,l}M$ the set of **real tensor fields on** M **of type** (k,l), *i.e.*, the set of fields $\mathscr{A}$ such that at each point $p \in M$, we have a tensor $\mathscr{A}_p$ of type (k,l). Thus, we have a k-linear map:

$$\mathscr{A}_p : (T_pM)^k \times \left(T_p^*M\right)^l \longrightarrow \mathbb{R}$$

using the equality:

$$\mathscr{T}^{k,l}T_pM = (T_pM)^{\otimes k} \otimes \left(T_p^*M\right)^{\otimes l}.$$

We denote $\mathscr{T}M$ the set of tensor fields on M:

$$\mathscr{T}M := \bigsqcup_{k,l\geq 0} \mathscr{T}^{k,l}M.$$

All the operations defined on tensors in the previous subsection extend naturally to tensor fields.

Example 1.1.3.1: Examples of Tensor Fields

(1) A **scalar field** is a tensor field of type $(0,0)$.

(2) A **vector field** is a tensor field of type $(1,0)$.

(3) A **covector field** or 1**-form** is a tensor field of type $(0,1)$.

(4) A **linear map from** M **to** M (*i.e.*, for each $p \in M$, there is a linear map from T_pM) can be seen as a tensor of type $(1,1)$.

Definition 1.1.3.2: Examples of Tensor Fields

A **spacetime metric (of signature** $(1,3)$**)** is a tensor field g of type $(0,2)$ such that for every $p \in M$:

(i) g_p is **symmetric**, *i.e.*, for all vector fields X and Y and every $p \in M$, we have:

$$g_p(X(p),Y(p)) = g_p(Y(p),X(p)).$$

(ii) g_p is **nonsingular**, *i.e.*, for all vector fields X and Y and every $p \in M$, we have:

$$g_p(X(p),Y(p)) = 0_{\mathbb{R}} \implies X(p) = 0_{T_pM} \ \vee \ Y(p) = 0_{T_pM}.$$

(iii) g_p has **signature** $(1,3)$, *i.e.*, there exists a basis $\mathscr{B}_p$ of T_pM such that

$$\mathscr{M}at_{\mathscr{B}_p}(g) = \begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{pmatrix}.$$

We then say that the pair (M, g) is a **spacetime manifold** or **Lorentzian manifold of signature** $(1,3)$.

From now on, we omit the letter "p". For example, we say that g is symmetric and write $g(X,Y) = g(Y,X)$ instead of $g_p(X(p),Y(p)) = g_p(Y(p),X(p))$.

Example 1.1.3.3: Spherically Symmetric Metrics

(1) **Metric on the unit 2-sphere**. Consider the unit 2-sphere in $\mathbb{R}^3$:

$$\mathbb{S}^2 := \left\{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+z^2=1\right\}.$$

The change of variables between Cartesian and spherical coordinates is given by:

$$\begin{array}{rccc} f: &]0,\pi[\times[0,2\pi[& \longrightarrow & \mathbb{S}^2 \subset \mathbb{R}^3 \\ & (\vartheta,\varphi) & \longmapsto & (x,y,z) := (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \end{array}$$

The Jacobian matrix of the change of variables is:

$$J := \begin{pmatrix} \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \vartheta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\vartheta\cos\varphi & -\sin\vartheta\sin\varphi \\ \cos\vartheta\sin\varphi & \sin\vartheta\cos\varphi \\ -\sin\vartheta & 0 \end{pmatrix}.$$

Let:

$$\delta := \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y + \mathrm{d}z\otimes\mathrm{d}z$$

and $h_\Omega := f^*\delta$ be the metric on the unit 2-sphere induced by δ through the parameterization f. Since:

$$\mathscr{M}at_{(\partial_x,\partial_y,\partial_z)}(\delta) = I_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have:

$$\begin{aligned} \mathscr{M}at_{(\partial_\varphi,\partial_\psi)}(h_\Omega) &= {}^tJ\mathscr{M}at_{(\partial_x,\partial_y,\partial_z)}(\delta)J \\ &= {}^tJI_3J \\ &= \begin{pmatrix} \cos\vartheta\cos\varphi & \cos\vartheta\sin\varphi & -\sin\vartheta \\ -\sin\vartheta\sin\varphi & \sin\vartheta\cos\varphi & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\vartheta\cos\varphi & -\sin\vartheta\sin\varphi \\ \cos\vartheta\sin\varphi & \sin\vartheta\cos\varphi \\ -\sin\vartheta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix} \end{aligned}$$

So we have:

$$h_\Omega := \mathrm{d}\vartheta\otimes\mathrm{d}\vartheta + \sin^2\vartheta\,\mathrm{d}\varphi\otimes\mathrm{d}\varphi$$

(2) **Transformation from Cartesian to Spherical Coordinates**. The metric η in the $\mathscr{C}^*$ basis is given by:

$$\eta = \mathrm{d}x^0\otimes\mathrm{d}x^0 - \mathrm{d}x^1\otimes\mathrm{d}x^1 - \mathrm{d}x^2\otimes\mathrm{d}x^2 - \mathrm{d}x^3\otimes\mathrm{d}x^3$$

So we have:

$$N_{\mathscr{C}} := \mathrm{Mat}_{\mathscr{C}}(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The change of variables from Cartesian to spherical coordinates is given by:

$$\begin{cases} x^0 := t \\ x^1 := r\sin\vartheta\cos\varphi \\ x^2 := r\sin\vartheta\sin\varphi \\ x^3 := r\cos\vartheta \end{cases}$$

The Jacobian matrix of the coordinate transformation is:

$$J := \begin{pmatrix} \frac{\partial x^0}{\partial t} & \frac{\partial x^0}{\partial r} & \frac{\partial x^0}{\partial \vartheta} & \frac{\partial x^0}{\partial \varphi} \\ \frac{\partial x^1}{\partial t} & \frac{\partial x^1}{\partial r} & \frac{\partial x^1}{\partial \vartheta} & \frac{\partial x^1}{\partial \varphi} \\ \frac{\partial x^2}{\partial t} & \frac{\partial x^2}{\partial r} & \frac{\partial x^2}{\partial \vartheta} & \frac{\partial x^2}{\partial \varphi} \\ \frac{\partial x^3}{\partial t} & \frac{\partial x^3}{\partial r} & \frac{\partial x^3}{\partial \vartheta} & \frac{\partial x^3}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ 0 & \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ 0 & \cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix}$$

Let's define:

$$\mathscr{B} := (\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi) \quad , \quad \mathscr{B}^* := (\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi)$$

For simplicity, let's define:

$$N_{\mathscr{B}} := \mathrm{Mat}_{\mathscr{B}}(\eta)\,,\; c(\vartheta) := \cos\vartheta\,,\; s(\vartheta) := \sin\vartheta\,,\; c(\varphi) := \cos\varphi\,,\; s(\varphi) := \sin\varphi$$

By change of basis, we have:

$$\begin{aligned}
& N_{\mathscr{B}} \\
&= {}^t J N_{\mathscr{C}} J \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s(\vartheta)c(\varphi) & s(\vartheta)s(\varphi) & c(\vartheta) \\ 0 & rc(\vartheta)c(\varphi) & rc(\vartheta)s(\varphi) & -rs(\vartheta) \\ 0 & -rs(\vartheta)s(\varphi) & rs(\vartheta)c(\varphi) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s(\vartheta)c(\varphi) & rc(\vartheta)c(\varphi) & -rs(\vartheta)s(\varphi) \\ 0 & s(\vartheta)s(\varphi) & rc(\vartheta)s(\varphi) & rs(\vartheta)c(\varphi) \\ 0 & c(\vartheta) & -rs(\vartheta) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -s(\vartheta)c(\varphi) & -s(\vartheta)s(\varphi) & -c(\vartheta) \\ 0 & -rc(\vartheta)c(\varphi) & -rc(\vartheta)s(\varphi) & rs(\vartheta) \\ 0 & rs(\vartheta)s(\varphi) & -rs(\vartheta)c(\varphi) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s(\vartheta)c(\varphi) & rc(\vartheta)c(\varphi) & -rs(\vartheta)s(\varphi) \\ 0 & s(\vartheta)s(\varphi) & rc(\vartheta)s(\varphi) & rs(\vartheta)c(\varphi) \\ 0 & c(\vartheta) & -rs(\vartheta) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\vartheta \end{pmatrix}
\end{aligned}$$

So we have:

$$\begin{aligned} \eta &= \mathrm{d}x^0 \otimes \mathrm{d}x^0 - \mathrm{d}x^1 \otimes \mathrm{d}x^1 - \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3 \\ &= \mathrm{d}t \otimes \mathrm{d}t - \mathrm{d}r \otimes \mathrm{d}r - r^2 \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta - r^2 \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi \end{aligned}$$

1.1.4 The diagonal metric of the text

Let's recall what has been defined.

- A differential manifold M of dimension 4.

- At each point p of M, we have a chart of M of the form:
$$(U, \varphi := (x^0, x^1, x^2, x^3))$$
on which we will perform calculations.
- $\mathscr{C} := (\partial_0, \partial_1, \partial_2, \partial_3)$ is the basis of vector fields on U associated with the coordinates $x := (x^0, x^1, x^2, x^3)$, *i.e.*, we have:
$$\partial_i := \frac{\partial}{\partial x^i}.$$
- $\mathscr{C}^* := (\mathrm{d}x^0, \mathrm{d}x^0, \mathrm{d}x^0, \mathrm{d}x^0)$ is the dual basis associated with $\mathscr{C}$ of covector fields, *i.e.*, we have:
$$\mathrm{d}x^i(\partial_j) = \delta^i_j.$$

We omit the "p" in the notations as well as the "U".

We have a metric g on M that is an spacetime metric (of signature $(1,3)$) and is locally given by the form:
$$\begin{aligned} g &= \eta_k \left(e^{u_k(x)}\mathrm{d}x^k\right) \otimes \left(e^{u_k(x)}\mathrm{d}x^k\right) \\ &= \eta_k e^{2u_k(x)} \mathrm{d}x^k \otimes \mathrm{d}x^k \\ &= e^{2u_0(x)}\mathrm{d}x^0 \otimes \mathrm{d}x^0 - e^{2u_1(x)}\mathrm{d}x^1 \otimes \mathrm{d}x^1 - e^{2u_2(x)}\mathrm{d}x^2 \otimes \mathrm{d}x^2 - e^{2u_3(x)}\mathrm{d}x^3 \otimes \mathrm{d}x^3 \end{aligned}$$
where
$$\eta_k := \begin{cases} 1 & \text{if } k := 0 \\ -1 & \text{otherwise} \end{cases} \quad , \quad \eta_{ij} := \delta^i_j \eta_i \quad , \quad \delta^i_j := \delta^{ij} := \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We also denote:

- the components of g as:
$$\mathrm{g}_{ij} := g(\partial_i, \partial_j) = \eta_{ij} e^{2u_i(x)}$$
- the metric:
$$\begin{aligned} \eta &= \eta_k \mathrm{d}x^k \otimes \mathrm{d}x^k \\ &= \mathrm{d}x^0 \otimes \mathrm{d}x^0 - \mathrm{d}x^1 \otimes \mathrm{d}x^1 - \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3 \end{aligned}$$
and thus we have:
$$\eta_{ij} := \eta(\partial_i, \partial_j).$$
- the derivatives:
$$u_{i,k} := \partial_k u_i := \frac{\partial u_i}{\partial x^k} \quad , \quad u_{i,kl} := \partial^2_{kl} u_i := \frac{\partial^2 u_i}{\partial x^k \partial x^l}.$$

We will omit the variable "x" in the functions u_i from now on.

Example 1.1.4.1: Example of the metric h

In this text, we will consider the following diagonal metric h as an example:
$$\begin{aligned} h &= e^{2u(t,r)}\mathrm{d}t \otimes \mathrm{d}t - e^{2v(t,r)}\mathrm{d}r \otimes \mathrm{d}r - r^2 e^{2b(t)} \left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right) \\ &= e^{2u(t,r)}\mathrm{d}t \otimes \mathrm{d}t - e^{2v(t,r)}\mathrm{d}r \otimes \mathrm{d}r - r^2 e^{2b(t)} h_\Omega \end{aligned}$$
where:
$$h_\Omega := \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi$$
is the metric on the unit 2-sphere in spherical coordinates (see point (2) of Example 1.1.3.3). We are

considering the case where:

$$(x^0, x^1, x^2, x^3) := (t, r, \vartheta, \phi).$$

and

$$\begin{aligned} u_0(t,r) &:= u(t,r) \\ u_1(t,r) &:= v(t,r) \\ u_2(t,r) &:= b(t) + \ln(r) \\ u_3(t,r,\vartheta) &:= b(t) + \ln(r) + \ln\sin(\vartheta) \end{aligned}$$

For indices or exponents, we use the notation:

$$(0,1,2,3) \equiv (t, r, \vartheta, \phi)$$

Derivatives with respect to t are denoted with a dot (e.g., $\dot{u}$ and $\ddot{u}$), and derivatives with respect to r are denoted with a prime (e.g., u' and u'').
This metric allows us to consider two common cases in the next chapter.

(a) The **interior and exterior Schwarzschild metrics (SCHW)**, and the **Reissner–Nordström metric (RN)** correspond to the case where:

$$b = 0.$$

(b) The **Friedmann–Lemaître–Robertson–Walker metric (FLRW)** corresponds to the case where:

$$\dot{u} = u' = 0 \quad , \quad v(t,r) = v(r) + b(t).$$

1.1.5 Orthonormalization

The basis $\mathscr{C}$ is not orthonormal for g in general, meaning that we generally have:

$$g(\partial_i, \partial_j) = \mathrm{g}_{ij} \neq \eta_{ij}.$$

We will construct an orthonormal basis $\mathscr{C}_\perp := (\theta_0, \theta_1, \theta_2, \theta_3)$ associated with $\mathscr{C}$. Let us define:

$$\theta_k := f^k \partial_k.$$

We want to have:

$$\eta_k = g(\theta_k, \theta_k) = g\left(f^k\partial_k, f^k\partial_k\right) = \left(f^k\right)^2 g(\partial_k, \partial_k) = \left(f^k\right)^2 \mathrm{g}_{kk}$$

Thus, we have:

$$f^k = \sqrt{\frac{\eta_k}{\mathrm{g}_{kk}}} = \frac{1}{\sqrt{|\mathrm{g}_{kk}|}} = e^{-u_k}$$

Therefore, we have:

$$\theta_k = f^k \partial_k = \frac{1}{\sqrt{|\mathrm{g}_{kk}|}}\partial_k = e^{-u_k}\partial_k \tag{1.1.5.1}$$

We denote $\mathscr{C}_\perp^* := (\theta^0, \theta^1, \theta^2, \theta^3)$ as the dual basis associated with $\mathscr{C}_\perp$, where:

$$\theta^k := f_k \mathrm{d}x^k.$$

Thus, we have:

$$1 = \theta^k(\theta_k) = f^k \mathrm{d}x^k\left(f_k\partial_k\right) = f^k f_k \mathrm{d}x^k\left(\partial_k\right) = f^k f_k$$

Therefore, we have:

$$f_k = \left(f^k\right)^{-1} = e^{u_k}.$$

Thus, we have:

$$\theta^k = f_k \mathrm{d}x^k = e^{u_k} \mathrm{d}x^k. \tag{1.1.5.2}$$

Consequently, we have:

$$g = \theta^0 \otimes \theta^0 - \theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2 - \theta^3 \otimes \theta^3 = \eta_k \theta^k \otimes \theta^k = \eta_{ij} \theta^i \otimes \theta^j.$$

Example 1.1.5.1: Extension 1 of the metric h

Let's continue with the example of the metric h (see 1.1.4.1) with:

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

We denote:

$$\mathscr{B}_\perp := (\theta_0, \theta_1, \theta_2, \theta_3) := (\theta_t, \theta_r, \theta_\vartheta, \theta_\phi)$$

the orthonormal basis associated with:

$$\mathscr{B} := (\partial_0, \partial_1, \partial_2, \partial_3) := (\partial_t, \partial_r, \partial_\vartheta, \partial_\phi)$$

and

$$\mathscr{B}_\perp^* := (\theta^0, \theta^1, \theta^2, \theta^3) := (\theta^t, \theta^r, \theta^\vartheta, \theta^\phi)$$

the dual basis associated with $\mathscr{B}_\perp$. We have:

$$\theta_t = \theta_0 = e^{-u}\partial_t \qquad \theta_r = \theta_1 = e^{-v}\partial_t$$
$$\theta_\vartheta = \theta_2 = r^{-1}e^{-b}\partial_\vartheta \qquad \theta_\phi = \theta_3 = r^{-1}\sin^{-1}\vartheta e^{-b}\partial_\phi$$
$$\theta^t = \theta^0 = e^u \mathrm{d}t \qquad \theta^r = \theta^1 = e^v \mathrm{d}r$$
$$\theta^\vartheta = \theta^2 = re^a \mathrm{d}\vartheta \qquad \theta^\phi = \theta^3 = r\sin\vartheta e^a \mathrm{d}\phi$$

Thus, we have:

$$h = \theta^t \otimes \theta^t - \theta^r \otimes \theta^r - \theta^\vartheta \otimes \theta^\vartheta - \theta^\phi \otimes \theta^\phi.$$

1.1.6 The bases $\mathscr{E}$ and $\mathscr{E}^*$

For the rest of the discussion, we define:

- $\mathscr{E} := (e_0, e_1, e_2, e_3)$ as a basis of vector fields on M;
- $\mathscr{E}^* := (e^0, e^1, e^2, e^3)$ as the dual basis associated with $\mathscr{E}$, *i.e.*, we have:

$$e^i(e_j) = \delta^i_j.$$

The bases $\mathscr{E}$ and $\mathscr{E}^*$ will be used to handle the general case. Let's define:

$$\mathbf{g}_{ij} := g(e_i, e_j)$$

which means that we can write:

$$g = \mathbf{g}_{ij} e^i \otimes e^j.$$

It is important to note that unlike $\mathscr{C}$ and $\mathscr{C}_\perp$, the metric g is not diagonal in $\mathscr{E}$, *i.e.*, we can have:

$$i \neq j \ \wedge \ \mathbf{g}_{ij} \neq 0.$$

1.1.7 Tensor Notations

Let $\mathscr{A}$ be a tensor field of type (k,l). We denote:

$$
\begin{aligned}
\mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_l} &:= \mathscr{A}\left(e^{j_1},\dots,e^{j_k},e_{i_1},\dots,e_{i_l}\right)\\
\mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l} &:= \mathscr{A}\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_{i_l}\right)\\
\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l} &:= \mathscr{A}\left(\theta^{j_1},\dots,\theta^{j_k},\theta_{i_1},\dots,\theta_{i_l}\right)
\end{aligned}
$$

We will use specific letters depending on the basis used to calculate the components of the tensor fields. Here is a table explaining the letter conventions used:

Tensor Fields	**In $\mathscr{E}$ and $\mathscr{E}^*$ Bases**	**In $\mathscr{C}$ and $\mathscr{C}^*$ Bases**	**In $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$ Bases**
$\mathscr{R},\ \mathscr{G},\ \mathscr{T},\dots$	$\mathbf{R},\ \mathbf{G},\ \mathbf{T},\dots$	$\mathrm{R},\ \mathrm{G},\ \mathrm{T},\dots$	$\mathbb{R},\ \mathbb{G},\ \mathbb{T},\dots$

Thus, we have the following decompositions:

$$
\begin{aligned}
\mathscr{A} &= \mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ e_{j_1}\otimes\cdots\otimes e_{j_k}\otimes e^{i_1}\otimes\cdots\otimes e^{i_l}\\
\mathscr{A} &= \mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ \partial_{j_1}\otimes\cdots\otimes \partial_{j_k}\otimes \mathrm{d}x^{i_1}\otimes\cdots\otimes \mathrm{d}x^{i_l}\\
\mathscr{A} &= \mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ \theta_{j_1}\otimes\cdots\otimes \theta_{j_k}\otimes \theta^{i_1}\otimes\cdots\otimes \theta^{i_l}
\end{aligned}
$$

Proposition 1.1.7.1: Relations between components

We have:

$$
\mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l} = \exp\left(\sum_{n=1}^{l} u_{i_n} - \sum_{n=1}^{k} u_{j_n}\right)\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}.
$$

Proof. We have:

$$
\begin{aligned}
&\mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ \partial_{j_1}\otimes\cdots\otimes \partial_{j_k}\otimes \mathrm{d}x^{i_1}\otimes\cdots\otimes \mathrm{d}x^{i_l}\\
=&\mathscr{A}\\
=&\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ \theta_{j_1}\otimes\cdots\otimes \theta_{j_k}\otimes \theta^{i_1}\otimes\cdots\otimes \theta^{i_l}\\
=&\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ e^{-u_{j_1}}\partial_{j_1}\otimes\cdots\otimes e^{-u_{j_k}}\partial_{j_k}\otimes e^{u_{i_1}}\mathrm{d}x^{i_1}\otimes\cdots\otimes e^{u_{i_l}}\mathrm{d}x^{i_l}\\
=&\exp\left(\sum_{n=1}^{l} u_{i_n} - \sum_{n=1}^{k} u_{j_n}\right)\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ \partial_{j_1}\otimes\cdots\otimes \partial_{j_k}\otimes \mathrm{d}x^{i_1}\otimes\cdots\otimes \mathrm{d}x^{i_l}
\end{aligned}
$$

Thus, by the uniqueness of the decomposition of $\mathscr{A}$ in the $\mathscr{C}$ and $\mathscr{C}^*$ bases, we have:

$$
\mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l} = \exp\left(\sum_{n=1}^{l} u_{i_n} - \sum_{n=1}^{k} u_{j_n}\right)\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}.
$$

$\square$

In the following example, we provide the main tensor fields studied in this chapter.

Example 1.1.7.2: Common examples

(1) The curvature tensor decomposes as follows:

$$\mathscr{R} = \mathbf{R}^i{}_{jkl}\ e_i \otimes e^j \otimes e^k \otimes e^l$$
$$\mathscr{R} = \mathrm{R}^i{}_{jkl}\ \partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l$$
$$\mathscr{R} = \mathbb{R}^i{}_{jkl}\ \theta_i \otimes \theta^j \otimes \theta^k \otimes \theta^l$$

Thus, by the previous proposition 1.1.7.1, we have:

$$\mathrm{R}^i{}_{jkl} = e^{-u_i+u_j+u_k+u_l}\mathbb{R}^i{}_{jkl}.$$

(2) The Riemann tensor decomposes as follows:

$$\mathscr{R}\mathrm{m} = \mathbf{R}_{ijkl}\ e^i \otimes e^j \otimes e^k \otimes e^l$$
$$\mathscr{R}\mathrm{m} = \mathrm{R}_{ijkl}\ \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l$$
$$\mathscr{R}\mathrm{m} = \mathbb{R}_{ijkl}\ \theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l$$

Thus, by the previous proposition 1.1.7.1, we have:

$$\mathrm{R}_{ijkl} = e^{u_i+u_j+u_k+u_l}\mathbb{R}_{ijkl}.$$

(3) The Ricci tensor decomposes as follows:

$$\mathscr{R}\mathrm{ic} = \mathbf{R}_{jl}\ e^j \otimes e^l$$
$$\mathscr{R}\mathrm{ic} = \mathrm{R}_{jl}\ \mathrm{d}x^j \otimes \mathrm{d}x^l$$
$$\mathscr{R}\mathrm{ic} = \mathbb{R}_{jl}\ \theta^j \otimes \theta^l$$

Thus, by the previous proposition 1.1.7.1, we have:

$$\mathrm{R}_{jl} = e^{u_j+u_l}\mathbb{R}_{jl}.$$

(4) The Einstein tensor decomposes as follows:

$$\mathscr{G} = \mathbf{G}_{jl}\ e^j \otimes e^l$$
$$\mathscr{G} = \mathrm{G}_{jl}\ \mathrm{d}x^j \otimes \mathrm{d}x^l$$
$$\mathscr{G} = \mathbb{G}_{jl}\ \theta^j \otimes \theta^l$$

Thus, by the previous proposition 1.1.7.1, we have:

$$\mathrm{G}_{jl} = e^{u_j+u_l}\mathbb{G}_{jl}.$$

1.1.8 Lowering indices with the tensor g

In this subsection, we define an operation that lowers indices. We recall that:

$$g = \mathbf{g}_{ij}e^i \otimes e^j$$

where:

$$\mathbf{g}_{ij} := g\left(e_i, e_j\right).$$

Example 1.1.8.1: A simple calculation

Consider two vector fields $X := X^k e_k$ and $Y := Y^l e_l$. Let's calculate $g(X, Y)$ using the given decomposition of g. We have:

$$\begin{aligned} g(X,Y) &= \mathbf{g}_{ij} e^i \otimes e^j \left(X^k e_k, Y^l e_l\right) \\ &= \mathbf{g}_{ij} e^i \left(X^k e_k\right) e^j \left(Y^l e_l\right) \\ &= \mathbf{g}_{ij} X^k Y^l e^i \left(e_k\right) e^j \left(e_l\right) \\ &= \mathbf{g}_{ij} X^k Y^l \delta^i_k \delta^j_l \\ &= \mathbf{g}_{kl} X^k Y^l \end{aligned}$$

Definition 1.1.8.2: Lowering of the i-th index

The **natural isomorphism associated with** g is the flat musical application $\mathfrak{a}_g$ defined as:

$$\begin{array}{rccc} \mathfrak{a}_g : & TM & \longrightarrow & T^*M \\ & X & \longmapsto & Y \longmapsto g(X,Y) \end{array}$$

We also denote, for any vector field X:

$$X^\flat := \mathfrak{a}_g(X).$$

Notation 1.1.8.3: Common notations

Let $X := \mathbf{X}^i e_i$ be a vector field. We define:

$$\mathbf{X}_i := X^\flat(e_i)$$

i.e., we have:

$$X^\flat = \mathbf{X}_i e^i.$$

Since:

$$\mathfrak{a}_g(e_i)(e_j) = g(e_i, e_j) = \mathbf{g}_{ij}$$

we have:

$$\mathfrak{a}_g(e_i) = \mathbf{g}_{ij} e^j$$

and by linearity, we have:

$$\begin{aligned} X^\flat &= \mathfrak{a}_g(X) \\ &= \mathfrak{a}_g(\mathbf{X}^i e_i) \\ &= \mathbf{X}^i \mathfrak{a}_g(e_i) \\ &= \mathbf{X}^i \mathbf{g}_{ij} e^j. \end{aligned}$$

Thus, by the uniqueness of the decomposition, we have:

$$\mathbf{X}_j = \mathbf{X}^i \mathbf{g}_{ij}.$$

In matrix form, we have:

$$G := \mathscr{M}at_{\mathscr{C},\mathscr{C}^*}(\mathfrak{a}_g) = \mathscr{M}at_{\mathscr{C}}(g) = (\mathbf{g}_{ij})_{ij}.$$

Example 1.1.8.4: Example of $\mathfrak{a}_g$ and g in the bases $\mathscr{C}$ and $\mathscr{C}_\perp$

(1) In $\mathscr{C}$, we have:

$$g_{ij} = \eta_{ij} e^{-u_j}.$$

Thus, we have:

$$\mathscr{M}at_{\mathscr{C},\mathscr{C}^*}(\mathfrak{a}_g) = \mathscr{M}at_{\mathscr{C}}(g) = \begin{pmatrix} e^{u_0} & 0 & 0 & 0 \\ 0 & -e^{u_1} & 0 & 0 \\ 0 & 0 & -e^{u_2} & 0 \\ 0 & 0 & 0 & -e^{u_3} \end{pmatrix}.$$

(2) In $\mathscr{C}_\perp$, we have:

$$\mathscr{M}at_{\mathscr{C}_\perp,\mathscr{C}_\perp^*}(\mathfrak{a}_g) = \mathscr{M}at_{\mathscr{C}_\perp}(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Definition 1.1.8.5: Flat musical isomorphism

Let $\mathscr{A}$ be a tensor field of type (k,l) with $k \geq 1$ and $n \in \{1,\dots,k\}$. The **n-lowered tensor field of** $\mathscr{A}$ is the tensor field $[\mathscr{A}]_n$ of type $(k-1, l+1)$ defined for all covector fields $\alpha_1,\dots,\alpha_{n-1},\alpha_{n+1},\dots,\alpha_k$ and vector fields $X_0,\dots,X_l$ as:

$$[\mathscr{A}]_n(\alpha_1,\dots,\alpha_{n-1},\alpha_{n+1},\dots,\alpha_k,X_0,\dots,X_l) \coloneqq g\left(X_0, \mathscr{A}(\alpha_1,\dots,\alpha_{n-1},\bullet,\alpha_{n+1},\dots,\alpha_k,X_1,\dots,X_l)\right).$$

We have the following relation:

$$[\mathscr{A}]_n(\alpha_1,\dots,\alpha_{n-1},\alpha_{n+1},\dots,\alpha_k,X_0,\dots,X_l) = \mathfrak{a}_g\left(\mathscr{A}(\alpha_1,\dots,\alpha_{n-1},\bullet,\alpha_{n+1},\dots,\alpha_k,X_1,\dots,X_l)\right)(X_0).$$

The following proposition shows why the term "lowered" has been used.

Proposition 1.1.8.6: Lowering of the n-th index

Let $\mathscr{A}$ be a tensor field of type (k,l) with $k \geq 1$ and $n \in \{1,\dots,k\}$. Define:

$$\begin{aligned} \mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_l} &\coloneqq \mathscr{A}\left(e^{j_1},\dots,e^{j_k},e_{i_1},\dots,e_{i_l}\right) \\ \mathbf{A}^{j_1\cdots j_{n-1}j_{n+1}\cdots j_k}_{j_n i_1\cdots i_l} &\coloneqq [\mathscr{A}]_n\left(e^{j_1},\dots,e^{j_{n-1}},e^{j_{n+1}},\dots,e^{j_k},e_{j_n},e_{i_1},\dots,e_{i_l}\right) \end{aligned}$$

Then we have:

$$\mathbf{A}^{j_1\cdots j_{n-1}j_{n+1}\cdots j_k}_{j_n i_1\cdots i_l} = \mathbf{A}^{j_1\cdots j_{n-1}m j_{n+1}\cdots j_k}_{i_1\cdots i_l}\mathbf{g}_{m j_n}.$$

Proof. We have:

$$\begin{aligned} \mathbf{A}^{j_1\cdots j_{n-1}j_{n+1}\cdots j_k}_{j_n i_1\cdots i_l} &= [\mathscr{A}]_n\left(e^{j_1},\dots,e^{j_{n-1}},e^{j_{n+1}},\dots,e^{j_k},e_{j_n},e_{i_1},\dots,e_{i_l}\right) \\ &= g\left(e_{j_n}, \mathscr{A}\left(e^{j_1},\dots,e^{j_{n-1}},\bullet,e^{j_{n+1}},\dots,e^{j_k},e_{i_1},\dots,e_{i_l}\right)\right) \\ &= g\left(e_{j_n}, \mathbf{A}^{j_1\cdots j_{n-1}m j_{n+1}\cdots j_k}_{i_1\cdots i_l} e_m\right) \\ &= \mathbf{A}^{j_1\cdots j_{n-1}m j_{n+1}\cdots j_k}_{i_1\cdots i_l} g\left(e_{j_n}, e_m\right) \\ &= \mathbf{A}^{j_1\cdots j_{n-1}m j_{n+1}\cdots j_k}_{i_1\cdots i_l}\mathbf{g}_{m j_n} \end{aligned}$$

□

Since g is diagonal in the bases $\mathscr{C}$ and $\mathscr{C}_\perp$, we have the following result.

Corollary 1.1.8.7: Lowering of the n-th index

Let $\mathscr{A}$ be a tensor field of type (k,l). Then we have:

$$\begin{aligned}\mathrm{A}_{j_n i_1\cdots i_l}^{j_1\cdots j_{n-1}j_{n+1}\cdots j_k} &= \eta_{j_n} e^{2u_{j_n}} \mathrm{A}_{i_1\cdots i_l}^{j_1\cdots j_k}\\ \mathbb{A}_{j_n i_1\cdots i_l}^{j_1\cdots j_{n-1}j_{n+1}\cdots j_k} &= \eta_{j_n} \mathbb{A}_{i_1\cdots i_l}^{j_1\cdots j_k}\end{aligned}$$

Example 1.1.8.8: Two simple examples

(1) Let $\mathscr{A} := \mathbf{A}^i{}_j e_i \otimes e^j$ be a tensor field of type $(1,1)$. We denote:

$$\mathbf{A}_{kl} := [\mathscr{A}]_1 (e_k, e_l)$$

which means:

$$[\mathscr{A}]_1 = \mathbf{A}_{kl}\ e^k \otimes e^l.$$

Since:

$$[\mathscr{A}]_1 (X,Y) = g(X, \mathscr{A}(\bullet, Y))$$

and

$$\begin{aligned}\mathscr{A}(\bullet, e_l) &= \mathbf{A}^i{}_j e^j (e_l) e_i\\ &= \mathbf{A}^i{}_j \delta^j_l e_i\\ &= \mathbf{A}^i{}_l\ e_i\end{aligned}$$

we have:

$$\begin{aligned}\mathbf{A}_{kl} &= [\mathscr{A}]_1 (e_k, e_l)\\ &= g(e_k, \mathscr{A}(\bullet, e_l))\\ &= g\left(e_k, \mathbf{A}^i{}_l\ e_i\right)\\ &= \mathbf{A}^i{}_l g(e_k, e_i)\\ &= \mathbf{A}^i{}_l \mathbf{g}_{ki}\end{aligned}$$

For example, we have:

$$\begin{aligned}\mathbb{A}_{kl} &= \eta_{ki}\mathbb{A}^i{}_l\\ &= \eta_k \mathbb{A}^k{}_l\\ \mathrm{A}_{kl} &= \mathrm{g}_{ki}\mathrm{A}^i{}_l\\ &= \mathrm{g}_{kk}\mathrm{A}^k{}_l\\ &= \eta_k e^{2u_k}\mathrm{A}^k{}_l\end{aligned}$$

(2) Let $X := \mathbf{X}^i e_i$ be a vector field, *i.e.*, X is a tensor field of type $(1,0)$. We denote:

$$\mathbf{X}_k := [X]_1 (e_k)$$

which means:

$$X^\flat = [X]_1 = \mathbf{X}_k e^k$$

Thus, we have:

$$\begin{aligned}\mathbf{X}_k &= [X]_1 (e_k)\\ &= g(e_k, X)\\ &= g\left(e_k, \mathbf{X}^i e_i\right)\\ &= \mathbf{X}^i g(e_k, e_i)\\ &= \mathbf{X}^i \mathbf{g}_{ki}\end{aligned}$$

For example, we have:

$$\begin{aligned}\mathbb{X}_k &= \mathbb{X}^i \eta_{ki} \\ &= \eta_k \mathbb{X}^k \\ \mathrm{X}_k &= \mathrm{X}^i \mathrm{g}_{ki} \\ &= \mathrm{X}^k \mathrm{g}_{kk} \\ &= \eta_k e^{2u_k} \mathrm{X}^k\end{aligned}$$

1.1.9 Raising indices with the tensor g^*

In this subsection, we define an operation that raises indices.

Definition 1.1.9.1: The sharp musical isomorphism

The **sharp musical isomorphism associated with** g is the inverse mapping $\mathfrak{a}_g^{-1}$ of $\mathfrak{a}_g$ defined by:

$$\mathfrak{a}_g^{-1} : \quad T^*M \quad \longrightarrow \quad TM$$

We also denote, for any covector field α:

$$\alpha^\sharp := \mathfrak{a}_g^{-1}(\alpha).$$

Notation 1.1.9.2: Definition of $\mathfrak{a}_g^{-1}$ and g^*

(1) We denote:

$$\mathbf{g}^{ij} := \mathfrak{a}_g^{-1}(e^i)(e^j)$$

Hence:

$$\mathfrak{a}_g^{-1}(e^i) = \mathbf{g}^{ij} e_j.$$

Since:

$$\mathfrak{a}_g^{-1} \circ \mathfrak{a}_g = \mathrm{Id}_{TM} \quad , \quad \mathfrak{a}_g \circ \mathfrak{a}_g^{-1} = \mathrm{Id}_{T^*M}$$

we have:

$$\begin{aligned}\mathrm{Id}_{TM}(e_k) &= e_k \\ &= \delta_k^j e_j \\ \mathfrak{a}_g^{-1}\left(\mathfrak{a}_g(e_k)\right) &= \mathfrak{a}_g^{-1}\left(\mathbf{g}_{ki} e^i\right) \\ &= \mathbf{g}_{ki} \mathfrak{a}_g^{-1}\left(e^i\right) \\ &= \mathbf{g}_{ki} \mathbf{g}^{ij} e_j\end{aligned}$$

Therefore, by the uniqueness of the decomposition, we have:

$$\mathbf{g}_{ki} \mathbf{g}^{ij} = \delta_k^j.$$

(2) Let $\alpha := \boldsymbol{\alpha}_i e^i$ be a covector field. We define:

$$\boldsymbol{\alpha}^j := \alpha^\sharp(e_j)$$

Thus:

$$\alpha^\sharp = \boldsymbol{\alpha}^j e_j.$$

Since:

$$\mathfrak{a}_g^{-1}(e^i) = \mathbf{g}^{ij} e_j$$

we have, by linearity:

$$\begin{aligned}\alpha^\sharp &= \mathfrak{a}_g^{-1}(\alpha)\\ &= \mathfrak{a}_g^{-1}(\boldsymbol{\alpha}_i e^i)\\ &= \boldsymbol{\alpha}_i \mathfrak{a}_g^{-1}(e^i)\\ &= \boldsymbol{\alpha}_i \mathbf{g}^{ij} e_j.\end{aligned}$$

Hence, by the uniqueness of the decomposition, we have:

$$\boldsymbol{\alpha}^j = \boldsymbol{\alpha}_i \mathbf{g}^{ij}.$$

(3) We associate the $(2,0)$-type tensor g^* to $\mathfrak{a}_g^{-1}$, defined for any covector fields $\alpha := \boldsymbol{\alpha}_i e^i$ and $\beta := \boldsymbol{\beta}_j e^j$ as:

$$g^*(\alpha, \beta) := \beta(\mathfrak{a}_g^{-1}(\alpha)).$$

Thus:

$$\begin{aligned}g^*(e^i, e^j) &= e^j(\mathfrak{a}_g^{-1}(e^i))\\ &= e^j(\mathbf{g}^{ik} e_k)\\ &= \mathbf{g}^{ik}\delta_k^j\\ &= \mathbf{g}^{ij}\end{aligned}$$

and by bilinearity:

$$\begin{aligned}g^*(\alpha, \beta) &= g^*(\boldsymbol{\alpha}_i e^i, \boldsymbol{\beta}_j e^j)\\ &= g^*(e^i, e^j)\boldsymbol{\alpha}_i \boldsymbol{\beta}_j\\ &= \mathbf{g}^{ij}\boldsymbol{\alpha}_i \boldsymbol{\beta}_j\\ &= \boldsymbol{\alpha}^j \boldsymbol{\beta}_j\\ &= \boldsymbol{\alpha}_i \boldsymbol{\beta}^i\end{aligned}$$

with:

$$\boldsymbol{\alpha}^i e_i := \alpha^\sharp \quad , \quad \boldsymbol{\beta}^j e_j := \beta^\sharp.$$

Therefore, we have:

$$g^* = \mathbf{g}^{ij} e_i \otimes e_j$$

Matricially, we have:

$$G^{-1} = \mathscr{M}at_{\mathscr{C}^*,\mathscr{C}}(\mathfrak{a}_g^{-1}) = \mathscr{M}at_{\mathscr{C}^*}(g^*) = \left(\mathbf{g}^{ij}\right)_{ij}.$$

Example 1.1.9.3: Example of $\mathfrak{a}_g^{-1}$ and g^* in the bases $\mathscr{C}^*$ and $\mathscr{C}_\perp^*$

(1) In $\mathscr{C}^*$, we have:

$$\begin{aligned}\mathrm{g}_{ki}\mathrm{g}^{ij} &= \delta_k^j\\ \mathrm{g}_{ki} &= \eta_{ik} e^{u_k}\end{aligned}$$

So we have:

$$\mathrm{g}^{ij} = \eta_{ij} e^{-u_k}.$$

Thus we have:

$$\mathscr{M}at_{\mathscr{C}^*,\mathscr{C}}(\mathfrak{a}_g^{-1}) = \mathscr{M}at_{\mathscr{C}^*}(g^*) = \begin{pmatrix} e^{-u_0} & 0 & 0 & 0 \\ 0 & -e^{-u_1} & 0 & 0 \\ 0 & 0 & -e^{-u_2} & 0 \\ 0 & 0 & 0 & -e^{-u_3} \end{pmatrix}.$$

(2) In $\mathscr{C}_\perp^*$, we have:

$$\mathfrak{g}_{ki}\mathfrak{g}^{ij} = \delta_k^j$$
$$\mathfrak{g}_{ki} = \eta_{ki}$$

So we have:

$$\mathfrak{g}^{ij} = \eta_{ij}.$$

Thus we have:

$$\mathscr{M}at_{\mathscr{C}_\perp^*,\mathscr{C}_\perp}(\mathfrak{a}_g^{-1}) = \mathscr{M}at_{\mathscr{C}_\perp^*}(g^*) = \mathscr{M}at_{\mathscr{C}_\perp}(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Definition 1.1.9.4: Raising the n-th index

Let $\mathscr{A}$ be a tensor field of type (k,l) with $l \geq 1$ and $n \in \{1,\ldots,l\}$. The **n-th raised tensor field of $\mathscr{A}$** is the tensor field $[\mathscr{A}]^n$ of type $(k+1, l-1)$ defined for any covector fields $\alpha_0, \alpha_1, \ldots, \alpha_k$ and vector fields $X_1, \ldots, X_{n-1}, X_{n+1}, \ldots, X_l$ by:

$$[\mathscr{A}]^n(\alpha_0,\ldots,\alpha_k,X_1,\ldots,X_{n-1},X_{n+1},\ldots,X_l) \coloneqq g^*(\alpha_0, \mathscr{A}(\alpha_1,\ldots,\alpha_k,X_1\ldots,X_{n-1},\bullet,X_{n+1},\ldots,X_l)).$$

These two concepts are then related by:

$$[\mathscr{A}]^n(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_{n-1},X_{n+1},\ldots,X_l) = \mathfrak{a}_g^{-1}(\mathscr{A}(\alpha_1,\ldots,\alpha_k,X_1\ldots,X_{n-1},\bullet,X_{n+1},\ldots,X_l))(\alpha_0).$$

The following proposition shows why the term "raised" has been used.

Proposition 1.1.9.5: Raising the n-th index

Let $\mathscr{A}$ be a tensor field of type (k,l) and $n \in \{1,\ldots,l\}$. Let's define:

$$\mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_l} \coloneqq \mathscr{A}\left(e^{j_1},\ldots,e^{j_k},e_{i_1},\ldots,e_{i_l}\right)$$
$$\mathbf{A}^{i_n j_1\cdots j_k}_{i_1\cdots i_{n-1}i_{n+1}\cdots i_l} \coloneqq [\mathscr{A}]^n\left(e^{i_n},e^{j_1},\ldots,e^{j_k},e_{i_1},\ldots,e_{i_{n-1}},e_{i_{n+1}},\ldots,e_{i_l}\right)$$

Then we have:

$$\mathbf{A}^{i_n j_1\cdots j_k}_{i_1\cdots i_{n-1}i_{n+1}\cdots i_l} = \mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_{n-1}m i_{n+1}\cdots i_l}\mathbf{g}^{m i_n}.$$

Proof. We have:

$$\begin{aligned}\mathbf{A}^{i_n j_1\cdots j_k}_{i_1\cdots i_{n-1}i_{n+1}\cdots i_l} &= [\mathscr{A}]^n\left(e^{i_n},e^{j_1},\ldots,e^{j_k},e_{i_1},\ldots,e_{i_{n-1}},e_{i_{n+1}},\ldots,e_{i_l}\right) \\ &= g^*\left(e^{i_n},\mathscr{A}\left(e^{j_1},\ldots,e^{j_k},e_{i_1},\ldots,e_{i_{n-1}},\bullet,e_{i_{n+1}},\ldots,e_{i_l}\right)\right) \\ &= g^*\left(e^{i_n},\mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_{n-1}m i_{n+1}\cdots i_l}e^m\right)\end{aligned}$$

$$\begin{aligned}&= \mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_{n-1}mi_{n+1}\cdots i_l} g^*\left(e^{i_n}, e^m\right)\\ &= \mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_{n-1}mi_{n+1}\cdots i_l} \mathbf{g}^{mi_n}\end{aligned}$$

□

As g^* is diagonal in $\mathscr{C}^*$ and $\mathscr{C}^*_\perp$, we have the following result.

Corollary 1.1.9.6: Raising the i-th index

Let $\mathscr{A}$ be a tensor field of type (k,l) with $l \geq 1$. Then we have:

$$\begin{aligned}\mathrm{A}^{i_n j_1\cdots j_k}_{i_1\cdots i_{n-1}i_{n+1}\cdots i_l} &= \mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\eta_{i_n}e^{-2u_{i_n}}\\ \mathbb{A}^{i_n j_1\cdots j_k}_{i_1\cdots i_{n-1}i_{n+1}\cdots i_l} &= \mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\eta_{i_n}e^{-2u_{i_n}}\end{aligned}$$

Therefore, we see that the defined operation raises the indices of a tensor field. Let's continue the calculations with two simple examples.

Example 1.1.9.7: Two simple examples

(1) Let $\mathscr{A} := \mathbf{A}^i{}_j e_i \otimes e^j$ be a tensor field of type $(1,1)$. We denote:

$$\mathbf{A}^{kl} := [\mathscr{A}]^1\left(e^k, e^l\right).$$

Since:

$$\begin{aligned}[\mathscr{A}]^1(\alpha,\beta) &= g^*(\alpha, \mathscr{A}(\beta,\bullet))\\ \mathscr{A}\left(e^l,\bullet\right) &= \mathbf{A}^i{}_j e_i\left(e^l\right)e^j\\ &= \mathbf{A}^i{}_j\delta^l_i e^j\\ &= \mathbf{A}^l{}_j\ e^j\end{aligned}$$

we have:

$$\begin{aligned}\mathbf{A}^{kl} &= [\mathscr{A}]^1\left(e^k, e^l\right)\\ &= g^*\left(e^k, \mathscr{A}\left(e^l,\bullet\right)\right)\\ &= g^*\left(e^k, \mathbf{A}^l{}_j\ e^j\right)\\ &= \mathbf{A}^l{}_j g^*\left(e^k, e^j\right)\\ &= \mathbf{A}^l{}_j \mathbf{g}^{kj}\end{aligned}$$

Therefore, for example:

$$\begin{aligned}\mathbb{A}^{kl} &= \eta^{kj}\mathbb{A}^l{}_j\\ &= \eta_k\mathbb{A}^l{}_k\\ \mathrm{A}^{kl} &= \mathrm{g}^{kj}\mathrm{A}^l{}_j\\ &= \eta_k e^{-2u_k}\mathrm{A}^l{}_k\end{aligned}$$

(2) Let $\alpha := \boldsymbol{\alpha}_i e^i$ be a covector field, *i.e.*, α is a tensor of type $(0,1)$. We denote:

$$\boldsymbol{\alpha}^k := [\alpha]^1\left(e^k\right)$$

Hence, we have:

$$\alpha^\sharp = [\alpha]_1 = \boldsymbol{\alpha}^k e_k$$

Thus, we have:

$$\begin{aligned}\boldsymbol{\alpha}^k &= [\alpha]^1 \left(e^k\right) \\ &= g^* \left(e^k, \alpha\right) \\ &= g^* \left(e^k, \boldsymbol{\alpha}_i e^i\right) \\ &= \boldsymbol{\alpha}_i g^* \left(e^k, e^i\right) \\ &= \boldsymbol{\alpha}_i \mathbf{g}^{ki}\end{aligned}$$

Therefore, for example:

$$\begin{aligned}\alpha^k &= \alpha_i \eta^{ki} \\ &= \eta^{kk} \alpha_k \\ \alpha^k &= \alpha_i \mathrm{g}^{ki} \\ &= \alpha_k \mathrm{g}^{kk} \\ &= \eta^{kk} e^{-2u_k} \alpha^k\end{aligned}$$

1.2 Connections associated with curvature

1.2.1 Lie Derivative and Lie Bracket

We start by recalling the definition of the Lie derivative and the Lie bracket along a vector field. They are useful for defining the Levi-Civita connection. We will focus on the derivative of scalar fields. By the definition of the tangent vector (see 1.1.1.1), we have defined the object "$X(f)$," which can be seen as an action of X on f. This is also known as the Lie derivative of f along X.

Definition 1.2.1.1: Lie Derivative

Let X be a vector field given by $X := X^j \partial_j$, and let f be a scalar field. The **Lie derivative along** X is defined as:

$$X(f) := X^i \partial_i(f).$$

Remark 1.2.1.2: Remark on the Lie derivative

We can extend the definition of the Lie derivative along X to ΩM (the algebra of differential forms on M) as the unique linear map:

$$\mathscr{L}_X : \Omega M \to \Omega M$$

such that:

(i) for any scalar field f, we have:

$$\mathscr{L}_X(f) := X(f) := \mathrm{d}f\,(X)\,;$$

(ii) $\mathscr{L}_X$ is a derivation of the algebra ΩM, *i.e.*:

$$\forall \alpha, \beta \in \Omega M,\ \mathscr{L}_X(\alpha \wedge \beta) = \mathscr{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathscr{L}_X(\beta);$$

(iii) the maps $\mathscr{L}_X$ and d commute.

We will only use the definition for scalar fields in the following.

Lemma 1.2.1.3: Alternative expression for the Lie derivative

Let X be a vector field. We have:

$$X(f) = \mathrm{d}f(X).$$

Proof. Since:

$$\mathrm{d}f = \partial_i(f)\mathrm{d}x^i$$

we have:

$$\begin{aligned}
\mathrm{d}f(X) &= \mathrm{d}f\left(X^j\partial_j\right) \\
&= X^j \mathrm{d}f(\partial_j) \\
&= X^j \partial_i(f)\mathrm{d}x^i(\partial_j) \\
&= X^j \partial_i(f)\delta^i_j \\
&= X^i \partial_i(f) \\
&= X(f)
\end{aligned}$$

□

This leads us to the definition of the Lie bracket.

Definition 1.2.1.4: Lie Bracket

Let X and Y be two vector fields. The **Lie bracket of X and Y** is the map $[X,Y]$ that associates to any scalar field f the scalar field:

$$[X,Y](f) := X(Y(f)) - Y(X(f)).$$

In particular, by the Schwartz theorem, we have:

$$\begin{aligned}
[\partial_i, \partial_j](f) &= \partial_i(\partial_j(f)) - \partial_j(\partial_i(f)) \\
&= \partial_{i,j}(f) - \partial_{j,i}(f) \\
&= 0
\end{aligned}$$

i.e., we have:

$$[\partial_i, \partial_j] = 0. \tag{1.2.1.1}$$

Lemma 1.2.1.5: Local descriptions

Let X and Y be two vector fields and f be a scalar field.

(i) We have:

$$Y(X(f)) = Y^j\partial_j X^i \partial_i f + Y^j X^i \partial^2_{ji} f$$

(ii) We have:

$$[X,Y](f) = \left(X^j\partial_j Y^i - Y^j \partial_j X^i\right)\partial_i f.$$

Proof. (i) Since:

$$X(f) = X^i\partial_i f$$

we have:

$$\begin{aligned}
Y(X(f)) &= Y^j\partial_j\left(X^i\partial_i f\right) \\
&= Y^j\partial_j X^i\partial_i f + Y^j X^i \partial^2_{ji} f
\end{aligned}$$

(ii) Since:

$$\partial^2_{ji} f - \partial^2_{ij} f = 0$$

we have:

$$\begin{aligned}
[X,Y](f) &= X(Y(f)) - Y(X(f)) \\
&= X^j \partial_j Y^i \partial_i f + X^j Y^i \partial^2_{ji} f - Y^j \partial_j X^i \partial_i f - Y^j X^i \partial^2_{ji} f \\
&= \left(X^j \partial_j Y^i - Y^j \partial_j X^i\right) \partial_i f + X^j Y^i \left(\partial^2_{ji} f - \partial^2_{ij} f\right) \\
&= \left(X^j \partial_j Y^i - Y^j \partial_j X^i\right) \partial_i f
\end{aligned}$$

□

Finally, we have the Jacobi identity.

Proposition 1.2.1.6: Jacobi Identity

Let X, Y, and Z be three vector fields. We have:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Proof. We have:

$$\begin{aligned}
[X,[Y,Z]](f) &= X((YZ - ZY)(f)) - (YZ - ZY)(X(f)) \\
&= X(Y(Z(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f)))
\end{aligned}$$

By removing the parentheses and the f, and permuting the variables X, Y, and Z, we have:

$$\begin{aligned}
[X,[Y,Z]] &= XYZ - XZY - YZX + ZYX \\
[Y,[Z,X]] &= YZX - YXZ - ZXY + XZY \\
[Z,[X,Y]] &= ZXY - ZYX - XYZ + YXZ
\end{aligned}$$

By summing the three equations, we get:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

□

1.2.2 Levi-Civita Connection

We start with a simplified definition of a connection that highlights the two essential properties that will be used in calculations.

Definition 1.2.2.1: Koszul Connection

A **connection** ∇ is a linear map that associates a vector field $\nabla_X Y$ to every pair of vector fields X and Y, satisfying the following two properties:

(i) (**Linearity in** X) For any scalar field f and vector fields X and Y, we have:

$$\nabla_{fX} Y = f \nabla_X Y.$$

(ii) (**Leibniz Rule**) For any scalar field f and vector fields X and Y, we have:

$$\nabla_X (fY) = \mathrm{d}f(X) Y + f \nabla_X Y = \mathrm{d}f(X) Y + \nabla_{fX} Y.$$

We then define the Levi-Civita connection as the unique torsion-free and g-parallel connection.

Proposition 1.2.2.2: The connection associated with g

The Levi-Civita connection is the unique connection ∇ on TM such that:

(i) ∇ is **torsion-free**, *i.e.*, for any vector fields X and Y, we have:

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

(ii) g is **parallel**, *i.e.*, for any vector fields X, Y, and Z, we have:

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Proof. We present a proof by analysis-synthesis.

- **Analysis-Uniqueness**: Let X and Y be two vector fields on M. Since the metric g is a symmetric non-degenerate bilinear form, to compute $\nabla_X Y$, it suffices to have an expression for $2g(\nabla_X Y, Z)$ for all vector fields Z. Using bilinearity and symmetry of g, for any vector field Z, we have:

$$\begin{aligned}
2g(\nabla_X Y, Z) &= g(2\nabla_X Y, Z)\\
&= g(\nabla_X Y, Z) + g(\nabla_X Z, Y) + g(\nabla_Y Z, X) + g(\nabla_Y X, Z) - g(\nabla_Z X, Y) - g(\nabla_Z Y, X)\\
&\quad + g(\nabla_X Y - \nabla_Y X, Z) + g(\nabla_Z X - \nabla_X Z, Y) - g(\nabla_Y Z - \nabla_Z Y, X)\\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)\\
&\quad + g([X,Y], Z) + g([Z,X], Y) - g([Y,Z], X)\\
&= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)
\end{aligned}$$

  Thus, the value of $\nabla_X Y$ is uniquely determined.

- **Synthesis-Existence**: For any vector fields X and Y, we define $\nabla_X Y$ as the unique vector field on M such that for any vector field Z, we have:

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X).$$

  Let us show that ∇ is indeed a connection on M. For any vector field Z, we have:

  (i) For any vector field Z and any scalar field $f \in \mathscr{C}^p(X, \mathbb{R})$, we have:

$$\begin{aligned}
2g(\nabla_X(fY), Z) &= X(g(fY,Z)) + (fY)(g(Z,X)) - Z(g(X,fY))\\
&\quad + g([X,fY],Z) + g([Z,X],fY) - g([fY,Z],X)\\
&= \mathrm{d}f(X)g(Y,Z) - \mathrm{d}f(Z)g(X,Y) + \mathrm{d}f(X)g(Y,Z)\\
&\quad + \mathrm{d}f(Z)g(X,Y) + f2g(\nabla_X Y, Z)\\
&= 2g(\mathrm{d}f(X)Y + f\nabla_X Y, Z),
\end{aligned}$$

  and we also have:

$$\begin{aligned}
2g(\nabla_{fX} Y, Z) =&(fX)(g(Y,Z)) + Y(g(Z,fX)) - Z(g(fX,Y))\\
&+ g([fX,Y],Z) + g([Z,fX],Y) - g([Y,Z],fX)\\
=&\mathrm{d}f(Y).g(Z,X) - \mathrm{d}f(Z).g(X,Y) - \mathrm{d}f(Y).g(X,Z)\\
&+ \mathrm{d}f(Z).g(X,Y) + f.2g(\nabla_X Y, Z)\\
=&2g(f\nabla_X Y, Z).
\end{aligned}$$

  Hence, the result follows from the degeneracy of g.

(ii) Let's show that ∇ is torsion-free. For any vector field Z, we have:

$$\begin{aligned}2g(\nabla_X Y - \nabla_Y X, Z) =& X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z)\\ &+ g([Z,X],Y) - g([Y,Z],X) - Y(g(X,Z)) - X(g(Z,Y))\\ &+ Z(g(Y,X)) - g([Y,X],Z) - g([Z,Y],X) + g([X,Z],Y)\\ =& 2g([X,Y],Z).\end{aligned}$$

Hence, the result follows from the degeneracy of g.

(iii) Let's show that g is parallel with respect to ∇. For any vector field Z, we have:

$$\begin{aligned}2g(\nabla_X Y, Z) + 2g(Y, \nabla_X Z) =& X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z)\\ &+ g([Z,X],Y) - g([Y,Z],X) + X(g(Z,Y)) + Z(g(Y,X))\\ &- Y(g(X,Z)) + g([X,Z],Y) + g([Y,X],Z) - g([Z,Y],X)\\ =& 2X(g(Y,Z)).\end{aligned}$$

Hence, the result follows from the degeneracy of g.

Thus, ∇ is the unique Koszul connection satisfying the theorem. □

Therefore, we have shown that ∇ satisfies:

$$2g(\nabla_X Y, Z) = X\left(g(Y,Z)\right) + Y\left(g(Z,X)\right) - Z\left(g(X,Y)\right) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X).$$

We extend the definition of the connection to arbitrary tensor fields. For this purpose, we define the action of ∇ on 1-forms and with respect to the operations + and $\otimes$.

Definition 1.2.2.3: Connection on tensors

Let f be a scalar field, A and B be tensor fields, X and Y be vector fields, and α be a covector field.

(i) We denote:

$$\nabla_X f := X\left(f\right).$$

(ii) We denote:

$$\left(\nabla_X \alpha\right)\left(Y\right) := X\left(\alpha(Y)\right) - \alpha\left(\nabla_X Y\right).$$

(iii) We denote:

$$\nabla_X\left(A \otimes B\right) := \left(\nabla_X A\right) \otimes B + A \otimes \left(\nabla_X B\right).$$

(iv) Suppose A and B are of the same type. We denote:

$$\nabla_X\left(A + B\right) := \nabla_X A + \nabla_X B.$$

We then have the following result, which gives an explicit formula for the action of ∇ on tensor fields.

Proposition 1.2.2.4: Formula for the connection on tensors

Let A be a tensor field of type (k,l). For any covector fields $\alpha_1, \ldots, \alpha_k$ and any vector fields $X_1, \ldots, X_l, Y$, we have:

$$\begin{aligned}&\left(\nabla_Y A\right)\left(\alpha_1, \ldots, \alpha_k, X_1, \ldots, X_l\right)\\ =& Y\left(A\left(\alpha_1, \ldots, \alpha_k, X_1, \ldots, X_l\right)\right)\\ &- A\left(\nabla_Y \alpha_1, \ldots, \alpha_k, X_1, \ldots, X_l\right) - \cdots - A\left(\alpha_1, \ldots, \nabla_Y \alpha_k, X_1, \ldots, X_l\right)\\ &- A\left(\alpha_1, \ldots, \alpha_k, \nabla_Y X_1, \ldots, X_l\right) - \cdots - A\left(\alpha_1, \ldots, \alpha_k, X_1, \ldots, \nabla_Y X_l\right)\end{aligned}$$

Proof. We fix Y as a vector field. We prove the result by a double induction on $(k,l) \in \mathbb{N}^2$. Let $\mathscr{Q}(k,l)$ be the proposition.

(i) **Case** $\mathscr{Q}(0,0)$ **true**: The result follows directly from the definition of ∇ on vector fields (see point (i) of Definition 1.2.2.3).

(ii) **Case** $\mathscr{Q}(0,l)$ **true** $\Rightarrow \mathscr{Q}(0,l+1)$ **true**: Let $l \in \mathbb{N}$ and A be a tensor field of type $(0,l+1)$. Assume that $\mathscr{Q}(k,l)$ is true. Then, by linearity of ∇ (see point (iv) of Definition 1.2.2.3), we can assume that A contains only one term of the form:

$$A := \mathbf{A} e^{i_1} \otimes \cdots \otimes e^{i_{l+1}}.$$

Let's define:

$$B := \mathbf{A} e^{i_1} \otimes \cdots \otimes e^{i_l}.$$

Therefore, we have:

$$A = B \otimes e^{i_{l+1}}.$$

Then, by point (iii) of Definition 1.2.2.3, we have:

$$\begin{aligned}\nabla_Y A &= \nabla_Y \left(B \otimes e^{i_{l+1}}\right)\\ &= (\nabla_Y B) \otimes e^{i_{l+1}} + B \otimes \left(\nabla_Y e^{i_{l+1}}\right)\end{aligned}$$

Using the Leibniz formula, we have:

$$\begin{aligned}Y\left(A\left(X_1,\ldots,X_{l+1}\right)\right) &= Y\left(B\left(X_1,\ldots,X_l\right) e^{i_{l+1}}\left(X_{l+1}\right)\right)\\ &= Y\left(B\left(X_1,\ldots,X_l\right)\right) e^{i_{l+1}}\left(X_{l+1}\right) + B\left(X_1,\ldots,X_l\right) Y\left(e^{i_{l+1}}\left(X_{l+1}\right)\right).\end{aligned}$$

Hence, by $\mathscr{Q}(0,l)$ and point (iii) of Definition 1.2.2.3, we have:

$$\begin{aligned}&\left(\nabla_Y A\right)\left(X_1,\ldots,X_{l+1}\right)\\ &= \left(\nabla_Y B\right) \otimes e^{i_{l+1}}\left(X_1,\ldots,X_{l+1}\right) + B \otimes \left(\nabla_Y e^{i_{l+1}}\right)\left(X_1,\ldots,X_{l+1}\right)\\ &= \left(\nabla_Y B\right)\left(X_1,\ldots,X_l\right) e^{i_{l+1}}\left(X_{l+1}\right) + B\left(X_1,\ldots,X_l\right)\left(\nabla_Y e^{i_{l+1}}\right)\left(X_{l+1}\right)\\ &= Y\left(B\left(X_1,\ldots,X_l\right)\right) e^{i_{l+1}}\left(X_{l+1}\right) - B\left(\nabla_Y X_1,\ldots,X_l\right) e^{i_{l+1}}\left(X_{l+1}\right) - \cdots - B\left(X_1,\ldots,\nabla_Y X_l\right) e^{i_{l+1}}\left(X_{l+1}\right)\\ &\quad + B\left(X_1,\ldots,X_l\right)\left(Y\left(e^{i_{l+1}}\left(X_{l+1}\right)\right) - e^{i_{l+1}}\left(\nabla_Y X_{l+1}\right)\right)\\ &= Y\left(A\left(X_1,\ldots,X_{l+1}\right)\right) - A\left(\nabla_Y X_1,\ldots,X_{l+1}\right) - \cdots - A\left(X_1,\ldots,\nabla_Y X_{l+1}\right)\end{aligned}$$

Thus, $\mathscr{Q}(0,l+1)$ is true.

(iii) **Cas** $\mathscr{Q}(k,l)$ **vraie** $\Rightarrow \mathscr{Q}(k+1,l)$ **vraie**. Soit $(k,l) \in \mathbb{N}^2$ et un champ tensoriel A de type (k,l). Supposons que $\mathscr{Q}(k,l)$ soit vraie. Alors par linéarité de ∇ (voir le point (iv) de la définition 1.2.2.3), on peut supposer que A ne contient qu'un seul terme du type :

$$A := \mathbf{A} e_{j_0} \otimes \cdots \otimes e^{j_k} \otimes e^{i_1} \otimes \cdots \otimes e^{i_l}.$$

Posons :

$$B := \mathbf{A} e_{j_1} \otimes \cdots \otimes e^{j_k} \otimes e^{i_1} \otimes \cdots \otimes e^{i_l}.$$

Donc on a :

$$A = e_{j_0} \otimes B.$$

On a alors par le point (iii) de la définition 1.2.2.3 :

$$\begin{aligned}\nabla_Y A &= \nabla_Y \left(e_{j_0} \otimes B\right)\\ &= \left(\nabla_Y e_{j_0}\right) \otimes B + e_{j_0} \otimes \left(\nabla_Y B\right)\end{aligned}$$

On a par la formule de Leibniz :

$$\begin{aligned}
&Y\left(A\left(\alpha_0,\ldots,\alpha_k,X_1,\ldots,X_l\right)\right)\\
=&Y\left(e_{j_0}\left(\alpha_0\right)B\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)\right)\\
=&Y\left(e_{j_0}\left(\alpha_0\right)\right)B\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)+Y\left(B\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)\right)e_{j_0}\left(\alpha_0\right).
\end{aligned}$$

Ainsi on a par $\mathscr{Q}(k,l)$ et par le point (iii) de la définition 1.2.2.3 :

$$\begin{aligned}
&\left(\nabla_Y A\right)\left(\alpha_0,\ldots,\alpha_k,X_1,\ldots,X_l\right)\\
=&\left(\nabla_Y e_{j_0}\right)\otimes B\left(\alpha_0,\ldots,\alpha_k,X_1,\ldots,X_l\right)+e_{j_0}\otimes\left(\nabla_Y B\right)\left(\alpha_0,\ldots,\alpha_k,X_1,\ldots,X_l\right)\\
=&\left(\nabla_Y e_{j_0}\right)\left(\alpha_0\right)B\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)+e_{j_0}\left(\alpha_0\right)\left(\nabla_Y B\right)\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)\\
=&Y\left(e_{j_0}\left(\alpha_0\right)\right)B\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)+e_{j_0}\left(\alpha_0\right)\left[Y\left(B\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)\right)\right.\\
&-B\left(\nabla_Y\alpha_1,\ldots,\alpha_k,X_1,\ldots,X_l\right)-\cdots-B\left(\alpha_1,\ldots,\nabla_Y\alpha_k,X_1,\ldots,X_l\right)\\
&\left.-B\left(\alpha_1,\ldots,\alpha_k,\nabla_Y X_1,\ldots,X_l\right)-\cdots-B\left(\alpha_1,\ldots,\alpha_k,X_1,\ldots,\nabla_Y X_l\right)\right]\\
=&Y\left(A\left(\alpha_0,\ldots,\alpha_k,X_1,\ldots,X_l\right)\right)\\
&-A\left(\nabla_Y\alpha_0,\ldots,\alpha_k,X_1,\ldots,X_l\right)-\cdots-A\left(\alpha_0,\ldots,\nabla_Y\alpha_k,X_1,\ldots,X_l\right)\\
&-A\left(\alpha_0,\ldots,\alpha_k,\nabla_Y X_1,\ldots,X_l\right)-\cdots-A\left(\alpha_0,\ldots,\alpha_k,X_1,\ldots,\nabla_Y X_l\right)
\end{aligned}$$

So $\mathscr{Q}(k+1,l)$ is true.

(iv) By the double induction principle, $\mathscr{Q}(k,l)$ is true for all $(k,l)\in\mathbb{N}^2$.

□

Notation 1.2.2.5: Notations for components

Let $\mathscr{A}$ be a tensor field of type (k,l). We will use a component notation for the action of ∇ on $\mathscr{A}$.

(1) Suppose the local description of $\mathscr{A}$ is given by:

$$\mathscr{A}=\mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ e_{j_1}\otimes\cdots\otimes e_{j_k}\otimes e^{i_1}\otimes\cdots\otimes e^{i_l}.$$

We then write:

$$\nabla_m\mathbf{A}^{j_1\cdots j_k}_{i_1\cdots i_l}:=\nabla_{e_m}\mathscr{A}\left(e^{j_1},\ldots,e^{j_k},e_{i_1},\ldots,e_{i_l}\right).$$

(2) Suppose the local description of $\mathscr{A}$ is given by:

$$\mathscr{A}=\mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ \partial_{j_1}\otimes\cdots\otimes\partial_{j_k}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_l}.$$

We then write:

$$\nabla_m\mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l}:=\nabla_{\partial_m}\mathscr{A}\left(\mathrm{d}x^{j_1},\ldots,\mathrm{d}x^{j_k},\partial_{i_1},\ldots,\partial_{i_l}\right).$$

(3) Suppose the local description of $\mathscr{A}$ is given by:

$$\mathscr{A}=\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\ \theta_{j_1}\otimes\cdots\otimes\theta_{j_k}\otimes\theta^{i_1}\otimes\cdots\otimes\theta^{i_l}.$$

We then write:

$$\nabla_m\mathbb{A}^{j_1\cdots j_k}_{i_1\cdots i_l}:=\nabla_{\theta_m}\mathscr{A}\left(\theta^{j_1},\ldots,\theta^{j_k},\theta_{i_1},\ldots,\theta_{i_l}\right).$$

Thus, the writing style of the component of $\mathscr{A}$ indicates whether m refers to e_m, ∂_m, or θ_m.

1.2.3 Trace and divergence

Throughout this subsection, we fix a tensor field $\mathscr{A}$ of type (k, l). We use the notations introduced in tensor field contraction (see definition 1.1.2.5). We begin with a standard notation for the action of nabla on tensor fields.

Notation 1.2.3.1: The tensor field $\nabla\mathscr{A}$

We denote by $\nabla\mathscr{A}$ the tensor field of type $(k, l+1)$ defined by:

$$\nabla\mathscr{A}\,(\alpha_1, \dots, \alpha_k, X_1, \dots, X_{l+1}) \coloneqq (\nabla_{X_{l+1}}\mathscr{A})\,(\alpha_1, \dots, \alpha_k, X_1, \dots, X_l)$$

Definition 1.2.3.2: Trace of a tensor field

(i) Suppose $k, l \geq 1$. Let $m \in \{1, \dots, k\}$ and $n \in \{1, \dots, l\}$. The **trace of $\mathscr{A}$ along** (n, m) is given by:

$$\mathrm{tr}_{(m,n)}\mathscr{A} \coloneqq [\mathscr{A}]^m_n\,.$$

(a) If $k \coloneqq 1$ (thus $m \coloneqq 1$), it is simply denoted as:

$$\mathrm{tr}_{(n)}\mathscr{A} \coloneqq [\mathscr{A}]^1_n\,.$$

(b) If $l \coloneqq 1$ (thus $n \coloneqq 1$), it is simply denoted as:

$$\mathrm{tr}_{(m)}\mathscr{A} \coloneqq [\mathscr{A}]^m_1\,.$$

(ii) Suppose $l \geq 2$ and $k \coloneqq 0$. Let $m \in \{1, \dots, l-1\}$ and $n \in \{1, \dots, l\}$. The **g-trace of $\mathscr{A}$ along** (m, n) is given by:

$$\mathrm{tr}_{g,(m,n)}\mathscr{A} \coloneqq [[\mathscr{A}]^n]^1_m\,.$$

Let's examine the type of tensor fields appearing in the definitions.

(i) We have:

- $\mathscr{A}$ is a tensor field of type (k, l).
- $\mathrm{tr}_{(m,n)}\mathscr{A}$ is a tensor field of type $(k-1, l-1)$.

(ii) We have:

- $\mathscr{A}$ is a tensor field of type $(0, l)$.
- $[\mathscr{A}]^n$ is a tensor field of type $(1, l-1)$.
- $\mathrm{tr}_{g,(m,n)}\mathscr{A}$ is a tensor field of type $(0, l-2)$.

Example 1.2.3.3: Example of the trace of a tensor field of type $(1, 1)$

Let $\mathscr{A}$ be a tensor field of type $(0, 2)$. Suppose that $\mathscr{A}$ is **symmetric**, meaning that for all vector fields X and Y, we have $\mathscr{A}(X, Y) = \mathscr{A}(Y, X)$. Then we have:

$$[\mathscr{A}]^1 = [\mathscr{A}]^2.$$

Therefore, we have:

$$\mathrm{tr}_{g,(1,1)}\mathscr{A} = \mathrm{tr}_{g,(2,1)}\mathscr{A}\,.$$

In this case, we simply write:

$$\mathrm{tr}_g\mathscr{A} \coloneqq \mathrm{tr}_{g,(i,1)}\mathscr{A}\,.$$

Definition 1.2.3.4: Divergence of a tensor field

(i) Suppose $k \geq 1$. Let $n \in \{1, \ldots, k\}$. The **divergence of $\mathscr{A}$ along n** is given by:

$$\mathrm{div}_{(n)}\mathscr{A} := [\nabla\mathscr{A}]^{n}_{l+1}\,.$$

If $k := 1$ (thus $n := 1$), it is simply denoted as:

$$\mathrm{div}\mathscr{A} := \mathrm{div}_{(1)}\mathscr{A} := [\nabla\mathscr{A}]^{1}_{l+1}\,.$$

(ii) Suppose $l \geq 1$. Let $n \in \{1, \ldots, l\}$. The **g-divergence of $\mathscr{A}$ along n** is given by:

$$\mathrm{div}_{g,(n)}\mathscr{A} := [\nabla\,[\mathscr{A}]^{n}]^{1}_{l+1}\,.$$

If $l := 1$ (thus $n := 1$), it is simply denoted as:

$$\mathrm{div}_{g}\mathscr{A} := \mathrm{div}_{g,(1)}\mathscr{A} := [\nabla\mathscr{A}]^{1}_{l+1}\,.$$

Let's examine the type of tensor fields appearing in the definitions.

(i) We have:

- $\mathscr{A}$ is a tensor field of type (k, l).
- $\nabla\mathscr{A}$ is a tensor field of type $(k, l+1)$.
- $\mathrm{div}_{(n)}\mathscr{A}$ is a tensor field of type $(k-1, l)$.

(ii) We have:

- $\mathscr{A}$ is a tensor field of type (k, l).
- $[\mathscr{A}]^{n}$ is a tensor field of type $(k+1, l-1)$.
- $\nabla[\mathscr{A}]^{n}$ is a tensor field of type $(k+1, l)$.
- $\mathrm{div}_{g,(n)}\mathscr{A}$ is a tensor field of type $(k, l-1)$.

Example 1.2.3.5: Divergences of a vector field and covector field in $\mathscr{C}$ and $\mathscr{C}^*$

(1) Let X be a vector field given by $X := X^{j}\partial_{j}$. By the definition of a connection, we have:

$$\begin{aligned}
\mathrm{div}(X) &= [\nabla X]^{1}_{1} \\
&= \mathrm{d}x^{i}\left(\nabla_{\partial_i} X\right) \\
&= \mathrm{d}x^{i}\left(\nabla_{\partial_i}\left(X^{j}\partial_{j}\right)\right) \\
&= \mathrm{d}x^{i}\left(\mathrm{d}X^{j}\left(\partial_{i}\right)\partial_{j} + X^{j}\nabla_{\partial_i}\partial_{j}\right) \\
&= \mathrm{d}x^{i}\left(\partial_{i}X^{j}\partial_{j} + X^{j}\nabla_{\partial_i}\partial_{j}\right) \\
&= \partial_{i}X^{j}\delta^{i}_{j} + X^{j}\Gamma^{i}_{ij} \\
&= \partial_{i}X^{i} + X^{j}\Gamma^{i}_{ij}
\end{aligned}$$

(2) Let α be a covector field given by $\alpha := \alpha_{j}\mathrm{d}x^{j}$. The **divergence of α** is defined as:

$$\mathrm{div}_{g}\alpha := \mathrm{div}\alpha^{\sharp} := \mathrm{div}[\alpha]^{1}.$$

(3) Let $\mathscr{A}$ be a tensor field of type $(0, 2)$. Suppose that $\mathscr{A}$ is **symmetric**, meaning that for all vector fields X and Y, we have $\mathscr{A}(X, Y) = \mathscr{A}(Y, X)$. Then we have:

$$[\mathscr{A}]^{1} = [\mathscr{A}]^{2}$$

Therefore, we have:

$$\mathrm{div}_{g,(1)}\mathscr{A} = \mathrm{div}_{g,(2)}\mathscr{A}$$

In this case, we simply write:

$$\mathrm{div}_{g}\mathscr{A} := \mathrm{div}_{g,(i)}\mathscr{A}.$$

1.2.4 Definition of the connection in the bases $\mathscr{E}$ and $\mathscr{E}^*$

The elements $\nabla_{e_k} e_j$ are vector fields and thus they can be expressed in the $\mathscr{E}$ basis. Hence the following definitions.

Definition 1.2.4.1: Rotation coefficients

Let $i, j, k \in \{0, 1, 2, 3\}$.

(i) The **rotation coefficients** $\mathbf{\Gamma}^i{}_{kj}$ are defined as:

$$\mathbf{\Gamma}^i{}_{kj} := e^i\left(\nabla_{e_k} e_j\right).$$

(ii) The **matrix of 1-forms of rotation**

$$\boldsymbol{\omega} := \left(\boldsymbol{\omega}^i{}_j\right)_{ij} \in \mathscr{M}_4\left(\Lambda^1(U)\right)$$

is defined as:

$$\boldsymbol{\omega}^i{}_j := \mathbf{\Gamma}^i{}_{kj} e^k.$$

Thus, we have:

$$\boldsymbol{\omega}^i{}_j\left(e_k\right) = \mathbf{\Gamma}^i{}_{kj}$$
$$\mathbf{\Gamma}^i{}_{kj} e_i = \nabla_{e_k} e_j = \boldsymbol{\omega}^i{}_j\left(e_k\right) e_i$$

Example 1.2.4.2: Christoffel symbols and tetradic coefficients

Let $i, j, k \in \{0, 1, 2, 3\}$.

(1) The elements $\nabla_{\partial_k}\partial_j$ are vector fields and thus they can be expressed in the $\mathscr{C}$ basis.

(a) The **Christoffel symbols** $\Gamma^i{}_{kj}$ are defined as:

$$\Gamma^i{}_{kj} := \mathrm{d}x^i\left(\nabla_{\partial_k}\partial_j\right).$$

(b) The **matrix of 1-forms of Christoffel**

$$\omega := \left(\omega^i{}_j\right)_{ij} \in \mathscr{M}_4\left(\Lambda^1(U)\right)$$

is defined as:

$$\omega^i{}_j := \Gamma^i{}_{kj}\mathrm{d}x^k.$$

Thus, we have:

$$\omega^i{}_j\left(\partial_k\right) = \Gamma^i{}_{kj}$$
$$\Gamma^i{}_{kj}\partial_i = \nabla_{\partial_k}\partial_j = \omega^i{}_j\left(\partial_k\right)\theta_i$$

(2) The elements $\nabla_{\theta_k}\theta_j$ are vector fields and thus they can be expressed in the $\mathscr{C}_\perp$ basis.

(a) The **tetradic coefficients of $\mathbb{\Gamma}^i{}_{kj}$** are defined as:

$$\mathbb{\Gamma}^i{}_{kj} := \theta^i \left(\nabla_{\theta_k} \theta_j \right).$$

(b) The **matrix of tetradic 1-forms**

$$\varpi := \left(\varpi^i{}_j \right)_{ij} \in \mathscr{M}_4 \left(\Lambda^1(U) \right)$$

is defined as:

$$\varpi^i{}_j := \mathbb{\Gamma}^i{}_{kj} \theta^k.$$

Thus, we have:

$$\begin{aligned} \varpi^i{}_j \left(\theta_k \right) &= \mathbb{\Gamma}^i{}_{kj} \\ \mathbb{\Gamma}^i{}_{kj} \theta_i &= \nabla_{\theta_k} \theta_j = \varpi^i{}_j \left(\theta_k \right) \theta_i \end{aligned}$$

The following lemma establishes the relationship between $\Gamma^j{}_{ki}$ and $\mathbb{\Gamma}^j{}_{ki}$.

Lemma 1.2.4.3: Relationship between $\Gamma^j{}_{ki}$ and $\mathbb{\Gamma}^j{}_{ki}$

Let $i, j, k \in \{0, 1, 2, 3\}$ such that $i \neq j$.

(i) (a) We have:

$$\mathbb{\Gamma}^j{}_{kj} = e^{-u_k} \left(-u_{j,k} + \Gamma^j{}_{kj} \right) \qquad \Gamma^j{}_{kj} = u_{j,k} + e^{u_k} \mathbb{\Gamma}^j{}_{kj}$$

(b) We have:

$$\mathbb{\Gamma}^i{}_{kj} = e^{u_i - u_j - u_k} \Gamma^i{}_{kj} \qquad \Gamma^i{}_{kj} = e^{-u_i + u_j + u_k} \mathbb{\Gamma}^i{}_{kj}$$

(ii) (a) We have:

$$\varpi^j{}_j = \omega^j{}_j - u_{j,k} e^{-u_k} \theta^k \qquad \omega^j{}_j = \varpi^j{}_j + u_{j,k} \mathrm{d}x^k$$

(b) We have:

$$\varpi^i{}_j = e^{u_i - u_j} \omega^i{}_j \qquad \omega^i{}_j = e^{-u_i + u_j} \varpi^i{}_j$$

Proof. (i) For all scalar fields f and g, and vector fields X and Y, we have:

$$\nabla_{gX}(fY) = g\nabla_X(fY) = g\mathrm{d}f(X)Y + fg\nabla_X Y.$$

Thus, we have:

$$\begin{aligned} \nabla_{\theta_k} \theta_j &= \nabla_{e^{-u_k}\partial_k} \left(e^{-u_j} \partial_j \right) \\ &= e^{-u_k} \mathrm{d} \left(e^{-u_j} \right) (\partial_k) \partial_j + e^{-u_j - u_k} \nabla_{\partial_k} \partial_j \\ &= -u_{j,k} e^{-u_j - u_k} \partial_j + e^{-u_j - u_k} \Gamma^i{}_{kj} \partial_i \\ &= e^{-u_j - u_k} \left[\left(-u_{j,k} + \Gamma^j{}_{kj} \right) \partial_j + \sum_{i \neq j} \Gamma^i{}_{kj} \partial_i \right] \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} \nabla_{\theta_k} \theta_j &= \mathbb{\Gamma}^i{}_{kj} \theta_i \\ &= \mathbb{\Gamma}^i{}_{kj} e^{-u_i} \partial_i \end{aligned}$$

By uniqueness of the decomposition, we have for $i \neq j$:

$$\begin{aligned}\mathbb{\Gamma}^j{}_{kj} e^{-u_j} &= e^{-u_j - u_k} \left(-u_{j,k} + \Gamma^j{}_{kj}\right) \\ \mathbb{\Gamma}^i{}_{kj} e^{-u_i} &= e^{-u_j - u_k} \Gamma^i{}_{kj}\end{aligned}$$

and also:

$$\begin{aligned}\Gamma^j{}_{kj} &= u_{j,k} + e^{u_k} \mathbb{\Gamma}^j{}_{kj} \\ \Gamma^i{}_{kj} &= e^{-u_i + u_j + u_k} \mathbb{\Gamma}^i{}_{kj}\end{aligned}$$

Hence, the points (a) and (b).

(ii) (a) Using (i.a), we have:

$$\begin{aligned}\omega^j{}_j &= \Gamma^j{}_{kj} \mathrm{d}x^k \\ &= \left(u_{j,k} + e^{u_k} \mathbb{\Gamma}^j{}_{kj}\right) \mathrm{d}x^k \\ &= u_{j,k} \mathrm{d}x^k + \mathbb{\Gamma}^j{}_{kj} e^{u_k} \mathrm{d}x^k \\ &= u_{j,k} \mathrm{d}x^k + \mathbb{\Gamma}^j{}_{kj} \theta^k \\ &= \varpi^j{}_j + u_{j,k} \mathrm{d}x^k\end{aligned}$$

Thus, we have:

$$\varpi^j{}_j = \omega^j{}_j - u_{j,k} e^{-u_k} \theta^k \qquad\qquad \omega^j{}_j = \varpi^j{}_j + u_{j,k} \mathrm{d}x^k$$

(b) We have:

$$\begin{aligned}\omega^i{}_j &= \Gamma^i{}_{kj} \mathrm{d}x^k \\ &= e^{-u_i + u_j + u_k} \mathbb{\Gamma}^i{}_{kj} e^{-u_k} \theta^k \\ &= e^{-u_i + u_j} \varpi^i{}_j\end{aligned}$$

Thus, we have:

$$\varpi^i{}_j = e^{u_i - u_j} \omega^i{}_j \qquad\qquad \omega^i{}_j = e^{-u_i + u_j} \varpi^i{}_j$$

□

1.2.5 Practical Calculations of Connection in $\mathscr{C}$ and $\mathscr{C}^*$ bases

In this subsection, we calculate the Christoffel symbols using a classical approach that relates $\Gamma^i{}_{jk}$ to the tensor fields g and g^*. We begin with a proposition that provides an explicit formula for the calculation of ∇ on a tensor field using the Christoffel symbols.

Let's start with an example.

Example 1.2.5.1: Calculation Example

(1) We have:

$$\begin{aligned}\left(\nabla_{\partial_k} \mathrm{d}x^i\right)(\partial_j) &= \partial_k \left(\mathrm{d}x^i(\partial_j)\right) - \mathrm{d}x^i \left(\nabla_{\partial_k} \partial_j\right) \\ &= \partial_k \left(\delta^i_j\right) - \Gamma^i{}_{kj} \\ &= -\Gamma^i{}_{kj}\end{aligned}$$

(2) Let α be a covector field given by $\alpha := \alpha_i \mathrm{d}x^i$. Then, using (1), we have:

$$\begin{aligned}(\nabla_{\partial_k}\alpha)(\partial_j) &= \left(\nabla_{\partial_k}\left(\alpha_i \mathrm{d}x^i\right)\right)(\partial_j)\\ &= \partial_k\left(\alpha_i \mathrm{d}x^i(\partial_j)\right) - \alpha_i \mathrm{d}x^i\left(\nabla_{\partial_k}\partial_j\right)\\ &= \partial_k\alpha_j - \alpha_i {\Gamma^i}_{kj}\end{aligned}$$

(3) Let A be a tensor field given by $A := A_{il}\mathrm{d}x^i \otimes \mathrm{d}x^l$. Then, using (1) and (2), we have:

$$\begin{aligned}(\nabla_{\partial_k}A)(\partial_j,\partial_p) &= \nabla_{\partial_k}\left(A_{il}\mathrm{d}x^i\otimes \mathrm{d}x^l\right)(\partial_j,\partial_p)\\ &= \left(\nabla_{\partial_k}\left(A_{il}\mathrm{d}x^i\right)\otimes \mathrm{d}x^l\right)(\partial_j,\partial_p) + \left(A_{il}\mathrm{d}x^i\otimes\nabla_{\partial_k}\left(\mathrm{d}x^l\right)\right)(\partial_j,\partial_p)\\ &= \nabla_{\partial_k}\left(A_{il}\mathrm{d}x^i\right)(\partial_j)\mathrm{d}x^l(\partial_p) + A_{il}\mathrm{d}x^i(\partial_j)\nabla_{\partial_k}\left(\mathrm{d}x^l\right)(\partial_p)\\ &= \left(\partial_k A_{jl} - A_{il}{\Gamma^i}_{kj}\right)\delta^l_p - A_{il}\delta^i_j\Gamma^l_{kp}\\ &= \partial_k A_{jp} - A_{ip}{\Gamma^i}_{kj} - A_{jl}\Gamma^l_{kp}\end{aligned}$$

Proposition 1.2.5.2: Formula for the Connection on Tensors

Let $\mathscr{A}$ be a tensor field of type (k,l) with the following local coordinate representation:

$$\mathscr{A} = \mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\,\partial_{j_1}\otimes\cdots\otimes\partial_{j_k}\otimes \mathrm{d}x^{i_1}\otimes\cdots\otimes \mathrm{d}x^{i_l}.$$

Then we have:

$$\begin{aligned}\nabla_m \mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l} =& \partial_m \mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l}\\ &+ {\Gamma^{j_1}}_{nm}\mathrm{A}^{nj_2\cdots j_k}_{i_1\cdots i_l} + \cdots + {\Gamma^{j_k}}_{nm}\mathrm{A}^{j_1\cdots j_{k-1}n}_{i_1\cdots i_l}\\ &- {\Gamma^n}_{i_1 m}\mathrm{A}^{j_1\cdots j_k}_{ni_2\cdots i_l} - \cdots - {\Gamma^n}_{i_l m}\mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_{l-1}n}\end{aligned}$$

Proof. Using the notations 1.2.2.5, we have:

$$\nabla_m \mathrm{A}^{j_1\cdots j_k}_{i_1\cdots i_l} = \nabla_{\partial_m}\mathscr{A}\left(\mathrm{d}x^{j_1},\ldots,\mathrm{d}x^{j_k},\partial_{i_1},\ldots,\partial_{i_l}\right).$$

We have three relations:

- For any scalar field f, we have:

$$\nabla_{\partial_m} f = \partial_m f$$

- By the definition of ${\Gamma^n}_{im}$, we have:

$$\nabla_{\partial_m}\partial_i = {\Gamma^n}_{im}\partial_n$$

- From point (1) of Example 1.2.5.1, we have:

$$\nabla_{\partial_m}\mathrm{d}x^j = -{\Gamma^j}_{nm}\mathrm{d}x^n.$$

Therefore, using Proposition 1.2.2.4, we obtain:

$$
\begin{aligned}
&\nabla_m A^{j_1\cdots j_k}_{i_1\cdots i_l}\\
=&\left(\nabla_{\partial_m} A\right)\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_{i_l}\right)\\
=&\nabla_{\partial_m}\left(A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_{i_l}\right)\right)\\
&-A\left(\nabla_{\partial_m}\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_{i_l}\right)-\dots-A\left(\mathrm{d}x^{j_1},\dots,\nabla_{\partial_m}\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_{i_l}\right)\\
&-A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\nabla_{\partial_m}\partial_{i_1},\dots,\partial_{i_l}\right)-\dots-A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\nabla_{\partial_m}\partial_{i_l}\right)\\
=&\nabla_{\partial_m}A^{j_1\cdots j_k}_{i_1\cdots i_l}\\
&-A\left(-\Gamma^{j_1}{}_{nm}\mathrm{d}x^n,\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_{i_l}\right)-\dots-A\left(\mathrm{d}x^{j_1},\dots,-\Gamma^{j_k}{}_{nm}\mathrm{d}x^n,\partial_{i_1},\dots,\partial_{i_l}\right)\\
&-A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\Gamma^{n}{}_{i_1 m}\partial_n,\dots,\partial_{i_l}\right)-\dots-A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\Gamma^{n}{}_{i_l m}\partial_n\right)\\
=&\nabla_{\partial_m}A^{j_1\cdots j_k}_{i_1\cdots i_l}\\
&+\Gamma^{j_1}{}_{nm}A\left(\mathrm{d}x^n,\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_{i_l}\right)+\dots+\Gamma^{j_k}{}_{nm}A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^n,\partial_{i_1},\dots,\partial_{i_l}\right)\\
&-\Gamma^{n}{}_{i_1 m}A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_n,\dots,\partial_{i_l}\right)-\dots-\Gamma^{n}{}_{i_l m}A\left(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_k},\partial_{i_1},\dots,\partial_n\right)\\
=&\partial_m A^{j_1\cdots j_k}_{i_1\cdots i_l}\\
&+\Gamma^{j_1}{}_{nm}A^{nj_2\cdots j_k}_{i_1\cdots i_l}+\dots+\Gamma^{j_k}{}_{nm}A^{j_1\cdots j_{k-1}n}_{i_1\cdots i_l}\\
&-\Gamma^{n}{}_{i_1 m}A^{j_1\cdots j_k}_{ni_2\cdots i_l}-\dots-\Gamma^{n}{}_{i_l m}A^{j_1\cdots j_k}_{i_1\cdots i_{l-1}n}
\end{aligned}
$$

□

We could generalize this result in the bases $\mathscr{E}$ and $\mathscr{E}^*$, but the result is not practical.
The connection ∇ is g-parallel, *i.e.*,

$$\nabla_{\partial_j} g = 0.$$

From this, we deduce the following usual result.

Lemma 1.2.5.3: Expression of Christoffel symbols in terms of the metric

We have:

$$\Gamma^i{}_{kl} = \frac{1}{2}\mathrm{g}^{im}\left(\mathrm{g}_{mk,l}+\mathrm{g}_{ml,k}-\mathrm{g}_{kl,m}\right).$$

Proof. Using point (3) of Example 1.2.5.1, we have:

$$
\begin{aligned}
0 &= \nabla_{\partial_i} g\left(\partial_m,\partial_n\right)\\
&= \mathrm{g}_{mn,i}-\mathrm{g}_{jn}\Gamma^j_{im}-\mathrm{g}_{ml}\Gamma^l_{in}\\
&= \mathrm{g}_{mn,i}-\mathrm{g}_{pn}\Gamma^p_{im}-\mathrm{g}_{mp}\Gamma^p_{in}
\end{aligned}
$$

Hence, we have:

$$\mathrm{g}_{mn,i}=\mathrm{g}_{pn}\Gamma^p_{im}+\mathrm{g}_{mp}\Gamma^p_{in}.$$

By permuting the indices and exponents, we obtain:

$$
\begin{aligned}
(1):&\quad \mathrm{g}_{mk,l}=\mathrm{g}_{pk}\Gamma^p_{lm}+\mathrm{g}_{mp}\Gamma^p_{lk}\\
(2):&\quad \mathrm{g}_{ml,k}=\mathrm{g}_{pl}\Gamma^p_{km}+\mathrm{g}_{mp}\Gamma^p_{kl}\\
(3):&\quad \mathrm{g}_{kl,m}=\mathrm{g}_{pl}\Gamma^p_{mk}+\mathrm{g}_{kp}\Gamma^p_{ml}
\end{aligned}
$$

By taking (1) + (2) − (3), we have:

$$\begin{aligned}
\frac{1}{2}g^{im}(g_{mk,l} + g_{ml,k} - g_{kl,m}) &= \frac{1}{2}g^{im}(g_{pk}\Gamma^p_{lm} + g_{mp}\Gamma^p_{lk} + g_{pl}\Gamma^p_{km} + g_{mp}\Gamma^p_{kl} - g_{pl}\Gamma^p_{mk} - g_{kp}\Gamma^p_{ml}) \\
&= \frac{1}{2}g^{im}2\Gamma^p_{lk}g_{mp} \\
&= \Gamma^p_{lk}g^{im}g_{mp} \\
&= \Gamma^p_{lk}\delta^i_p \\
&= \Gamma^i{}_{kl}
\end{aligned}$$

□

From this, we obtain the exact values of $\Gamma^j{}_{ki}$.

Theorem 1.2.5.4: Exact values of $\Gamma^j{}_{ki}$

Let $i, j, k \in \{0, 1, 2, 3\}$ **distinct**, and l arbitrary.

(i) We have:

$$\Gamma^i{}_{li} = \Gamma^i{}_{il} = u_{i,l}.$$

(ii) We have:

$$\Gamma^j{}_{ii} = -\eta_i\eta_j u_{i,j}e^{2u_i - 2u_j}.$$

(iii) We have:

$$\Gamma^j{}_{ki} = 0.$$

Proof. (i) Since g is symmetric, we have:

$$\begin{aligned}
\Gamma^i{}_{li} &= \frac{1}{2}g^{im}(g_{ml,i} + g_{mi,j} - g_{li,m}) \\
&= \frac{1}{2}g^{im}(g_{mi,l} + g_{ml,i} - g_{il,m}) \\
&= \Gamma^i{}_{il}
\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}
\Gamma^i{}_{li} &= \frac{1}{2}g^{im}(g_{ml,i} + g_{mi,l} - g_{li,m}) \\
&= \frac{1}{2}g^{ii}(g_{il,i} + g_{ii,l} - g_{li,i}) \\
&= \frac{1}{2}g^{ii}g_{ii,l} \\
&= u_{i,l}
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\Gamma^j{}_{ii} &= \frac{1}{2}g^{jm}(g_{mi,i} + g_{mi,i} - g_{ii,m}) \\
&= \frac{1}{2}g^{jj}(g_{ji,i} + g_{ji,i} - g_{ii,j}) \\
&= -\frac{1}{2}g^{jj}g_{ii,j} \\
&= -\eta_i\eta_j u_{i,j}e^{2u_i - 2u_j}
\end{aligned}$$

(iii) Since g is diagonal, we directly have:

$$\begin{aligned}
\Gamma^j{}_{ki} &= \frac{1}{2}\mathrm{g}^{jm}(\mathrm{g}_{mk,i} + \mathrm{g}_{mi,k} - \mathrm{g}_{ki,m}) \\
&= \frac{1}{2}\mathrm{g}^{jj}(\mathrm{g}_{jk,i} + \mathrm{g}_{ji,k} - 0) \\
&= \frac{1}{2}\mathrm{g}^{jj}(0 + 0) \\
&= 0
\end{aligned}$$

□

We denote:

$$\Gamma^i := \begin{pmatrix} \Gamma^i{}_{00} & \Gamma^i{}_{01} & \Gamma^i{}_{02} & \Gamma^i{}_{03} \\ \Gamma^i{}_{10} & \Gamma^i{}_{11} & \Gamma^i{}_{12} & \Gamma^i{}_{13} \\ \Gamma^i{}_{20} & \Gamma^i{}_{21} & \Gamma^i{}_{22} & \Gamma^i{}_{23} \\ \Gamma^i{}_{30} & \Gamma^i{}_{31} & \Gamma^i{}_{32} & \Gamma^i{}_{33} \end{pmatrix}.$$

- Case $i := 0$. We have:

$$\Gamma^0 = \begin{pmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} \\ u_{0,1} & u_{1,0}e^{-2u_0+2u_1} & 0 & 0 \\ u_{0,2} & 0 & u_{2,0}e^{-2u_0+2u_2} & 0 \\ u_{0,3} & 0 & 0 & u_{3,0}e^{-2u_0+2u_3} \end{pmatrix}.$$

- Case $i := 1$. We have:

$$\Gamma^1 = \begin{pmatrix} u_{0,1}e^{-2u_1+2u_0} & u_{1,0} & 0 & 0 \\ u_{1,0} & u_{1,1} & u_{1,2} & u_{1,3} \\ 0 & u_{1,2} & -u_{2,1}e^{-2u_1+2u_2} & 0 \\ 0 & u_{1,3} & 0 & -u_{3,1}e^{-2u_1+2u_3} \end{pmatrix}.$$

- Case $i := 2$. We have:

$$\Gamma^2 = \begin{pmatrix} u_{0,2}e^{-2u_2+2u_0} & 0 & u_{2,0} & 0 \\ 0 & -u_{1,2}e^{-2u_2+2u_1} & u_{2,1} & 0 \\ u_{2,0} & u_{2,1} & u_{2,2} & u_{2,3} \\ 0 & 0 & u_{2,3} & -u_{3,2}e^{-2u_2+2u_3} \end{pmatrix}.$$

- Case $i := 3$. We have:

$$\Gamma^3 = \begin{pmatrix} u_{0,3}e^{-2u_3+2u_0} & 0 & 0 & u_{3,0} \\ 0 & -u_{1,3}e^{-2u_3+2u_1} & 0 & u_{3,1} \\ 0 & 0 & -u_{2,3}e^{-2u_3+2u_2} & u_{3,2} \\ u_{3,0} & u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix}.$$

Example 1.2.5.5: Sequence 2 of the metric h

We resume the example with the metric h (see 1.1.4.1 and 1.1.5.1) with:

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

We have:

$$\Gamma^0 = \begin{pmatrix} \dot{u} & u' & 0 & 0 \\ u' & \dot{v}e^{-2u+2v} & 0 & 0 \\ 0 & 0 & r^2\dot{b}e^{2b-2u} & 0 \\ 0 & 0 & 0 & r^2\dot{b}\sin^2\vartheta e^{2b-2u} \end{pmatrix}$$

$$\Gamma^1 = \begin{pmatrix} u'e^{2u-2v} & \dot{v} & 0 & 0 \\ \dot{v} & v' & 0 & 0 \\ 0 & 0 & -re^{2b-2v} & 0 \\ 0 & 0 & 0 & -r\sin^2\vartheta e^{2b-2v} \end{pmatrix}$$

$$\Gamma^2 = \begin{pmatrix} 0 & 0 & \dot{b} & 0 \\ 0 & 0 & r^{-1} & 0 \\ \dot{b} & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & -\cos\vartheta\sin\vartheta \end{pmatrix}$$

$$\Gamma^3 = \begin{pmatrix} 0 & 0 & 0 & \dot{b} \\ 0 & 0 & 0 & r^{-1} \\ 0 & 0 & 0 & \cot\vartheta \\ \dot{b} & r^{-1} & \cot\vartheta & 0 \end{pmatrix}$$

Thus, we have:

$$\omega = \begin{pmatrix} \dot{u}\mathrm{d}x^0 + u'\mathrm{d}x^1 & u'\mathrm{d}x^0 + \dot{v}e^{-2u+2v}\mathrm{d}x^1 & r^2\dot{b}e^{2b-2u}\mathrm{d}x^2 & r^2\dot{b}\sin^2\vartheta e^{-2b-2u}\mathrm{d}x^3 \\ u'e^{2u-2v}\mathrm{d}x^0 + \dot{v}\mathrm{d}x^1 & \dot{v}\mathrm{d}x^0 + v'\mathrm{d}x^1 & -re^{2b-2v}\mathrm{d}x^2 & -r\sin^2\vartheta e^{2b-2v}\mathrm{d}x^3 \\ \dot{b}\mathrm{d}x^2 & r^{-1}\mathrm{d}x^2 & \dot{b}\mathrm{d}x^0 + r^{-1}\mathrm{d}x^1 & -\cos\vartheta\sin\vartheta\mathrm{d}x^3 \\ \dot{b}\mathrm{d}x^3 & r^{-1}\mathrm{d}x^3 & \cot\vartheta\mathrm{d}x^3 & \dot{b}\mathrm{d}x^0 + r^{-1}\mathrm{d}x^1 + \cot\vartheta\mathrm{d}x^2 \end{pmatrix}.$$

We conclude this subsection with another formula for computing the sum $\sum_{i=0}^{3}\Gamma^i_{ij}$.

Proposition 1.2.5.6: Another formula for Γ^i_{ij}

For $j \in \{0,1,2,3\}$, we have:

$$\sum_{i=0}^{3}\Gamma^i_{ij} = \partial_j\left(\ln\sqrt{|\det g|}\right).$$

Proof. We prove the result in two steps:

(1) For any $A \in \mathrm{GL}_n(\mathbb{R})$, we have:

$$d_A\det : H \mapsto \det(A)\mathrm{tr}(A^{-1}H).$$

(2) We have:

$$\sum_{i=0}^{3}\Gamma^i_{ij} = \partial_j\left(\ln\sqrt{|\det g|}\right).$$

(1) We start with a result from linear algebra. We equip $\mathscr{M}_n(\mathbb{R})$ with the **Frobenius norm**:

$$\forall M \in \mathscr{M}_n(\mathbb{R}),\ \|M\| := \sqrt{\mathrm{tr}({}^tMM)}.$$

This norm is a sub-multiplicative norm on $\mathscr{M}_n(\mathbb{R})$, *i.e.*, we have:

$$\forall A,B \in \mathscr{M}_n(\mathbb{R}),\ \|AB\| \le \|A\|.\|B\|.$$

Let us show that for any $A \in \mathrm{GL}_n(\mathbb{R})$, we have:

$$d_A\det : H \mapsto \det(A)\mathrm{tr}(A^{-1}H).$$

To do this, let us first calculate the differential of det at I_n. Since det is of class $\mathscr{C}^\infty$, it suffices to calculate the derivative of det at I_n in an arbitrary direction $H \in \mathscr{M}_n(\mathbb{R})$. Let H be a real matrix $H \in \mathscr{M}_n(\mathbb{R})$ with eigenvalues $\lambda_1, \dots, \lambda_n$. For any real number t in the vicinity of 0, we have:

$$\det(I_n + tH) = \prod_{i=1}^{n}(1 + t\lambda_n) = 1 + t\sum_{i=1}^{n}\lambda_i + O(t^2) = 1 + t\ \mathrm{tr}(H) + O(t^2).$$

Since the map $H \mapsto \mathrm{tr}(H)$ is linear, we have:

$$d_{I_n}\det : H \mapsto \mathrm{tr}(H).$$

Let $A \in \mathrm{GL}_n(\mathbb{R})$. We deduce that for any matrix $H \in \mathscr{M}_n(\mathbb{R})$:

$$\begin{aligned}
\det(A + H) &= \det(A(I_n + A^{-1}H) \\
&= \det(A)\det(I_n + A^{-1}H) \\
&= \det(A)(\det(I_n) + d_{I_n}(A^{-1}H) + o(\|A^{-1}H\|)) \\
&= \det(A) + \det(A)\mathrm{tr}(A^{-1}H) + o(\|A^{-1}H\|)\det(A)
\end{aligned}$$

Now, $\|.\|$ is a sub-matrix norm, so we have $\|A^{-1}H\| \le \|A^{-1}\|.\|H\|$. Thus:

$$o(\|A^{-1}H\|)\det(A) = o(\|H\|).$$

Since the map $H \mapsto \det(A)\mathrm{tr}(A^{-1}H)$ is linear, we have:

$$d_A\det : H \mapsto \det(A)\mathrm{tr}(A^{-1}H).$$

(2) Since:

$$\Gamma^k_{ij} = \frac{1}{2}\mathrm{g}^{kl}\left(\mathrm{g}_{lj,i} + \mathrm{g}_{li,j} - \mathrm{g}_{ij,l}\right)$$

we have:

$$\begin{aligned}
\sum_{i=0}^{3}\Gamma^i_{ij} &= \sum_{i,l=0}^{3}\frac{1}{2}\left(\mathrm{g}^{il}\mathrm{g}_{lj,i} + \mathrm{g}^{il}\mathrm{g}_{il,j} - \mathrm{g}^{il}\mathrm{g}_{ij,l}\right) \\
&= \frac{1}{2}\left[\sum_{i,l=0}^{3}\mathrm{g}^{il}\mathrm{g}_{lj,i} + \sum_{i,l=0}^{3}\mathrm{g}^{il}\mathrm{g}_{il,j} - \sum_{i,l=0}^{3}\mathrm{g}^{il}\mathrm{g}_{ij,l}\right] \\
&= \frac{1}{2}\left[\sum_{i,l=0}^{3}\mathrm{g}^{il}\mathrm{g}_{lj,i} + \sum_{i,l=0}^{3}\mathrm{g}^{il}\mathrm{g}_{il,j} - \sum_{i,l=0}^{3}\mathrm{g}^{il}\mathrm{g}_{lj,i}\right] \\
&= \frac{1}{2}\sum_{i,l=0}^{3}\mathrm{g}^{il}\mathrm{g}_{il,j}
\end{aligned}$$

Using (1), we have:

$$\begin{aligned}
\partial_j\left(\det g\right)\mathrm{d}x^j &= \mathrm{d}(\det g) \\
&= (\det g)\,\mathrm{g}^{il}\mathrm{d}\mathrm{g}_{il} \\
&= (\det g)\,\mathrm{g}^{il}\mathrm{g}_{il,j}\mathrm{d}x^j
\end{aligned}$$

Thus:

$$\mathrm{g}^{il}\mathrm{g}_{il,j} = \frac{1}{\det g}\partial_j\left(\det g\right)$$

Therefore, we have:

$$\begin{aligned}
\sum_{i=0}^{3}\Gamma^i_{ij} &= \frac{1}{2\det g}\partial_j\left(\det g\right) \\
&= \frac{1}{\sqrt{|\det g|}}\partial_j\sqrt{|\det g|} \\
&= \partial_j\left(\ln\sqrt{|\det g|}\right).
\end{aligned}$$

□

Corollary 1.2.5.7: Alternative formulas for Γ^i_{ij}

For $j \in \{0,1,2,3\}$, we have:

$$\sum_{i=0}^{3} \Gamma^i_{ij} = \sum_{i=0}^{3} u_{i,j}.$$

Proof. Since g is diagonal, we have:

$$\begin{aligned}\det g &= \prod_{i=0}^{3} \mathrm{g}_{ii} \\ &= -e^{2u_0+2u_1+2u_2+2u_3}\end{aligned}$$

Thus, we have:

$$\ln\sqrt{|\det g|} = u_0 + u_1 + u_2 + u_3.$$

And therefore we have:

$$\sum_{i=0}^{3} \Gamma^i_{ij} = \sum_{i=0}^{3} u_{i,j}.$$

□

1.2.6 Practical calculations of the connection in the $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$ bases

In this subsection, we calculate the tetradic coefficients ${\mathbb{F}^j}_{ki}$ using the tetrad method. We start with some simple properties of the symbols ${\mathbb{F}^j}_{ki}$ and the 1-forms ${\varpi^i}_j$.

Lemma 1.2.6.1: Symmetries of the symbols ${\mathbb{F}^j}_{ki}$ and ${\varpi^i}_j$

For $i, j, k, l \in \{0,1,2,3\}$, we have:

(i) We have:

$${\varpi^l}_k = -\eta_l \eta_k {\varpi^k}_l.$$

(ii) We have:

$${\mathbb{F}^j}_{ki} = -\eta_j \eta_i {\mathbb{F}^i}_{kj}.$$

(iii) We have:

$${\varpi^j}_j = 0.$$

(iv) We have:

$${\mathbb{F}^j}_{kj} = 0.$$

Proof. (i) Since η_{kl} is constant, for any vector field X, we have:

$$\begin{aligned}0 &= \nabla_X \eta_{kl}\\ &= \nabla_X g(\theta_k, \theta_l)\\ &= g(\nabla_X \theta_k, \theta_l) + g(\theta_k, \nabla_X \theta_l)\\ &= g\left(\varpi^i{}_k(X)\theta_i, \theta_l\right) + g\left(\theta_k, \varpi^i{}_l(X)\theta_i\right)\\ &= \varpi^i{}_k(X) g(\theta_i, \theta_l) + \varpi^i{}_l(X) g(\theta_k, \theta_i)\\ &= \varpi^i{}_k(X)\eta_{il} + \varpi^i{}_l(X)\eta_{ki}\\ &= \varpi^l{}_k(X)\eta_l + \varpi^k{}_l(X)\eta_k\\ &= \left(\eta_l \varpi^l{}_k + \eta_k \varpi^k{}_l\right)(X)\end{aligned}$$

Since this is true for any vector field X, we have:

$$\eta_l \varpi^l{}_k + \eta_k \varpi^k{}_l = 0$$

Thus, we have:

$$\varpi^l{}_k = -\eta_l \eta_k \varpi^k{}_l.$$

(ii) We have by (i):

$$\Gamma^i{}_{kj} = \varpi^i{}_j(\theta_k) = -\eta_i \eta_j \varpi^j{}_i(\theta_k) = -\eta_i \eta_j \Gamma^j{}_{ki}.$$

(iii) We have by (i):

$$\varpi^j{}_j = -\eta_j \eta_j \varpi^j{}_j = -\varpi^j{}_j.$$

Therefore, we have $\varpi^j{}_j = 0$.

(iv) We have by (iii):

$$\Gamma^j{}_{kj} = \varpi^j{}_j(\theta_k) = 0.$$

□

The following fundamental result establishes the connection between the 1-forms $\varpi^i{}_j$ and the functions u_i appearing in the metric. This will allow us to compute the symbols $\Gamma^j{}_{ki}$ in terms of these functions. To do this, we will calculate the 2-forms in two different ways:

$$\mathrm{d}\theta^i \in \Lambda^2(U)$$

We denote:

$$\theta^{j,i} := \theta^j \wedge \theta^i$$

Thus, we have:

$$\theta^{j,i} = -\theta^{i,j}$$

and:

$$\theta^{i,j}(\theta_k, \theta_l) = \theta^i \wedge \theta^j(\theta_k, \theta_l) = \theta^i \otimes \theta^j(\theta_k, \theta_l) - \theta^j \otimes \theta^i(\theta_k, \theta_l) = \delta^i_k \delta^j_l - \delta^j_k \delta^i_l \tag{1.2.6.1}$$

Proposition 1.2.6.2: Calculation of $\mathrm{d}\theta^i$

For $i \in \{0, 1, 2, 3\}$, we have:

(i) We have:

$$\mathrm{d}\theta^i = u_{i,j} e^{-u_j} \theta^{j,i}.$$

(ii) We have:

$$\mathrm{d}\theta^i = -\varpi^i{}_j \wedge \theta^j.$$

Proof. (i) We have:

$$\begin{aligned}
\mathrm{d}\theta^i &= \mathrm{d}(f_i \mathrm{d}x^i) \\
&= \partial_j (f_i) \,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
&= \partial_j (f_i)\, f^j \theta^j \wedge f^i \theta^i \\
&= \partial_j (e^{u_i})\, f^i f^j \theta^j \wedge \theta^i \\
&= u_{i,j} e^{u_i} e^{-u_j - u_i} \theta^j \wedge \theta^i \\
&= u_{i,j} e^{-u_j} \theta^j \wedge \theta^i \\
&= u_{i,j} e^{-u_j} \theta^{j,i}
\end{aligned}$$

(ii) We have $f_k f^k = 1$, which implies:

$$f^k \mathrm{d}f_k + f^k \mathrm{d}f_k = 0$$

Thus, we have $f^k \mathrm{d}f_k = -f_k \mathrm{d}f^k$. Therefore, we have:

$$\begin{aligned}
\mathrm{d}\theta^i &= \mathrm{d}\left(f_i \mathrm{d}x^i\right) = \mathrm{d}f_i \wedge \mathrm{d}x^i \\
&= \mathrm{d}f_i \wedge f^i \theta^i \\
&= f^i \mathrm{d}f_i \wedge \theta^i \\
&= -f_i \mathrm{d}f^i \wedge \theta^i \\
&= -f_i \partial_j \left(f^i\right) \mathrm{d}x^j \wedge \theta^i \\
&= -f_i \partial_j \left(f^i\right) f^j \theta^j \wedge \theta^i
\end{aligned}$$

Using equation (1.2.6.1), we have:

$$\begin{aligned}
\mathrm{d}\theta^i \left(\theta_k, \theta_l\right) \theta_i &= f_i f^j \partial_j \left(f^i\right) \theta^j \wedge \theta^i \left(\theta_k, \theta_l\right) \theta_i \\
&= f_i f^j \partial_j \left(f^i\right) \left(\delta^j_k \delta^i_l + \delta^j_l \delta^i_k\right) \theta_i \\
&= f_l f^k \partial_k \left(f^l\right) \theta_l - f_k f^l \partial_l \left(f^k\right) \theta_k
\end{aligned}$$

Using the identity:

$$\begin{aligned}
\nabla_{\theta_k} \theta_l &= \nabla_{f^k \partial_k} \left(f^l \partial_l\right) \\
&= f^k \nabla_{\partial_k} \left(f^l \partial_l\right) \\
&= f^k \mathrm{d}\left(f^l\right) \left(\partial_k\right) \partial_l + f^k f^l \nabla_{\partial_k} \left(\partial_l\right)
\end{aligned}$$

and since $[\partial_k, \partial_l] = 0$, we have:

$$\begin{aligned}
\varpi^i{}_j \wedge \theta^j \left(\theta_k, \theta_l\right) \theta_i &= \varpi^i{}_j \otimes \theta^j \left(\theta_k, \theta_l\right) \theta_i - \theta^j \otimes \varpi^i{}_j \left(\theta_k, \theta_l\right) \theta_i \\
&= \theta^j \left(\theta_l\right) \varpi^i{}_j \left(\theta_k\right) \theta_i - \theta^j \left(\theta_k\right) \varpi^i{}_j \left(\theta_l\right) \theta_i \\
&= \delta^j_l \nabla_{\theta_k} \theta_j - \delta^j_k \nabla_{\theta_l} \theta_j \\
&= \nabla_{\theta_k} \theta_l - \nabla_{\theta_l} \theta_k \\
&= f^k \mathrm{d}\left(f^l\right) \left(\partial_k\right) \partial_l + f^k f^l \nabla_{\partial_k} \left(\partial_l\right) - f^l \mathrm{d}\left(f^k\right) \left(\partial_l\right) \partial_k - f^k f^l \nabla_{\partial_l} \left(\partial_k\right) \\
&= f^k \partial_k \left(f^l\right) \partial_l - f^l \partial_l \left(f^k\right) \partial_k + f^k f^l \left[\partial_k, \partial_l\right] \\
&= f^k \partial_k \left(f^l\right) \partial_l - f^l \partial_l \left(f^k\right) \partial_k \\
&= -\mathrm{d}\theta^i \left(\theta_k, \theta_l\right) \theta_i
\end{aligned}$$

Hence the result, as this holds for any vector fields $\theta_k, \theta_l, \theta_i$.

□

So we have:

$$\begin{aligned}
\mathrm{d}\theta^0 &= u_{0,j}e^{-u_j}\theta^{j,0} = u_{0,1}e^{-u_1}\theta^{1,0} + u_{0,2}e^{-u_2}\theta^{2,0} + u_{0,3}e^{-u_3}\theta^{3,0} \\
\mathrm{d}\theta^1 &= u_{1,j}e^{-u_j}\theta^{j,1} = u_{1,0}e^{-u_0}\theta^{0,1} + u_{1,2}e^{-u_2}\theta^{2,1} + u_{1,3}e^{-u_3}\theta^{3,1} \\
\mathrm{d}\theta^2 &= u_{2,j}e^{-u_j}\theta^{j,2} = u_{2,0}e^{-u_0}\theta^{0,2} + u_{2,1}e^{-u_1}\theta^{1,2} + u_{2,3}e^{-u_3}\theta^{3,2} \\
\mathrm{d}\theta^3 &= u_{3,j}e^{-u_j}\theta^{j,3} = u_{3,0}e^{-u_0}\theta^{0,3} + u_{3,1}e^{-u_1}\theta^{1,3} + u_{3,2}e^{-u_2}\theta^{2,3}
\end{aligned}$$

Example 1.2.6.3: Sequence 3 of the metric h

We revisit the example with the metric h (see 1.1.4.1, 1.1.5.1, and 1.2.5.5) with:

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

We have:

$$\begin{aligned}
\mathrm{d}\theta^t &= u_{0,1}e^{-u_1}\theta^{1,0} \\
&= u'e^{v}\theta^{r,t} \\
\mathrm{d}\theta^r &= u_{1,0}e^{-u_0}\theta^{0,1} \\
&= \dot{v}e^{-u}\theta^{t,r} \\
\mathrm{d}\theta^\vartheta &= u_{2,0}e^{-u_0}\theta^{0,2} + u_{2,1}e^{-u_1}\theta^{1,2} \\
&= \dot{b}e^{-u}\theta^{t,\vartheta} + r^{-1}e^{-v}\theta^{r,\vartheta} \\
\mathrm{d}\theta^\phi &= u_{3,0}e^{-u_0}\theta^{0,3} + u_{3,1}e^{-u_1}\theta^{1,3} + u_{3,2}e^{-u_2}\theta^{2,3} \\
&= \dot{b}e^{-u}\theta^{t,\phi} + r^{-1}e^{-v}\theta^{r,\phi} + r^{-1}\cot\vartheta e^{-b}\theta^{\vartheta,\phi}
\end{aligned}$$

We can then deduce the exact values of the symbols $\Gamma^j{}_{ki}$ and $\varpi^i{}_j$ in terms of the elements of the functions u_i appearing in the metric.

Theorem 1.2.6.4: Exact values of $\Gamma^j{}_{ki}$ and $\varpi^i{}_j$

Let $i, j, k \in \{0,1,2,3\}$ **distinct**.

(i) We have:

$$\begin{aligned}
\Gamma^i{}_{ij} &= u_{i,j}e^{-u_j} \\
\Gamma^j{}_{ii} &= -\eta_i\eta_j u_{i,j}e^{-u_j} \\
\Gamma^j{}_{ki} &= 0
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\varpi^j{}_j &= 0 \\
\varpi^i{}_j &= u_{i,j}e^{-u_j}\theta^i - \eta_i\eta_j u_{j,i}e^{-u_i}\theta^j
\end{aligned}$$

Proof. Let's define the following for the proof:

$$a^i{}_{kj} := \Gamma^i{}_{kj} - \Gamma^i{}_{jk} \quad , \quad b_{ji} := u_{i,j}e^{-u_j}.$$

Using the notations and the previous proposition 1.2.6.2, we have:

$$\begin{aligned} b_{ji}\theta^{ji} &= u_{i,j}e^{-u_j}\theta^{j,i} \\ &= \mathrm{d}\theta^i \\ &= -\varpi^i{}_j \wedge \theta^j \\ &= -\Gamma^i{}_{kj}\theta^k \wedge \theta^j \\ &= \sum_{0\leq j<k\leq 3} \left(\Gamma^i{}_{kj} - \Gamma^i{}_{jk}\right)\theta^j \wedge \theta^k \\ &= \sum_{0\leq j<k\leq 3} a^i{}_{kj}\theta^{j,k}. \end{aligned}$$

Thus, we have:

(I) for all distinct i, j, k:

$$0 = a^i{}_{kj} = \Gamma^i{}_{kj} - \Gamma^i{}_{jk} = 0 \quad \Rightarrow \quad \Gamma^i{}_{kj} = \Gamma^i{}_{jk}.$$

(II) for all distinct i, j:

$$b_{ji} = -a^i{}_{ji} = a^i{}_{ij}.$$

Let's prove the theorem.

(i) We have the symmetries:

$$\Gamma^j{}_{ki} = \Gamma^j{}_{ik} \ , \ \Gamma^i{}_{kj} = \Gamma^i{}_{jk} \ , \ \Gamma^k{}_{ij} = \Gamma^j{}_{ji}$$

We will show that $\Gamma^j{}_{ki} = 0$ by demonstrating that $\Gamma^j{}_{ki} = -\Gamma^j{}_{ki}$. Using (I) and point (ii) of lemma 1.2.6.1, we have:

$$\begin{aligned} \Gamma^j{}_{ki} &= -\eta_i\eta_j\Gamma^i{}_{kj} \\ &= -\eta_i\eta_j\Gamma^i{}_{jk} \\ &= \eta_i\eta_j\eta_k\eta_i\Gamma^k{}_{ji} \\ &= \eta_j\eta_k\Gamma^k{}_{ij} \\ &= -\Gamma^j{}_{ik} = -\Gamma^j{}_{ki} \end{aligned}$$

Hence, the result follows from symmetry.

(ii) Since $\Gamma^i{}_{ji} = 0$, we have:

$$u_{i,j}e^{-u_j} = b_{ji} = -a^i{}_{ji} = -\Gamma^i{}_{ji} + \Gamma^i{}_{ij} = \Gamma^i{}_{ij}$$

Therefore, we have:

$$\Gamma^i{}_{ij} = u_{i,j}e^{-u_j}.$$

(iii) We have by (ii):

$$\begin{aligned} \Gamma^j{}_{ii} &= -\eta_j\eta_i\Gamma^i{}_{ij} \\ &= -\eta_i\eta_j u_{i,j}e^{-u_j}. \end{aligned}$$

(iv) We have:

$$\begin{aligned} \varpi^i{}_j &= \Gamma^i{}_{kj}\theta^k \\ &= \Gamma^i{}_{ij}\theta^i + \Gamma^i{}_{jj}\theta^j \\ &= u_{i,j}e^{-u_j}\theta^i - \eta_i\eta_j u_{j,i}e^{-u_i}\theta^j \end{aligned}$$

□

We denote:

$$\Gamma^i = \begin{pmatrix} {\Gamma^i}_{00} & {\Gamma^i}_{01} & {\Gamma^i}_{02} & {\Gamma^i}_{03} \\ {\Gamma^i}_{10} & {\Gamma^i}_{11} & {\Gamma^i}_{12} & {\Gamma^i}_{13} \\ {\Gamma^i}_{20} & {\Gamma^i}_{21} & {\Gamma^i}_{22} & {\Gamma^i}_{23} \\ {\Gamma^i}_{30} & {\Gamma^i}_{31} & {\Gamma^i}_{32} & {\Gamma^i}_{33} \end{pmatrix}$$

- Case $i := 0$. We have:

$$\Gamma^0 = \begin{pmatrix} 0 & u_{0,1}e^{-u_1} & u_{0,2}e^{-u_2} & u_{0,3}e^{-u_3} \\ 0 & u_{1,0}e^{-u_0} & 0 & 0 \\ 0 & 0 & u_{2,0}e^{-u_0} & 0 \\ 0 & 0 & 0 & u_{3,0}e^{-u_0} \end{pmatrix}.$$

- Case $i := 1$. We have:

$$\Gamma^1 = \begin{pmatrix} u_{0,1}e^{-u_1} & 0 & 0 & 0 \\ u_{1,0}e^{-u_0} & 0 & u_{1,2}e^{-u_2} & u_{1,3}e^{-u_3} \\ 0 & 0 & -u_{2,1}e^{-u_1} & 0 \\ 0 & 0 & 0 & -u_{3,1}e^{-u_1} \end{pmatrix}.$$

- Case $i := 2$. We have:

$$\Gamma^2 = \begin{pmatrix} u_{0,2}e^{-u_2} & 0 & 0 & 0 \\ 0 & -u_{1,2}e^{-u_2} & 0 & 0 \\ u_{2,0}e^{-u_0} & u_{2,1}e^{-u_1} & 0 & u_{2,3}e^{-u_3} \\ 0 & 0 & 0 & -u_{3,2}e^{-u_2} \end{pmatrix}.$$

- Case $i := 3$. We have:

$$\Gamma^3 = \begin{pmatrix} u_{0,3}e^{-u_3} & 0 & 0 & 0 \\ 0 & -u_{1,3}e^{-u_3} & 0 & 0 \\ 0 & 0 & -u_{2,3}e^{-u_3} & 0 \\ u_{3,0}e^{-u_0} & u_{3,1}e^{-u_1} & u_{3,2}e^{-u_2} & 0 \end{pmatrix}.$$

By Lemma 1.2.4.3, we obtain the values ${\Gamma^i}_{jk}$ found in the previous subsection.

Example 1.2.6.5: Sequence 4 of the metric h

We revisit the example with the metric h (see 1.1.4.1, 1.1.5.1, 1.2.5.5, and 1.2.6.3) with:

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

Therefore, we have:

$$\Gamma^0 = \begin{pmatrix} 0 & u'e^{-v} & 0 & 0 \\ 0 & \dot{v}e^{-u} & 0 & 0 \\ 0 & 0 & \dot{b}e^{-u} & 0 \\ 0 & 0 & 0 & \dot{b}e^{-u} \end{pmatrix}$$

$$\Gamma^1 = \begin{pmatrix} u'e^{-v} & 0 & 0 & 0 \\ \dot{v}e^{-u} & 0 & 0 & 0 \\ 0 & 0 & -r^{-1}e^{-v} & 0 \\ 0 & 0 & 0 & -r^{-1}e^{-v} \end{pmatrix}$$

$$\Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dot{b}e^{-u} & r^{-1}e^{-v} & 0 & 0 \\ 0 & 0 & 0 & -r^{-1}\cot\vartheta e^{-b} \end{pmatrix}$$

$$\Gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dot{b}e^{-u} & r^{-1}e^{-v} & r^{-1}\cot\vartheta e^{-b} & 0 \end{pmatrix}$$

Thus, we have:

$$\varpi = \begin{pmatrix} 0 & u'e^{-v}\theta^0 + \dot{v}e^{-u}\theta^1 & \dot{b}e^{-u}\theta^2 & \dot{b}e^{-u}\theta^3 \\ u'e^{-v}\theta^0 + \dot{v}e^{-u}\theta^1 & 0 & -r^{-1}e^{-v}\theta^2 & -r^{-1}e^{-v}\theta^3 \\ \dot{b}e^{-u}\theta^2 & r^{-1}e^{-v}\theta^2 & 0 & -r^{-1}\cot\vartheta e^{-b}\theta^3 \\ \dot{b}e^{-u}\theta^3 & r^{-1}e^{-v}\theta^3 & r^{-1}\cot\vartheta e^{-b}\theta^3 & 0 \end{pmatrix}.$$

1.3 Curvature Tensor and Riemann Tensor

1.3.1 Generalities

Definition 1.3.1.1: Curvature mappings

(i) Let X and Y be two vector fields. The **curvature mapping associated with** X **and** Y is the mapping $\mathscr{R}(X,Y)$ defined by:

$$\mathscr{R}(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

That is, for any vector field Z, we associate the vector field:

$$\mathscr{R}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

(ii) The **Riemann mapping** is the mapping that associates a scalar field to all vector fields X, Y, Z, and W:

$$\mathscr{R}\mathrm{m}(W,Z,X,Y) := g\left(W, \mathscr{R}(X,Y)Z\right).$$

The mapping $\mathscr{R}$ is a tensor field of type $(1,3)$ and the mapping $\mathscr{R}\mathrm{m}$ is a tensor field of type $(0,4)$.

Proposition 1.3.1.2: Usual properties

Let X, Y, Z, W, and T be vector fields.

(i) (**First Bianchi identity**) We have:

$$\mathscr{R}(X,Y)Z + \mathscr{R}(Y,Z)X + \mathscr{R}(Z,X)Y = 0$$
$$\mathscr{R}\mathrm{m}(W,Z,X,Y) + \mathscr{R}\mathrm{m}(W,X,Y,Z) + \mathscr{R}\mathrm{m}(W,Y,Z,X) = 0$$

(ii) We have:

$$\mathscr{R}(X,Y)Z = -\mathscr{R}(Y,X)Z$$
$$\mathscr{R}\mathrm{m}(W,Z,X,Y) = -\mathscr{R}\mathrm{m}(W,Z,Y,X)$$

(iii) We have:

$$\mathscr{R}\mathrm{m}(W,Z,X,Y) = -\mathscr{R}\mathrm{m}(Z,W,X,Y).$$

(iv) We have:

$$\mathscr{R}\mathrm{m}(W,Z,X,Y) = \mathscr{R}\mathrm{m}(X,Y,W,Z).$$

(v) (**Second Bianchi identity**) We have:

$$\begin{aligned}
(\nabla_X \mathscr{R})(Y,Z) + (\nabla_Y \mathscr{R})(Z,X) + (\nabla_Z \mathscr{R})(X,Y) = 0 \\
\nabla_X \mathscr{R}\mathrm{m}(T,W,Z,Y) + \nabla_Y \mathscr{R}\mathrm{m}(T,W,X,Z) + \nabla_Z \mathscr{R}\mathrm{m}(T,W,Y,X) = 0 \\
\nabla \mathscr{R}\mathrm{m}(T,W,Z,Y,X) + \nabla \mathscr{R}\mathrm{m}(T,W,X,Z,Y) + \nabla \mathscr{R}\mathrm{m}(T,W,Y,X,Z) = 0
\end{aligned}$$

Proof. (i) By the Jacobi identity (see Proposition 1.2.1.6), we have:

$$\begin{aligned}
&\mathscr{R}(X,Y)Z + \mathscr{R}(Y,Z)X + \mathscr{R}(Z,X)Y \\
=&\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \\
&+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y \\
=&\nabla_X (\nabla_Y Z - \nabla_Z Y) + \nabla_Y (\nabla_Z X - \nabla_X Z) + \nabla_Z (\nabla_X Y - \nabla_Y X) \\
&- \left(\nabla_{[X,Y]} Z + \nabla_{[Y,Z]} X + \nabla_{[Z,X]} Y\right) \\
=&\left(\nabla_X [Y,Z] - \nabla_{[Y,Z]} X\right) + \left(\nabla_Y [Z,X] - \nabla_{[Z,X]} Y\right) + \left(\nabla_Z [X,Y] - \nabla_{[X,Y]} Z\right) \\
=&[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \\
=&0
\end{aligned}$$

Thus, by linearity of g, we have:

$$\begin{aligned}
&\mathscr{R}\mathrm{m}(W,Z,X,Y) + \mathscr{R}\mathrm{m}(W,X,Y,Z) + \mathscr{R}\mathrm{m}(W,Y,Z,X) \\
=&g(W,\mathscr{R}(X,Y)Z) + g(W,\mathscr{R}(Y,Z)X) + g(W,\mathscr{R}(Z,X)Y) \\
=&g(W,\mathscr{R}(X,Y)Z + \mathscr{R}(Y,Z)X + \mathscr{R}(Z,X)Y) \\
=&g(W,0) \\
=&0
\end{aligned}$$

(ii) As:

$$\begin{aligned}
-\nabla_{[X,Y]} Z &= \nabla_{-[X,Y]} Z \\
&= \nabla_{[Y,X]} Z
\end{aligned}$$

we have:

$$\begin{aligned}
\mathscr{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= -\left(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z\right) \\
&= -\mathscr{R}(Y,X)Z
\end{aligned}$$

And thus we have:

$$\begin{aligned}
\mathscr{R}\mathrm{m}(W,Z,X,Y) &= g(W,\mathscr{R}(X,Y)Z) \\
&= g(W,-\mathscr{R}(Y,X)Z) \\
&= -g(W,\mathscr{R}(Y,X)Z) \\
&= -\mathscr{R}\mathrm{m}(W,Z,Y,X)
\end{aligned}$$

(iii) We have:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

i.e., we have:

$$g(\nabla_X Y, Z) = X(g(Y,Z)) - g(Y, \nabla_X Z).$$

Thus, we have:

$$\begin{aligned}
\mathscr{R}\mathrm{m}(W,Z,X,Y) =& g(W, \mathscr{R}(X,Y)Z) \\
=& g(W, \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) \\
=& g(W, \nabla_X\nabla_Y Z) - g(W, \nabla_Y\nabla_X Z) - g(W, \nabla_{[X,Y]}Z) \\
=& [X(g(W,\nabla_Y Z)) - g(\nabla_X W, \nabla_Y Z)] \\
& - [Y(g(W,\nabla_X Z)) - g(\nabla_Y W, \nabla_X Z)] \\
& - [X,Y](g(W,Z)) - g(\nabla_{[X,Y]}W, Z) \\
=& [X(Y(g(W,Z))) - X(g(\nabla_Y W, Z)) - g(\nabla_X W, \nabla_Y Z)] \\
& - [Y(X(g(W,Z))) - Y(g(\nabla_X W, Z)) - g(\nabla_Y W, \nabla_X Z)] \\
& - [X,Y](g(W,Z)) - g(\nabla_{[X,Y]}W, Z) \\
=& (XY - YX)(g(W,Z)) - X(g(\nabla_Y W, Z)) - g(\nabla_X W, \nabla_Y Z) \\
& + Y(g(\nabla_X W, Z)) + g(\nabla_Y W, \nabla_X Z) - [X,Y](g(W,Z)) + g(\nabla_{[X,Y]}W, Z) \\
=& [X,Y](g(W,Z)) - g(\nabla_X\nabla_Y W, Z) - g(\nabla_X W, \nabla_Y Z) \\
& + g(\nabla_Y\nabla_X W, Z) + g(\nabla_Y W, \nabla_X Z) - [X,Y](g(W,Z)) + g(\nabla_{[X,Y]}W, Z) \\
=& - g(\nabla_X\nabla_Y W, Z) - g(\nabla_X W, \nabla_Y Z) + g(\nabla_Y\nabla_X W, Z) \\
& + g(\nabla_Y W, \nabla_X Z) + g(\nabla_{[X,Y]}W, Z) \\
=& - [g(\nabla_X\nabla_Y W, Z) - g(\nabla_Y\nabla_X W, Z) - g(\nabla_{[X,Y]}W, Z)] \\
=& - g(\nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W, Z) \\
=& - g(Z, \mathscr{R}(X,Y)W) \\
=& - \mathscr{R}\mathrm{m}(Z,W,X,Y)
\end{aligned}$$

(iv) We have by (i, ii, iii):

$$\begin{aligned}
& 2\mathscr{R}\mathrm{m}(Z,W,X,Y) - 2\mathscr{R}\mathrm{m}(X,Y,Z,W) \\
=& \mathscr{R}\mathrm{m}(Z,W,X,Y) + \mathscr{R}\mathrm{m}(W,Z,Y,X) + \mathscr{R}\mathrm{m}(X,Y,W,Z) + \mathscr{R}\mathrm{m}(Y,X,Z,W) \\
=& \mathscr{R}\mathrm{m}(Z,W,X,Y) + \mathscr{R}\mathrm{m}(Z,X,Y,W) + \mathscr{R}\mathrm{m}(Z,Y,W,X) \\
& + \mathscr{R}\mathrm{m}(W,Z,Y,X) + \mathscr{R}\mathrm{m}(W,Y,X,Z) + \mathscr{R}\mathrm{m}(W,X,Z,Y) \\
& + \mathscr{R}\mathrm{m}(X,Y,W,Z) + \mathscr{R}\mathrm{m}(X,W,Z,Y) + \mathscr{R}\mathrm{m}(X,Z,Y,W) \\
& + \mathscr{R}\mathrm{m}(Y,X,Z,W) + \mathscr{R}\mathrm{m}(Y,Z,W,X) + \mathscr{R}\mathrm{m}(Y,W,X,Z) \\
& - (\mathscr{R}\mathrm{m}(Z,X,Y,W) + \mathscr{R}\mathrm{m}(Z,Y,W,X)) - (\mathscr{R}\mathrm{m}(W,Y,X,Z) + \mathscr{R}\mathrm{m}(W,X,Z,Y)) \\
& - (\mathscr{R}\mathrm{m}(X,W,Z,Y) + \mathscr{R}\mathrm{m}(X,Z,Y,W)) \\
& - (\mathscr{R}\mathrm{m}(Y,Z,W,X) + \mathscr{R}\mathrm{m}(Y,W,X,Z)) \\
=& 0 + 0 + 0 + 0 \\
& - (\mathscr{R}\mathrm{m}(Z,X,Y,W) + \mathscr{R}\mathrm{m}(X,Z,Y,W)) - (\mathscr{R}\mathrm{m}(W,Y,X,Z) + \mathscr{R}\mathrm{m}(Y,W,X,Z)) \\
& - (\mathscr{R}\mathrm{m}(X,W,Z,Y) + \mathscr{R}\mathrm{m}(W,X,Z,Y)) \\
& - (\mathscr{R}\mathrm{m}(Y,Z,W,X) + \mathscr{R}\mathrm{m}(Z,Y,W,X)) \\
=& 0 - 0 - 0 - 0 \\
=& 0
\end{aligned}$$

i.e. we have:

$$\mathscr{R}\mathrm{m}(Z,W,X,Y) = \mathscr{R}\mathrm{m}(X,Y,Z,W).$$

(v) We have by the action of ∇ on tensor fields (see Proposition 1.2.2.4):

$$\begin{aligned}
&\nabla_X \mathscr{R}(Y,Z)W + \nabla_Y \mathscr{R}(Z,X)W + \nabla_Z \mathscr{R}(X,Y)W \\
=&\nabla_X(\mathscr{R}(Y,Z)W) + \nabla_Y(\mathscr{R}(Z,X)W) + \nabla_Z(\mathscr{R}(X,Y)W) \\
&-\mathscr{R}(\nabla_X Y,Z)W - \mathscr{R}(Y,\nabla_X Z)W - \mathscr{R}(\nabla_Y Z,X)W - \mathscr{R}(Z,\nabla_Y X)W - \mathscr{R}(\nabla_Z X,Y)W - \mathscr{R}(X,\nabla_Z Y)W \\
&-\mathscr{R}(Y,Z)\nabla_X W - \mathscr{R}(Z,X)\nabla_Y W - \mathscr{R}(X,Y)\nabla_Z W \\
=:&(\mathrm{I}) \\
&-(\mathrm{II}) \\
&-(\mathrm{III})
\end{aligned}$$

Let's calculate elements (II) and (I) separately. We have:

$$\begin{aligned}
(\mathrm{II}) =&\mathscr{R}(\nabla_X Y,Z) + \mathscr{R}(Y,\nabla_X Z) + \mathscr{R}(\nabla_Y Z,X) + \mathscr{R}(Z,\nabla_Y X) \\
&+\mathscr{R}(\nabla_Z X,Y) + \mathscr{R}(X,\nabla_Z Y) \\
=&\mathscr{R}(\nabla_X Y - \nabla_Y X,Z) + \mathscr{R}(\nabla_Y Z - \nabla_Z Y,X) \\
&+\mathscr{R}(\nabla_Z X - \nabla_X Z,Y) \\
=&\mathscr{R}([X,Y],Z) + \mathscr{R}([Y,Z],X) + \mathscr{R}([Z,X],Y) \\
=:&(\mathrm{IV})
\end{aligned}$$

By the Jacobi identity, we have:

$$\begin{aligned}
\nabla_{[X,[Y,Z]]}W + \nabla_{[Y,[Z,X]]}W + \nabla_{[Z,[X,Y]]}W &= \nabla_{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]}W \\
&= \nabla_0 W \\
&= 0
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
(\mathrm{I}) =&\nabla_X(\mathscr{R}(Y,Z)W) + \nabla_Y(\mathscr{R}(Z,X)W) + \nabla_Z(\mathscr{R}(X,Y)W) \\
=&\nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W + \nabla_Y\nabla_Z\nabla_X W \\
&-\nabla_Y\nabla_X\nabla_Z W + \nabla_Z\nabla_X\nabla_Y W - \nabla_Z\nabla_Y\nabla_X W \\
&-\nabla_X\nabla_{[Y,Z]}W - \nabla_Y\nabla_{[Z,X]}W - \nabla_Z\nabla_{[X,Y]}W \\
=&\nabla_Y\nabla_Z\nabla_X W - \nabla_Z\nabla_Y\nabla_X W + \nabla_Z\nabla_X\nabla_Y W \\
&-\nabla_X\nabla_Z\nabla_Y W + \nabla_X\nabla_Y\nabla_Z W - \nabla_Y\nabla_X\nabla_Z W \\
&-\nabla_{[Y,Z]}\nabla_X W - \mathscr{R}(X,[Y,Z]) - \nabla_{[X,[Y,Z]]}W \\
&-\nabla_{[Z,X]}\nabla_Y W - \mathscr{R}(Y,[Z,X]) - \nabla_{[Y,[Z,X]]}W \\
&-\nabla_{[X,Y]}\nabla_Z W - \mathscr{R}(Z,[X,Y]) - \nabla_{[Z,[X,Y]]}W \\
=&\mathscr{R}(Y,Z)\nabla_X W + \mathscr{R}(Z,X)\nabla_Y W + \mathscr{R}(X,Y)\nabla_Z W \\
&-\mathscr{R}(X,[Y,Z])W - \mathscr{R}(Y,[Z,X])W - \mathscr{R}(Z,[X,Y])W \\
=&(\mathrm{III}) + \mathscr{R}([Y,Z],X)W + \mathscr{R}([Z,X],Y)W + \mathscr{R}([X,Y],Z)W \\
=&(\mathrm{III}) + (\mathrm{IV})
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\nabla_X \mathscr{R}(Y,Z) + \nabla_Y \mathscr{R}(Z,X) + \nabla_Z \mathscr{R}(X,Y) &= (\mathrm{I}) - (\mathrm{II}) - (\mathrm{III}) \\
&= (\mathrm{III}) + (\mathrm{IV}) - (\mathrm{IV}) - (\mathrm{III}) \\
&= 0
\end{aligned}$$

We also have:

$$\begin{aligned}
\nabla_X \mathscr{R}\mathrm{m}(T,W,Z,Y) =& X(\mathscr{R}\mathrm{m}(T,W,Z,Y)) - \mathscr{R}\mathrm{m}(\nabla_X T,W,Z,Y) - \mathscr{R}\mathrm{m}(T,\nabla_X W,Z,Y)\\
& - \mathscr{R}\mathrm{m}(T,W,\nabla_X Z,Y) - \mathscr{R}\mathrm{m}(T,W,Z,\nabla_X Y)\\
=& X(g(T,\mathscr{R}(Z,Y)W)) - \mathscr{R}\mathrm{m}(\nabla_X T,W,Z,Y) - \mathscr{R}\mathrm{m}(T,\nabla_X W,Z,Y)\\
& - \mathscr{R}\mathrm{m}(T,W,\nabla_X Z,Y) - \mathscr{R}\mathrm{m}(T,W,Z,\nabla_X Y)\\
=& g(\nabla_X T,\mathscr{R}(Z,Y)W) + g(T,\nabla_X(\mathscr{R}(Z,Y)W)) - g(\nabla_X T,\mathscr{R}(Z,Y)W)\\
& - g(T,\mathscr{R}(Z,Y)\nabla_X W) - g(T,\mathscr{R}(\nabla_X Z,Y)W) - g(T,\mathscr{R}(Z,\nabla_X Y)W)\\
=& g(T,\nabla_X(\mathscr{R}(Z,Y)W) - \mathscr{R}(Z,Y)\nabla_X W - \mathscr{R}(\nabla_X Z,Y)W - \mathscr{R}(Z,\nabla_X Y)W)\\
=& g(T,\nabla_X \mathscr{R}(Z,Y)W)
\end{aligned}$$

Thus, by permuting X, Z, and Y, we have:

$$\begin{aligned}
& \nabla_X \mathscr{R}\mathrm{m}(T,W,Z,Y) + \nabla_Y \mathscr{R}\mathrm{m}(T,W,X,Z) + \nabla_Z \mathscr{R}\mathrm{m}(T,W,Y,X)\\
=& g(T,\nabla_X \mathscr{R}(Z,Y)W) + g(T,\nabla_Y \mathscr{R}(X,Z)W) + g(T,\nabla_Z \mathscr{R}(Y,X)W)\\
=& g(T,\nabla_X \mathscr{R}(Z,Y)W + \nabla_Y \mathscr{R}(Z,X)W + \nabla_Z \mathscr{R}(Y,X)W)\\
=& g(T,0)\\
=& 0
\end{aligned}$$

□

1.3.2 Computation of components in $\mathscr{E}$ and $\mathscr{E}^*$

Definition 1.3.2.1: Curvature and Riemann tensors

(i) The **components of the curvature tensor in** $\mathscr{E}$ **and** $\mathscr{E}^*$ are given by:

$$\mathbf{R}^i{}_{jkl} := e^i\left(\mathscr{R}\left(e_k,e_l\right)e_j\right).$$

(ii) The **components of the Riemann tensor in** $\mathscr{E}$ **and** $\mathscr{E}^*$ are given by:

$$\mathbf{R}_{ijkl} := \mathscr{R}\mathrm{m}\left(e_i,e_j,e_k,e_l\right).$$

We have the following decompositions:

$$\begin{aligned}
\mathscr{R} &= \mathbf{R}^i{}_{jkl}\ e_i \otimes e^j \otimes e^k \otimes e^l\\
\mathscr{R}\mathrm{m} &= \mathbf{R}_{ijkl}\ e^i \otimes e^j \otimes e^k \otimes e^l
\end{aligned}$$

There is a simple relation between these two components.

Lemma 1.3.2.2: Relation between the two components

We have:

$$\mathbf{R}_{ijkl} = \mathbf{g}_{in}\mathbf{R}^n{}_{jkl}.$$

Proof. This follows directly from proposition 1.1.8.6. We provide the proof here for completeness. We have:

$$\begin{aligned}
\mathbf{R}_{ijkl} &= \mathscr{R}\mathrm{m}\left(e_i,e_j,e_k,e_l\right)\\
&= g\left(e_i,\mathscr{R}\left(e_j,e_k\right)e_l\right)\\
&= g\left(e_i,\mathbf{R}^n{}_{jkl}\ e_n\right)\\
&= \mathbf{R}^n{}_{jkl}\, g\left(e_i,e_n\right)\\
&= \mathbf{g}_{in}\mathbf{R}^n{}_{jkl}
\end{aligned}$$

□

From proposition 1.3.2.3, we deduce the following proposition.

Proposition 1.3.2.3: Usual properties

Let $i, j, k, l \in \{0, 1, 2, 3\}$.

(i) (**First Bianchi identity**) We have:

$$\mathbf{R}^i{}_{jkl} + \mathbf{R}^i{}_{klj} + \mathbf{R}^i{}_{ljk} = 0$$
$$\mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk} = 0$$

(ii) We have:

$$\mathbf{R}^i{}_{jkl} = -\mathbf{R}^i{}_{jlk}$$
$$\mathbf{R}_{ijkl} = -\mathbf{R}_{ijlk}$$

(iii) We have:

$$\mathbf{R}_{ijkl} = -\mathbf{R}_{jikl}.$$

(iv) We have:

$$\mathbf{R}_{ijkl} = \mathbf{R}_{klij}.$$

(v) (**Second Bianchi identity**) We have:

$$\nabla_n \mathbf{R}_{ijkl} + \nabla_k \mathbf{R}_{ijln} + \nabla_l \mathbf{R}_{ijnk} = 0.$$

Proof. We use proposition 1.3.2.3.

(i) Since:

$$\mathscr{R}(e_k, e_l)e_j + \mathscr{R}(e_l, e_j)e_k + \mathscr{R}(e_j, e_k)e_l$$

we have:

$$\begin{aligned}
\mathbf{R}^i{}_{jkl} + \mathbf{R}^i{}_{klj} + \mathbf{R}^i{}_{ljk} &= e^i\left(\mathscr{R}(e_k, e_l)e_j\right) + e^i\left(\mathscr{R}(e_l, e_j)e_k\right) + e^i\left(\mathscr{R}(e_j, e_k)e_l\right) \\
&= e^i\left(\mathscr{R}(e_k, e_l)e_j + \mathscr{R}(e_l, e_j)e_k + \mathscr{R}(e_j, e_k)e_l\right) \\
&= e^i(0) \\
&= 0 \\
\mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk} &= \mathscr{R}\mathrm{m}(e_i, e_j, e_k, e_l) + \mathscr{R}\mathrm{m}(e_i, e_k, e_l, e_j) + \mathscr{R}\mathrm{m}(e_i, e_l, e_j, e_k) \\
&= 0
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\mathbf{R}^i{}_{jkl} &= e^i\left(\mathscr{R}(e_k, e_l)e_j\right) \\
&= e^i\left(-\mathscr{R}(e_l, e_k)e_j\right) \\
&= -e^i\left(\mathscr{R}(e_l, e_k)e_j\right) \\
&= -\mathbf{R}^i{}_{kjl} \\
\mathbf{R}_{ijkl} &= \mathscr{R}\mathrm{m}(e_i, e_j, e_k, e_l) \\
&= -\mathscr{R}\mathrm{m}(e_i, e_j, e_l, e_k) \\
&= -\mathbf{R}_{ijlk}
\end{aligned}$$

(iii) We have:

$$\begin{aligned}\mathbf{R}_{ijkl} &= \mathscr{R}\mathrm{m}(e_i, e_j, e_k, e_l)\\ &= -\mathscr{R}\mathrm{m}(e_j, e_i, e_k, e_l)\\ &= -\mathbf{R}_{jikl}\end{aligned}$$

(iv) We have:

$$\begin{aligned}\mathbf{R}_{ijkl} &= \mathscr{R}\mathrm{m}(e_i, e_j, e_k, e_l)\\ &= \mathscr{R}\mathrm{m}(e_k, e_l, e_i, e_j)\\ &= \mathbf{R}_{klij}\end{aligned}$$

(v) We have:

$$\begin{aligned}\nabla_n\mathbf{R}_{ijkl} + \nabla_k\mathbf{R}_{ijln} + \nabla_l\mathbf{R}_{ijnk} &= \nabla_{e_n}\mathscr{R}\mathrm{m}(e_i, e_j, e_k, e_l) + \nabla_{e_k}\mathscr{R}\mathrm{m}(e_i, e_j, e_l, e_n) + \nabla_{e_l}\mathscr{R}\mathrm{m}(e_i, e_j, e_n, e_k)\\ &= 0\end{aligned}$$

□

This implies the nullity of a class of components.

Corollary 1.3.2.4: Nullities and symmetries of certain components

Let $i, j, k, l \in \{0, 1, 2, 3\}$.

(i) We have:

$$\mathbf{R}_{jkii} = \mathbf{R}_{iijk} = 0.$$

(ii) We have:

$$\mathbf{R}_{ijki} = \mathbf{R}_{kiij} = -\mathbf{R}_{kiji} = -\mathbf{R}_{ikij} = -\mathbf{R}_{ijik} = -\mathbf{R}_{jiki} = \mathbf{R}_{jiik} = \mathbf{R}_{ikji}.$$

Proof. (i) We have from point (ii) of proposition 1.3.2.3:

$$\mathbf{R}_{ijll} = -\mathbf{R}_{ijll}$$

i.e., we have:

$$\mathbf{R}_{ijll} = 0.$$

We have from point (iii) of proposition 1.3.2.3:

$$\mathbf{R}_{iikl} = -\mathbf{R}_{iikl}$$

i.e., we have:

$$\mathbf{R}_{iikl} = 0.$$

(ii) From (i) and point (i) of proposition 1.3.2.3, we have:

$$\begin{aligned}0 &= \mathbf{R}_{ijll} + \mathbf{R}_{illj} + \mathbf{R}_{iljl}\\ &= 0 + \mathbf{R}_{illj} + \mathbf{R}_{iljl}\end{aligned}$$

i.e., we have:

$$\mathbf{R}_{illj} = -\mathbf{R}_{iljl}.$$

And we have:

$$\begin{aligned} 0 &= \mathbf{R}_{iikl} + \mathbf{R}_{ikli} + \mathbf{R}_{ilik} \\ &= 0 + \mathbf{R}_{ikli} + \mathbf{R}_{ilik} \end{aligned}$$

i.e., we have:

$$\mathbf{R}_{ikli} = -\mathbf{R}_{ilik}.$$

By point (iv) of proposition 1.3.2.3, we have directly:

$$\begin{aligned} \mathbf{R}_{ijil} &= \mathbf{R}_{ilij} \\ \mathbf{R}_{ijki} &= \mathbf{R}_{kiij} \end{aligned}$$

□

Example 1.3.2.5: Examples of components

(1) (a) The **components of the curvature tensor in the bases $\mathscr{C}$ and $\mathscr{C}^*$** are:

$$\mathrm{R}^i{}_{jkl} := \mathrm{d}x^i \left(\mathscr{R} \left(\partial_k, \partial_l \right) \partial_j \right).$$

(b) The **components of the Riemann tensor in the bases $\mathscr{C}$ and $\mathscr{C}^*$** are:

$$\mathrm{R}_{ijkl} := \mathscr{R}\mathrm{m} \left(\partial_i, \partial_j, \partial_k, \partial_l \right).$$

Therefore, we have the decompositions:

$$\begin{aligned} \mathscr{R} &= \mathrm{R}^i{}_{jkl}\ \partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l \\ \mathscr{R}\mathrm{m} &= \mathrm{R}_{ijkl}\ \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l \end{aligned}$$

By lemma 1.3.2.2, we have the relations:

$$\begin{aligned} \mathrm{R}_{ijkl} &= \mathrm{g}_{in} \mathrm{R}^n{}_{jkl} \\ &= \mathrm{g}_{ii} \mathrm{R}^i{}_{jkl} \\ &= \eta_i e^{u_i} \mathrm{R}^i{}_{jkl} \end{aligned}$$

(2) (a) The **components of the curvature tensor in the bases $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$** are:

$$\mathbb{R}^i{}_{jkl} := \theta^i \left(\mathscr{R} \left(\theta_k, \theta_l \right) \theta_j \right).$$

(b) The **components of the Riemann tensor in the bases $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$** are:

$$\mathbb{R}_{ijkl} := \mathscr{R}\mathrm{m} \left(\theta_i, \theta_j, \theta_k, \theta_l \right).$$

Therefore, we have the decompositions:

$$\begin{aligned} \mathscr{R} &= \mathbb{R}^i{}_{jkl}\ \theta_i \otimes \theta^j \otimes \theta^k \otimes \theta^l \\ \mathscr{R}\mathrm{m} &= \mathbb{R}_{ijkl}\ \theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l \end{aligned}$$

By lemma 1.3.2.2, we have the relations:

$$\begin{aligned} \mathbb{R}_{ijkl} &= \eta_{ii} \mathbb{R}^i{}_{jkl} \\ &= \eta_i \mathbb{R}^i{}_{jkl} \end{aligned}$$

1.3.3 Practical calculations in $\mathscr{C}$ and $\mathscr{C}^*$

We begin with a proposition that establishes a link between the components of the curvature tensor and the Christoffel symbols.

Proposition 1.3.3.1: Usual properties

Let $i, j, k, l \in \{0, 1, 2, 3\}$. We have:

$$\mathrm{R}^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{kj} + \Gamma^m_{jl}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}.$$

Proof. By definition of $\mathscr{R}$:

$$\begin{aligned}
\mathscr{R}\left(\partial_k, \partial_l\right)\partial_j &= \nabla_{\partial_k}\nabla_{\partial_l}\partial_j - \nabla_{\partial_l}\nabla_{\partial_k}\partial_j - \nabla_{[\partial_k,\partial_l]}\partial_j \\
&= \nabla_{\partial_k}\left(\Gamma^m_{jl}\partial_m\right) - \nabla_{\partial_l}\left(\Gamma^m_{kj}\partial_m\right) \\
&= \mathrm{d}\Gamma^m_{jl}\left(\partial_k\right)\partial_m + \Gamma^m_{jl}\nabla_{\partial_k}\partial_m - \mathrm{d}\Gamma^m_{kj}\left(\partial_l\right)\partial_m + \Gamma^m_{kj}\nabla_{\partial_l}\partial_m \\
&= \partial_k\Gamma^m_{jl}\partial_m + \Gamma^m_{jl}\Gamma^n_{km}\partial_n - \partial_l\Gamma^m_{kj}\partial_m - \Gamma^m_{kj}\Gamma^n_{lm}\partial_n
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\mathrm{R}^i{}_{jkl} &= \mathrm{d}x^i\left(\mathscr{R}\left(\partial_k, \partial_l\right)\partial_j\right) \\
&= \mathrm{d}x^i\left(\partial_k\Gamma^m_{jl}\partial_m + \Gamma^m_{jl}\Gamma^n_{km}\partial_n - \partial_l\Gamma^m_{kj}\partial_m - \Gamma^m_{kj}\Gamma^n_{lm}\partial_n\right) \\
&= \partial_k\Gamma^m_{jl}\delta^i_m + \Gamma^m_{jl}\Gamma^n_{km}\delta^i_n - \partial_l\Gamma^m_{kj}\delta^i_m - \Gamma^m_{kj}\Gamma^n_{lm}\delta^i_n \\
&= \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{kj} + \Gamma^m_{jl}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}
\end{aligned}$$

□

According to Corollary 1.3.2.4, there are only two types of components to calculate, as the others can be obtained through symmetry:

$$\mathrm{R}^i{}_{jil} \quad , \quad \mathrm{R}^i{}_{jij}.$$

Theorem 1.3.3.2: Values of the curvature and Riemann tensors

Let $i, j, k, l \in \{0, 1, 2, 3\}$ **distinct**, *i.e.*, $\{0, 1, 2, 3\} = \{i, j, k, l\}$.

(i) We have:

$$\begin{aligned}
\mathrm{R}^i{}_{jil} &= -u_{i,jl} + u_{i,j}\left(u_{j,l} - u_{i,l}\right) + u_{l,j}u_{i,l} \\
\mathrm{R}^j{}_{iil} &= -\eta_i\eta_j e^{2u_i - 2u_j}\mathrm{R}^i{}_{jil} \\
\mathrm{R}^j{}_{ili} &= -\mathrm{R}^j{}_{iil} \\
\mathrm{R}^i{}_{jli} &= -\mathrm{R}^i{}_{jil}
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\mathrm{R}^i{}_{jij} = &-\left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right) - \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right)e^{2u_j - 2u_i} \\
&- \eta_k\eta_j u_{i,k}u_{j,k}e^{2u_j - 2u_k} - \eta_l\eta_j u_{i,l}u_{j,l}e^{2u_j - 2u_l}
\end{aligned}$$

and:

$$\mathrm{R}^j{}_{iij} = -\eta_i\eta_j\mathrm{R}^i{}_{jij}.$$

(iii) All other components of the Riemann tensor are zero.

Proof. (i) We have:

$$\begin{aligned}
\mathrm{R}^i{}_{jil} =& \partial_i \Gamma^i_{jl} - \partial_l \Gamma^i_{ij} + \Gamma^m_{jl}\Gamma^i_{im} - \Gamma^m_{ij}\Gamma^i_{lm} \\
=& -\partial_l \Gamma^i_{ij} + \Gamma^j_{jl}\Gamma^i_{ij} + \Gamma^l_{jl}\Gamma^i_{il} - \Gamma^i_{ij}\Gamma^i_{li} \\
=& -\partial_l \left(u_{i,j}\right) + u_{j,l}u_{i,j} + u_{l,j}u_{i,l} - u_{i,j}u_{i,l} \\
=& -u_{i,jl} + u_{i,j}\left(u_{j,l} - u_{i,l}\right) + u_{l,j}u_{i,l} \\
=& -u_{i,jl} + u_{i,j}\left(u_{j,l} - u_{i,l}\right) + u_{l,j}u_{i,l}
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\mathrm{R}^i{}_{jij} =& \partial_i \Gamma^i_{jj} - \partial_j \Gamma^i_{ij} + \Gamma^m_{jj}\Gamma^i_{im} - \Gamma^m_{ij}\Gamma^i_{jm} \\
=& \partial_i \Gamma^i_{jj} - \partial_j \Gamma^i_{ij} + \Gamma^m_{jj}\Gamma^i_{im} - \Gamma^i_{ij}\Gamma^i_{ji} - \Gamma^j_{ij}\Gamma^i_{jj} \\
=& \partial_i \left(-\eta_i\eta_j u_{j,i} e^{2u_j - 2u_i}\right) - \partial_j \left(u_{i,j}\right) + \left(-\eta_m\eta_j u_{j,m} e^{2u_j - 2u_m}\right) u_{i,m} - u_{i,j}^2 - u_{j,i}\left(-\eta_i\eta_j u_{j,i} e^{2u_j - 2u_i}\right) \\
=& -\left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right) - \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right) e^{2u_j - 2u_i} \\
& - \eta_k\eta_j u_{i,k}u_{j,k} e^{2u_j - 2u_k} - \eta_l\eta_j u_{i,l}u_{j,l} e^{2u_j - 2u_l}
\end{aligned}$$

□

1.3.4 Practical computations of the curvature 2-form in $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$

Let X and Y be two vector fields. For any vector field ∂_j, the element $\mathscr{R}(X,Y)\partial_j$ is a vector field, so it can be decomposed in the $\mathscr{C}_\perp$ basis. This leads to the definition of the curvature 2-forms.

Definition 1.3.4.1: Curvature applications

Let X and Y be two vector fields, and $i,j \in \{0,1,2,3\}$. We define the scalar field $\Omega^i_j(X,Y)$ by:

$$\Omega^i_j(X,Y) := \theta^i\left(\mathscr{R}(X,Y)\theta_j\right).$$

Thus, we have a 2-form $\Omega^i_j \in \Lambda^2(U)$ that associates the scalar field $\Omega^i_j(X,Y)$ to every pair of vector fields X and Y. We have the decomposition:

$$\mathscr{R}(X,Y)\theta_j = \Omega^i_j(X,Y)\theta_i.$$

Lemma 1.3.4.2: Usual properties

Let $i,j,k \in \{0,1,2,3\}$.

(i) We have:

$$\Omega^i_j = \mathrm{d}\varpi^i{}_j + \varpi^i{}_k \wedge \varpi^k{}_j$$

(ii) We have:

$$\Omega^k{}_j = -\eta_k\eta_j\Omega^j{}_k.$$

(iii) We have:

$$\Omega^k{}_k = 0.$$

Proof. (i) For all vector fields X and Y, we have:

$$\begin{aligned}
\Omega^i_j(X,Y)\theta_i =& R(X,Y)\theta_j = \nabla_X\nabla_Y\theta_j - \nabla_Y\nabla_X\theta_j - \nabla_{[X,Y]}\theta_j \\
=& \nabla_X\left(\varpi^i{}_j(Y)\theta_i\right) - \nabla_Y\left(\varpi^i{}_j(X)\theta_i\right) - \varpi^i{}_j([X,Y])\theta_i \\
=& X\left(\varpi^k{}_j(Y)\right)\theta_k + \varpi^k{}_j(Y)\nabla_X\theta_k - Y\left(\varpi^k{}_j(X)\right)\theta_k - \varpi^k{}_j(X)\nabla_X\theta_k - \varpi^i{}_j([X,Y])\theta_i \\
=& X\left(\varpi^k{}_j(Y)\right)\theta_k + \varpi^k{}_j(Y)\varpi^i{}_k(X)\theta_i - Y\left(\varpi^k{}_j(X)\right)\theta_k - \varpi^k{}_j(X)\varpi^i{}_k(Y)\theta_i - \varpi^i{}_j([X,Y])\theta_i \\
=& X\left(\varpi^i{}_j(Y)\right)\theta_i + \varpi^k{}_j(Y)\varpi^i{}_k(X)\theta_i - Y\left(\varpi^i{}_j(X)\right)\theta_i - \varpi^k{}_j(X)\varpi^i{}_k(Y)\theta_i - \varpi^i{}_j([X,Y])\theta_i \\
=& \left(X\left(\varpi^i{}_j(Y)\right) + \varpi^k{}_j(Y)\varpi^i{}_k(X) - Y\left(\varpi^i{}_j(X)\right) - \varpi^k{}_j(X)\varpi^i{}_k(Y) - \varpi^i{}_j([X,Y])\right)\theta_i \\
=& \left(X\left(\varpi^i{}_j(Y)\right) - Y\left(\varpi^i{}_j(X)\right) - \varpi^i{}_j([X,Y]) + \varpi^k{}_j(Y)\varpi^i{}_k(X) - \varpi^k{}_j(X)\varpi^i{}_k(Y)\right)\theta_i \\
=& \left(\mathrm{d}\varpi^i{}_j(X,Y) + \left(\varpi^i{}_k\otimes\varpi^k{}_j - \varpi^k{}_j\otimes\varpi^i{}_k\right)(X,Y)\right)\theta_i \\
=& \left(\mathrm{d}\varpi^i{}_j + \varpi^i{}_k\wedge\varpi^k{}_j\right)(X,Y)\theta_i
\end{aligned}$$

Since this holds for all vector fields X, Y, and θ_i, we have the result.

(ii) We have:

$$\begin{aligned}
-\eta_k\eta_j\Omega^j{}_k &= -\eta_k\eta_j\left(\mathrm{d}\varpi^j{}_k + \varpi^j{}_i\wedge\varpi^i{}_k\right) \\
&= -\eta_k\eta_j\mathrm{d}\varpi^j{}_k - \eta_k\eta_j\varpi^j{}_i\wedge\varpi^i{}_k \\
&= \mathrm{d}\left(-\eta_k\eta_j\varpi^j{}_k\right) - \left(-\eta_i\eta_j\varpi^j{}_i\right)\wedge\left(-\eta_k\eta_i\varpi^i{}_k\right) \\
&= \mathrm{d}\varpi^k{}_j - \varpi^i{}_j\wedge\varpi^k{}_i \\
&= \mathrm{d}\varpi^k{}_j + \varpi^k{}_i\wedge\varpi^i{}_j \\
&= \Omega^k{}_j
\end{aligned}$$

(iii) We have by (ii):

$$\Omega^k{}_k = -\eta_k\eta_k\Omega^k{}_k = -\Omega^k{}_k$$

In other words, we have $\Omega^k{}_k = 0$.

□

Lemma 1.3.4.3: Lemma for the computation of the two forms Ω^i_j

Let $i,j,k,l\in\{0,1,2,3\}$ **distinct** such that $\{0,1,2,3\} = \{i,j,k,l\}$, and $p,q\neq i,j$.

(i) We have:

$$\varpi^i{}_p\wedge\varpi^p{}_j = u_{i,p}u_{p,j}e^{-u_p-u_j}\theta^{i,p} - \eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}\theta^{i,j} + \eta_i\eta_j u_{p,i}u_{j,p}e^{-u_i-u_p}\theta^{p,j}.$$

(ii) We have:

$$\mathrm{d}\varpi^i{}_j = \left(u_{i,jq} + u_{i,j}\left(u_{i,q} - u_{j,q}\right)\right)e^{-u_j-u_q}\theta^{q,i} - \eta_i\eta_j\left(u_{j,iq} + u_{j,i}\left(u_{j,q} - u_{i,q}\right)\right)e^{-u_i-u_q}\theta^{q,j}.$$

Proof. (i) We have:

$$\begin{aligned}
\varpi^i{}_p\wedge\varpi^p{}_j &= \left(u_{i,p}e^{-u_p}\theta^i - \eta_i\eta_p u_{p,i}e^{-u_i}\theta^p\right)\wedge\left(u_{p,j}e^{-u_j}\theta^p - \eta_p\eta_j u_{j,p}e^{-u_p}\theta^j\right) \\
&= u_{i,p}u_{p,j}e^{-u_p-u_j}\theta^{i,p} - \eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}\theta^{i,j} + \eta_i\eta_j u_{p,i}u_{j,p}e^{-u_i-u_p}\theta^{p,j}
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\mathrm{d}\varpi^i{}_j =& \mathrm{d}\left(u_{i,j}e^{-u_j}\theta^i - \eta_i\eta_j u_{j,i}e^{-u_i}\theta^j\right)\\
=& \mathrm{d}\left(u_{i,j}e^{-u_j}\theta^i\right) - \eta_i\eta_j\mathrm{d}\left(u_{j,i}e^{-u_i}\theta^j\right)\\
=& \mathrm{d}\left(u_{i,j}e^{-u_j}\right)\wedge\theta^i + u_{i,j}e^{-u_j}\mathrm{d}\left(\theta^i\right) - \eta_i\eta_j\mathrm{d}\left(u_{j,i}e^{-u_i}\right)\wedge\theta^j - \eta_i\eta_j u_{j,i}e^{-u_i}\mathrm{d}\left(\theta^j\right)\\
=& \left(u_{i,jq} - u_{i,j}u_{j,q}\right)e^{-u_j}\mathrm{d}x^q\wedge\theta^i + u_{i,j}e^{-u_j}u_{i,q}e^{-u_q}\theta^{q,i}\\
&- \eta_i\eta_j\left(u_{j,iq} - u_{j,i}u_{i,q}\right)e^{-u_i}\mathrm{d}x^q\wedge\theta^j - \eta_i\eta_j u_{j,i}e^{-u_i}u_{j,q}e^{-u_q}\theta^{q,j}\\
=& \left(u_{i,jq} + u_{i,j}\left(u_{i,q} - u_{j,q}\right)\right)e^{-u_j-u_q}\theta^{q,i} - \eta_i\eta_j\left(u_{j,iq} + u_{j,i}\left(u_{j,q} - u_{i,q}\right)\right)e^{-u_i-u_q}\theta^{q,j}
\end{aligned}$$

□

Proposition 1.3.4.4: Exact values of the two forms Ω^i_j

We have:

$$\begin{aligned}
\Omega^i_j =& \sum_{p\neq i,j}\left[\left(u_{i,jp} + u_{i,j}\left(u_{i,p} - u_{j,p}\right) - u_{i,p}u_{p,j}\right)e^{-u_p-u_j}\theta^{p,i} + \eta_i\eta_j\left(u_{p,i}u_{j,p} - u_{j,ip} + u_{j,i}\left(u_{i,p} - u_{j,p}\right)\right)e^{-u_i-u_p}\theta^{p,j}\right]\\
&+ \left[\left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right)e^{-2u_j} + \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right)e^{-2u_i} + \eta_j\sum_{p\neq i,j}\eta_p u_{i,p}u_{j,p}e^{-2u_p}\right]\theta^{j,i}
\end{aligned}$$

Proof. We have:

$$\begin{aligned}
\Omega^i_j =& \mathrm{d}\varpi^i{}_j + \sum_{p\neq i,j}\varpi^i{}_p\wedge\varpi^p{}_j\\
=& \left(u_{i,jq} + u_{i,j}\left(u_{i,q} - u_{j,q}\right)\right)e^{-u_j-u_q}\theta^{q,i} - \eta_i\eta_j\left(u_{j,iq} + u_{j,i}\left(u_{j,q} - u_{i,q}\right)\right)e^{-u_i-u_q}\theta^{q,j}\\
&+ \sum_{p\neq i,j} u_{i,p}u_{p,j}e^{-u_p-u_j}\theta^{i,p} - \eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}\theta^{i,j} + \eta_i\eta_j u_{p,i}u_{j,p}e^{-u_i-u_p}\theta^{p,j}\\
=& \left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right)e^{-u_j-u_j}\theta^{j,i} + \left(u_{i,jk} + u_{i,j}\left(u_{i,k} - u_{j,k}\right)\right)e^{-u_j-u_k}\theta^{k,i}\\
&+ \left(u_{i,jl} + u_{i,j}\left(u_{i,l} - u_{j,l}\right)\right)e^{-u_j-u_l}\theta^{l,i} - \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right)e^{-u_i-u_i}\theta^{i,j}\\
&- \eta_i\eta_j\left(u_{j,ik} + u_{j,i}\left(u_{j,k} - u_{i,k}\right)\right)e^{-u_i-u_k}\theta^{k,j} - \eta_i\eta_j\left(u_{j,il} + u_{j,i}\left(u_{j,l} - u_{i,l}\right)\right)e^{-u_i-u_l}\theta^{l,j}\\
&+ u_{i,k}u_{k,j}e^{-u_k-u_j}\theta^{i,k} - \eta_k\eta_j u_{i,k}u_{j,k}e^{-2u_k}\theta^{i,j} + \eta_i\eta_j u_{k,i}u_{j,k}e^{-u_i-u_k}\theta^{k,j}\\
&+ u_{i,l}u_{l,j}e^{-u_l-u_j}\theta^{i,l} - \eta_l\eta_j u_{i,l}u_{j,l}e^{-2u_l}\theta^{i,j} + \eta_i\eta_j u_{l,i}u_{j,l}e^{-u_i-u_l}\theta^{l,j}\\
=& \sum_{p\neq i,j}\left[\left(u_{i,jp} + u_{i,j}\left(u_{i,p} - u_{j,p}\right) - u_{i,p}u_{p,j}\right)e^{-u_j-u_p}\theta^{p,i} - \eta_i\eta_j\left(u_{j,ip} + u_{j,i}\left(u_{j,p} - u_{i,p}\right) - u_{p,i}u_{j,p}\right)e^{-u_i-u_p}\theta^{p,j}\right]\\
&+ \left[\left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right)e^{-2u_j} + \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right)e^{-2u_i} + \eta_j\sum_{p\neq i,j}\eta_p u_{i,p}u_{j,p}e^{-2u_p}\right]\theta^{j,i}
\end{aligned}$$

□

Example 1.3.4.5: Sequence 5 of metric h

We continue the example with the metric h (see 1.1.4.1, 1.1.5.1, 1.2.5.5, 1.2.6.3, and 1.2.6.5) with :

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

- We have :

$$\Omega^1_0 = \left(u_{1,02} + u_{1,0}\left(u_{1,2} - u_{0,2}\right) - u_{1,2}u_{2,0}\right)e^{-u_2-u_0}\theta^{2,1}$$

$$
\begin{aligned}
&+\eta_1\eta_0\left(u_{2,1}u_{0,2}-u_{0,12}+u_{0,1}\left(u_{1,2}-u_{0,2}\right)\right)e^{-u_1-u_2}\theta^{2,0}\\
&+\left(u_{1,03}+u_{1,0}\left(u_{1,3}-u_{0,3}\right)-u_{1,3}u_{3,0}\right)e^{-u_3-u_0}\theta^{3,1}\\
&+\eta_1\eta_0\left(u_{3,1}u_{0,3}-u_{0,13}+u_{0,1}\left(u_{1,3}-u_{0,3}\right)\right)e^{-u_1-u_3}\theta^{3,0}\\
&+\left[\left(u_{1,00}+u_{1,0}\left(u_{1,0}-u_{0,0}\right)\right)e^{-2u_0}+\eta_1\eta_0\left(u_{0,11}+u_{0,1}\left(u_{0,1}-u_{1,1}\right)\right)e^{-2u_1}\right]\theta^{0,1}\\
&+\left[\eta_0\eta_2u_{1,2}u_{0,2}e^{-2u_2}+\eta_0\eta_3u_{1,3}u_{0,3}e^{-2u_3}\right]\theta^{0,1}\\
=&\left[\left(u_{1,00}+u_{1,0}\left(u_{1,0}-u_{0,0}\right)\right)e^{-2u_0}+\eta_1\eta_0\left(u_{0,11}+u_{0,1}\left(u_{0,1}-u_{1,1}\right)\right)e^{-2u_1}\right]\theta^{0,1}\\
=&\left[\left(\ddot{v}+\dot{v}\left(\dot{v}-\dot{u}\right)\right)e^{-2u}-\left(u''+u'\left(u'-v'\right)\right)e^{-2v}\right]\theta^{0,1}
\end{aligned}
$$

- We have :

$$
\begin{aligned}
\Omega^2_0=&\left(u_{2,01}+u_{2,0}\left(u_{2,1}-u_{0,1}\right)-u_{2,1}u_{1,0}\right)e^{-u_1-u_0}\theta^{1,2}\\
&+\eta_2\eta_0\left(u_{1,2}u_{0,1}-u_{0,21}+u_{0,2}\left(u_{2,1}-u_{0,1}\right)\right)e^{-u_2-u_1}\theta^{1,0}\\
&+\left(u_{2,03}+u_{2,0}\left(u_{2,3}-u_{0,3}\right)-u_{2,3}u_{3,0}\right)e^{-u_3-u_0}\theta^{3,2}\\
&+\eta_2\eta_0\left(u_{3,2}u_{0,3}-u_{0,23}+u_{0,2}\left(u_{2,3}-u_{0,3}\right)\right)e^{-u_2-u_3}\theta^{3,0}\\
&+\left[\left(u_{2,00}+u_{2,0}\left(u_{2,0}-u_{0,0}\right)\right)e^{-2u_0}+\eta_2\eta_0\left(u_{0,22}+u_{0,2}\left(u_{0,2}-u_{2,2}\right)\right)e^{-2u_2}\right]\theta^{0,2}\\
&+\left[\eta_0\eta_1u_{2,1}u_{0,1}e^{-2u_1}+\eta_0\eta_3u_{2,3}u_{0,3}e^{-2u_3}\right]\theta^{0,2}\\
=&\left(u_{2,0}\left(u_{2,1}-u_{0,1}\right)-u_{2,1}u_{1,0}\right)e^{-u_1-u_0}\theta^{1,2}+\left[\left(u_{2,00}+u_{2,0}\left(u_{2,0}-u_{0,0}\right)\right)e^{-2u_0}\right]\theta^{0,2}\\
&+\eta_0\eta_1u_{2,1}u_{0,1}e^{-2u_1}\theta^{0,2}\\
=&\left(\dot{b}\left(r^{-1}-u'\right)-r^{-1}\dot{v}\right)e^{-u-v}\theta^{1,2}+\left[\left(\ddot{b}+\dot{b}\left(\dot{b}-\dot{u}\right)\right)e^{-2u}-r^{-1}u'e^{-2v}\right]\theta^{0,2}
\end{aligned}
$$

- We have :

$$
\begin{aligned}
\Omega^3_0=&\left(u_{3,01}+u_{3,0}\left(u_{3,1}-u_{0,1}\right)-u_{3,1}u_{1,0}\right)e^{-u_1-u_0}\theta^{1,3}\\
&+\eta_3\eta_0\left(u_{1,3}u_{0,1}-u_{0,31}+u_{0,3}\left(u_{3,1}-u_{0,1}\right)\right)e^{-u_3-u_1}\theta^{1,0}\\
&+\left(u_{3,02}+u_{3,0}\left(u_{3,2}-u_{0,2}\right)-u_{3,2}u_{2,0}\right)e^{-u_2-u_0}\theta^{2,3}\\
&+\eta_3\eta_0\left(u_{2,3}u_{0,2}-u_{0,32}+u_{0,3}\left(u_{3,2}-u_{0,2}\right)\right)e^{-u_3-u_2}\theta^{2,0}\\
&+\left[\left(u_{3,00}+u_{3,0}\left(u_{3,0}-u_{0,0}\right)\right)e^{-2u_0}+\eta_3\eta_0\left(u_{0,33}+u_{0,3}\left(u_{0,3}-u_{3,3}\right)\right)e^{-2u_3}\right]\theta^{0,3}\\
&+\left[\eta_0\eta_1u_{3,1}u_{0,1}e^{-2u_1}+\eta_0\eta_2u_{3,2}u_{0,2}e^{-2u_2}\right]\theta^{0,3}\\
=&\left(u_{3,0}\left(u_{3,1}-u_{0,1}\right)-u_{3,1}u_{1,0}\right)e^{-u_1-u_0}\theta^{1,3}+\left(u_{3,0}u_{3,2}-u_{3,2}u_{2,0}\right)e^{-u_2-u_0}\theta^{2,3}\\
&+\left[\left(u_{3,00}+u_{3,0}\left(u_{3,0}-u_{0,0}\right)\right)e^{-2u_0}+\eta_0\eta_1u_{3,1}u_{0,1}e^{-2u_1}\right]\theta^{0,3}\\
=&\left(\dot{b}\left(r^{-1}-u'\right)-r^{-1}\dot{v}\right)e^{-u-v}\theta^{1,3}+\left[\left(\ddot{b}+\dot{b}\left(\dot{b}-\dot{u}\right)\right)e^{-2u}-r^{-1}u'e^{-2v}\right]\theta^{0,3}
\end{aligned}
$$

- We have :

$$
\begin{aligned}
\Omega^2_1=&\left(u_{2,10}+u_{2,1}\left(u_{2,0}-u_{1,0}\right)-u_{2,0}u_{0,1}\right)e^{-u_0-u_1}\theta^{0,2}\\
&+\eta_2\eta_1\left(u_{0,2}u_{1,0}-u_{1,20}+u_{1,2}\left(u_{2,0}-u_{1,0}\right)\right)e^{-u_2-u_0}\theta^{0,1}\\
&+\left(u_{2,13}+u_{2,1}\left(u_{2,3}-u_{1,3}\right)-u_{2,3}u_{3,1}\right)e^{-u_3-u_1}\theta^{3,2}\\
&+\eta_2\eta_1\left(u_{3,2}u_{1,3}-u_{1,23}+u_{1,2}\left(u_{2,3}-u_{1,3}\right)\right)e^{-u_2-u_3}\theta^{3,1}\\
&+\left[\left(u_{2,11}+u_{2,1}\left(u_{2,1}-u_{1,1}\right)\right)e^{-2u_1}+\eta_2\eta_1\left(u_{1,22}+u_{1,2}\left(u_{1,2}-u_{2,2}\right)\right)e^{-2u_2}\right]\theta^{1,2}\\
&+\left[\eta_1\eta_0u_{2,0}u_{1,0}e^{-2u_0}+\eta_1\eta_3u_{2,3}u_{1,3}e^{-2u_3}\right]\theta^{1,2}\\
=&\left(u_{2,1}\left(u_{2,0}-u_{1,0}\right)-u_{2,0}u_{0,1}\right)e^{-u_0-u_1}\theta^{0,2}\\
&+\left[\left(u_{2,11}+u_{2,1}\left(u_{2,1}-u_{1,1}\right)\right)e^{-2u_1}+\eta_1\eta_0u_{2,0}u_{1,0}e^{-2u_0}\right]\theta^{1,2}\\
=&\left(r^{-1}\left(\dot{b}-\dot{v}\right)-\dot{b}u'\right)e^{-u-v}\theta^{0,2}+\left[-r^{-1}v'e^{-2v}-\dot{v}\dot{b}e^{-2u}\right]\theta^{1,2}
\end{aligned}
$$

- We have :

$$\begin{aligned}
\Omega^3_1 &= \left(u_{3,10} + u_{3,1}\left(u_{3,0} - u_{1,0}\right) - u_{3,0}u_{0,1}\right) e^{-u_0-u_1}\theta^{0,3} \\
&\quad + \eta_3\eta_1\left(u_{0,3}u_{1,0} - u_{1,30} + u_{1,3}\left(u_{3,0} - u_{1,0}\right)\right) e^{-u_3-u_0}\theta^{0,1} \\
&\quad + \left(u_{3,12} + u_{3,1}\left(u_{3,2} - u_{1,2}\right) - u_{3,2}u_{2,1}\right) e^{-u_2-u_1}\theta^{2,3} \\
&\quad + \eta_3\eta_1\left(u_{2,3}u_{1,2} - u_{1,32} + u_{1,3}\left(u_{3,2} - u_{1,2}\right)\right) e^{-u_3-u_2}\theta^{2,1} \\
&\quad + \left[\left(u_{3,11} + u_{3,1}\left(u_{3,1} - u_{1,1}\right)\right) e^{-2u_1} + \eta_3\eta_1\left(u_{1,33} + u_{1,3}\left(u_{1,3} - u_{3,3}\right)\right) e^{-2u_3}\right]\theta^{1,3} \\
&\quad + \left[\eta_1\eta_0 u_{3,0}u_{1,0}e^{-2u_0} + \eta_1\eta_2 u_{3,2}u_{1,2}e^{-2u_2}\right]\theta^{1,3} \\
&= \left(u_{3,1}\left(u_{3,0} - u_{1,0}\right) - u_{3,0}u_{0,1}\right) e^{-u_0-u_1}\theta^{0,3} + \left(u_{3,1}u_{3,2} - u_{3,2}u_{2,1}\right) e^{-u_2-u_1}\theta^{2,3} \\
&\quad + \left[\left(u_{3,11} + u_{3,1}\left(u_{3,1} - u_{1,1}\right)\right) e^{-2u_1} + \eta_1\eta_0 u_{3,0}u_{1,0}e^{-2u_0}\right]\theta^{1,3} \\
&= \left(r^{-1}\left(\dot{b} - \dot{v}\right) - \dot{b}u'\right) e^{-u-v}\theta^{0,3} + \left[-r^{-1}v'e^{-2v} - \dot{v}\dot{b}e^{-2u}\right]\theta^{1,3}
\end{aligned}$$

- We have :

$$\begin{aligned}
\Omega^3_2 &= \left(u_{3,20} + u_{3,2}\left(u_{3,0} - u_{2,0}\right) - u_{3,0}u_{0,2}\right) e^{-u_0-u_2}\theta^{0,3} \\
&\quad + \eta_3\eta_2\left(u_{0,3}u_{2,0} - u_{2,30} + u_{2,3}\left(u_{3,0} - u_{2,0}\right)\right) e^{-u_3-u_0}\theta^{0,2} \\
&\quad + \left(u_{3,21} + u_{3,2}\left(u_{3,1} - u_{2,1}\right) - u_{3,1}u_{1,2}\right) e^{-u_1-u_2}\theta^{1,3} \\
&\quad + \eta_3\eta_2\left(u_{1,3}u_{2,1} - u_{2,31} + u_{2,3}\left(u_{3,1} - u_{2,1}\right)\right) e^{-u_3-u_1}\theta^{1,2} \\
&\quad + \left[\left(u_{3,22} + u_{3,2}\left(u_{3,2} - u_{2,2}\right)\right) e^{-2u_2} + \eta_3\eta_2\left(u_{2,33} + u_{2,3}\left(u_{2,3} - u_{3,3}\right)\right) e^{-2u_3}\right]\theta^{2,3} \\
&\quad + \left[\eta_2\eta_0 u_{3,0}u_{2,0}e^{-2u_0} + \eta_2\eta_1 u_{3,1}u_{2,1}e^{-2u_1}\right]\theta^{2,3} \\
&= u_{3,2}\left(u_{3,0} - u_{2,0}\right) e^{-u_0-u_2}\theta^{0,3} + u_{3,2}\left(u_{3,1} - u_{2,1}\right) e^{-u_1-u_2}\theta^{1,3} \\
&\quad + \left[\left(u_{3,22} + \left(u_{3,2}\right)^2\right) e^{-2u_2} + \eta_2\eta_0 u_{3,0}u_{2,0}e^{-2u_0} + \eta_2\eta_1 u_{3,1}u_{2,1}e^{-2u_1}\right]\theta^{2,3} \\
&= \left[-r^{-2}e^{-2b} - \left(\dot{b}\right)^2 e^{-2u} + r^{-2}e^{-2v}\right]\theta^{2,3}
\end{aligned}$$

1.3.5 Practical computations in $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$

We start with a link between $\mathbb{R}^i{}_{jkl}$ and Ω^i_j.

Lemma 1.3.5.1: Link with the 2-forms Ω^i_j

We have:

$$\mathbb{R}^i{}_{jkl} = \Omega^i_j\left(\theta_k, \theta_l\right).$$

Proof. By definition of $\mathbb{R}^i{}_{jkl}$, we have:

$$\begin{aligned}
\mathbb{R}^i{}_{jkl} &= \theta^i\left(R\left(\theta_k, \theta_l\right)\theta_j\right) \\
&= \theta^i\left(\Omega^l_j\left(\theta_k, \theta_l\right)\theta_l\right) \\
&= \Omega^l_j\left(\theta_k, \theta_l\right)\delta^i_l \\
&= \Omega^i_j\left(\theta_k, \theta_l\right)
\end{aligned}$$

□

Therefore, we have:

$$\begin{aligned}
\mathbb{R}^i{}_{jil} &= \Omega^i_j\left(\theta_i, \theta_l\right) \\
&= \mathrm{d}\varpi^i{}_j\left(\theta_i, \theta_l\right) + \varpi^i{}_p \wedge \varpi^p{}_j\left(\theta_i, \theta_l\right)
\end{aligned}$$

The following lemma gives the value of each term on the right-hand side.

Lemma 1.3.5.2: Preliminary lemma for the calculation of Riemann tensors

Let i,j,k,l such that $\{0,1,2,3\}=\{i,j,k,l\}$ and $p\neq i,j$.

(i) (a) We have:

$$\varpi^i{}_p\wedge\varpi^p{}_j\left(\theta_i,\theta_l\right)=\delta^p_l u_{i,l}u_{l,j}e^{-u_l-u_j}.$$

(b) We have:

$$\varpi^i{}_p\wedge\varpi^p{}_j\left(\theta_i,\theta_j\right)=-\eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}.$$

(ii) (a) We have:

$$\mathrm{d}\varpi^i{}_j\left(\theta_i,\theta_l\right)=-\left(u_{i,jl}+u_{i,j}\left(u_{i,l}-u_{j,l}\right)\right)e^{-u_j-u_l}.$$

(b) We have:

$$\mathrm{d}\varpi^i{}_j\left(\theta_i,\theta_j\right)=-\left(u_{i,jj}+u_{i,j}\left(u_{i,j}-u_{j,j}\right)\right)e^{-2u_j}-\eta_i\eta_j\left(u_{j,ii}+u_{j,i}\left(u_{j,i}-u_{i,i}\right)\right)e^{-2u_i}.$$

Proof. (i) (a) We have:

$$\begin{aligned}\varpi^i{}_p\wedge\varpi^p{}_j\left(\theta_i,\theta_l\right)=&u_{i,p}u_{p,j}e^{-u_p-u_j}\theta^{i,p}\left(\theta_i,\theta_l\right)-\eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}\theta^{i,j}\left(\theta_i,\theta_l\right)\\&+\eta_i\eta_j u_{p,i}u_{j,p}e^{-u_i-u_p}\theta^{p,j}\left(\theta_i,\theta_l\right)\\=&u_{i,p}u_{p,j}e^{-u_p-u_j}\theta^{i,p}\left(\theta_i,\theta_l\right)\\=&\delta^p_l u_{i,l}u_{l,j}e^{-u_l-u_j}\end{aligned}$$

(b) We have:

$$\begin{aligned}\varpi^i{}_p\wedge\varpi^p{}_j\left(\theta_i,\theta_j\right)=&u_{i,p}u_{p,j}e^{-u_p-u_j}\theta^{i,p}\left(\theta_i,\theta_j\right)-\eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}\theta^{i,j}\left(\theta_i,\theta_j\right)\\&+\eta_i\eta_j u_{p,i}u_{j,p}e^{-u_i-u_p}\theta^{p,j}\left(\theta_i,\theta_j\right)\\=&-\eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}\theta^{i,j}\left(\theta_i,\theta_j\right)\\=&-\eta_p\eta_j u_{i,p}u_{j,p}e^{-2u_p}\end{aligned}$$

(ii) (a) We have:

$$\begin{aligned}\mathrm{d}\varpi^i{}_j\left(\theta_i,\theta_l\right)=&\left(u_{i,jq}+u_{i,j}\left(u_{i,q}-u_{j,q}\right)\right)e^{-u_j-u_q}\theta^{q,i}\left(\theta_i,\theta_l\right)\\&-\eta_i\eta_j\left(u_{j,iq}+u_{j,i}\left(u_{j,q}-u_{i,q}\right)\right)e^{-u_i-u_q}\theta^{q,j}\left(\theta_i,\theta_l\right)\\=&\left(u_{i,jq}+u_{i,j}\left(u_{i,q}-u_{j,q}\right)\right)e^{-u_j-u_q}\theta^{q,i}\left(\theta_i,\theta_l\right)\\=&-\left(u_{i,jl}+u_{i,j}\left(u_{i,l}-u_{j,l}\right)\right)e^{-u_j-u_l}\end{aligned}$$

(b) We have:

$$\begin{aligned}\mathrm{d}\varpi^i{}_j\left(\theta_i,\theta_j\right)=&\left(u_{i,jq}+u_{i,j}\left(u_{i,q}-u_{j,q}\right)\right)e^{-u_j-u_q}\theta^{q,i}\left(\theta_i,\theta_j\right)\\&-\eta_i\eta_j\left(u_{j,iq}+u_{j,i}\left(u_{j,q}-u_{i,q}\right)\right)e^{-u_i-u_q}\theta^{q,j}\left(\theta_i,\theta_j\right)\\=&-\delta^q_j\left(u_{i,jq}+u_{i,j}\left(u_{i,q}-u_{j,q}\right)\right)e^{-u_j-u_q}-\delta^q_i\eta_i\eta_j\left(u_{j,iq}+u_{j,i}\left(u_{j,q}-u_{i,q}\right)\right)e^{-u_i-u_q}\\=&-\left(u_{i,jj}+u_{i,j}\left(u_{i,j}-u_{j,j}\right)\right)e^{-2u_j}-\eta_i\eta_j\left(u_{j,ii}+u_{j,i}\left(u_{j,i}-u_{i,i}\right)\right)e^{-2u_i}\end{aligned}$$

□

We deduce the exact values of the components of the Riemann tensor $\mathbb{R}^i{}_{jkl}$.

Theorem 1.3.5.3: Values of the Riemann tensor

Let i, j, k, l be such that $\{0,1,2,3\} = \{i,j,k,l\}$.

(i) We have:

$$\begin{aligned}
\mathbb{R}^i{}_{jil} &= -\left(u_{i,jl} + u_{i,j}\left(u_{i,l} - u_{j,l}\right) - u_{i,l}u_{l,j}\right)e^{-u_j - u_l} \\
\mathbb{R}^j{}_{iil} &= -\eta_i\eta_j\mathbb{R}^i{}_{jil} \\
\mathbb{R}^j{}_{ili} &= -\mathbb{R}^j{}_{iil} \\
\mathbb{R}^i{}_{jli} &= -\mathbb{R}^i{}_{jil}
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\mathbb{R}^i{}_{jij} = &-\left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right)e^{-2u_j} - \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right)e^{-2u_i} \\
&- \eta_k\eta_j u_{i,k}u_{j,k}e^{-2u_k} - \eta_l\eta_j u_{i,l}u_{j,l}e^{-2u_l}
\end{aligned}$$

and:

$$\mathbb{R}^j{}_{iij} = -\eta_i\eta_j\mathbb{R}^i{}_{jij}.$$

(iii) All other values of the Riemann tensor are zero.

Proof. (i) We have:

$$\begin{aligned}
\mathbb{R}^i{}_{jil} &= \Omega^i_j\left(\theta_i, \theta_l\right) \\
&= \mathrm{d}\varpi^i{}_j\left(\theta_i, \theta_l\right) + \varpi^i{}_p \wedge \varpi^p{}_j\left(\theta_i, \theta_l\right) \\
&= \mathrm{d}\varpi^i{}_j\left(\theta_i, \theta_l\right) + \varpi^i{}_k \wedge \varpi^k{}_j\left(\theta_i, \theta_l\right) + \varpi^i{}_l \wedge \varpi^l{}_j\left(\theta_i, \theta_l\right) \\
&= -\left(u_{i,jl} + u_{i,j}\left(u_{i,l} - u_{j,l}\right)\right)e^{-u_j - u_l} + u_{i,l}u_{l,j}e^{-u_l - u_j} \\
&= -\left(u_{i,jl} + u_{i,j}\left(u_{i,l} - u_{j,l}\right) - u_{i,l}u_{l,j}\right)e^{-u_j - u_l}
\end{aligned}$$

(ii) We have:

$$\begin{aligned}
\mathbb{R}^i{}_{jij} &= \Omega^i_j\left(\theta_i, \theta_j\right) \\
&= \mathrm{d}\varpi^i{}_j\left(\theta_i, \theta_j\right) + \varpi^i{}_p \wedge \varpi^p{}_j\left(\theta_i, \theta_j\right) \\
&= \mathrm{d}\varpi^i{}_j\left(\theta_i, \theta_j\right) + \varpi^i{}_k \wedge \varpi^k{}_j\left(\theta_i, \theta_j\right) + \varpi^i{}_l \wedge \varpi^l{}_j\left(\theta_i, \theta_j\right) \\
&= -\left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right)e^{-2u_j} - \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right)e^{-2u_i} \\
&\quad - \eta_k\eta_j u_{i,k}u_{j,k}e^{-2u_k} - \eta_l\eta_j u_{i,l}u_{j,l}e^{-2u_l}
\end{aligned}$$

(iii) Since $\Omega^i_i = 0$, we have for all p, q:

$$\mathbb{R}^i{}_{ipq} = 0.$$

Since Ω^i_j is an alternating form, we have:

$$\mathbb{R}^p{}_{qii} = 0.$$

Using the general expression found for Ω^i_j in Proposition 1.3.4.4, we have $\mathbb{R}^i{}_{jkl} = 0$.

□

The values can be deduced in the case of the metric h.

Example 1.3.5.4: Sequence 6 of the metric h

We resume the example with the metric h (see 1.1.4.1, 1.1.5.1, 1.2.5.5, 1.2.6.3, 1.2.6.5, and 1.3.4.5) with:

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

We consider two cases:

$$(1)\ \ \mathbb{R}^i{}_{jij} \quad , \quad (2)\ \ \mathbb{R}^i{}_{jil}.$$

(1) **Calculations of $\mathbb{R}^i{}_{jij}$.**

- Case where $i := 0$. We have:

$$\begin{aligned}
\mathbb{R}^0{}_{101} &= -\left(u_{0,11} + u_{0,1}\left(u_{0,1} - u_{1,1}\right)\right)e^{-2u_1} \\
&\quad - \eta_0\eta_1\left(u_{1,00} + u_{1,0}\left(u_{1,0} - u_{0,0}\right)\right)e^{-2u_0} \\
&\quad - \eta_2\eta_1 u_{0,2}u_{1,2}e^{-2u_2} - \eta_3\eta_1 u_{0,3}u_{1,3}e^{-2u_3} \\
&= -\left(u_{0,11} + u_{0,1}\left(u_{0,1} - u_{1,1}\right)\right)e^{-2u_1} - \eta_0\eta_1\left(u_{1,00} + u_{1,0}\left(u_{1,0} - u_{0,0}\right)\right)e^{-2u_0} \\
&= -\left(u'' + u'\left(u' - v'\right)\right)e^{-2v-2b} + \left(\ddot{v} + \dot{v}\left(\dot{v} - \dot{u}\right)\right)e^{-2b-2u}
\end{aligned}$$

- Case where $j := 2, 3$ $(k := 3, 2)$. We have:

$$\begin{aligned}
\mathbb{R}^0{}_{j0j} &= -\left(u_{0,jj} + u_{0,j}\left(u_{0,j} - u_{j,j}\right)\right)e^{-2u_j} \\
&\quad - \eta_0\eta_j\left(u_{j,00} + u_{j,0}\left(u_{j,0} - u_{0,0}\right)\right)e^{-2u_0} \\
&\quad - \eta_1\eta_j u_{0,1}u_{j,1}e^{-2u_1} - \eta_k\eta_j u_{0,k}u_{j,k}e^{-2u_k} \\
&= -\eta_0\eta_j\left(u_{j,00} + u_{j,0}\left(u_{j,0} - u_{0,0}\right)\right)e^{-2u_0} - \eta_1\eta_j u_{0,1}u_{j,1}e^{-2u_1} \\
&= \left(\ddot{b} + \dot{b}\left(\dot{b} - \dot{u}\right)\right)e^{-2u} - r^{-1}u'e^{-2v} \\
\mathbb{R}^1{}_{j1j} &= -\left(u_{1,jj} + u_{1,j}\left(u_{1,j} - u_{j,j}\right)\right)e^{-2u_j} \\
&\quad - \eta_1\eta_j\left(u_{j,11} + u_{j,1}\left(u_{j,1} - u_{1,1}\right)\right)e^{-2u_1} \\
&\quad - \eta_0\eta_j u_{1,0}u_{j,0}e^{-2u_0} - \eta_k\eta_j u_{1,k}u_{j,k}e^{-2u_k} \\
&= -\eta_1\eta_j\left(u_{j,11} + u_{j,1}\left(u_{j,1} - u_{1,1}\right)\right)e^{-2u_1} - \eta_0\eta_j u_{1,0}u_{j,0}e^{-2u_0} \\
&= -r^{-1}v'e^{-2v} + \dot{v}\dot{b}e^{-2u}
\end{aligned}$$

- Case where $i := 2$. We have:

$$\begin{aligned}
\mathbb{R}^2{}_{323} &= -\left(u_{2,33} + u_{2,3}\left(u_{2,3} - u_{3,3}\right)\right)e^{-2u_3} \\
&\quad - \eta_2\eta_3\left(u_{3,22} + u_{3,2}\left(u_{3,2} - u_{2,2}\right)\right)e^{-2u_2} \\
&\quad - \eta_0\eta_3 u_{2,0}u_{3,0}e^{-2u_0} - \eta_1\eta_3 u_{2,1}u_{3,1}e^{-2u_1} \\
&= -\eta_2\eta_3\left(u_{3,22} + \left(u_{3,2}\right)^2\right)e^{-2u_2} \\
&\quad - \eta_0\eta_3 u_{2,0}u_{3,0}e^{-2u_0} - \eta_1\eta_3 u_{2,1}u_{3,1}e^{-2u_1} \\
&= r^{-2}e^{-2b} + \left(\dot{b}\right)^2 e^{-2u} - r^{-2}e^{-2v}
\end{aligned}$$

The other terms are obtained by symmetry.

(2) **Calculations of $\mathbb{R}^i{}_{jil}$.** We have:

$$\begin{aligned}
\mathbb{R}^i{}_{jil} &= -\left(u_{i,jl} + u_{i,j}\left(u_{i,l} - u_{j,l}\right) + u_{i,l}u_{l,j}\right)e^{-u_j-u_l} \\
&= -\left(u_{i,j}\left(u_{i,l} - u_{j,l}\right) + u_{i,l}u_{l,j}\right)e^{-u_j-u_l}
\end{aligned}$$

- Case where $j, l = 3$. We have:

$$\mathbb{R}^i{}_{3il} = -\left(u_{i,3}\left(u_{i,l} - u_{3,l}\right) + u_{i,l}u_{l,3}\right)e^{-u_3-u_l} = 0$$
$$\mathbb{R}^i{}_{ji3} = -\left(u_{i,j}\left(u_{i,3} - u_{j,3}\right) + u_{i,3}u_{l,j}\right)e^{-u_j-u_3} = 0$$

- Case where $i \coloneqq 0, 1$ (thus $j, l = 1, 0 = 1 - i$). We have:

$$\mathbb{R}^i{}_{2il} = -\left(u_{i,2}\left(u_{i,l} - u_{2,l}\right) + u_{i,l}u_{l,2}\right)e^{-u_2-u_l} = 0$$
$$\mathbb{R}^i{}_{ji2} = -\left(u_{i,j}\left(u_{i,2} - u_{j,2}\right) + u_{i,2}u_{2,j}\right)e^{-u_j-u_2} = 0$$

- Case where $i \coloneqq 3$. For $j, l = 0, 1$ we have:

$$\begin{aligned}
\mathbb{R}^3{}_{230} &= -\left(u_{3,2}\left(u_{3,l} - u_{2,l}\right) - u_{3,l}u_{l,2}\right)e^{-u_2-u_l} \\
&= -\left(u_{3,2}\left(u_{2,l} - u_{2,l}\right)\right)e^{-u_2-u_l} \\
&= 0 \\
\mathbb{R}^3{}_{j32} &= -\left(u_{3,j}\left(u_{3,2} - u_{j,2}\right) - u_{3,2}u_{2,j}\right)e^{-u_j-u_2} \\
&= -\left(u_{3,j}u_{3,2} - u_{3,2}u_{2,j}\right)e^{-u_j-u_2} \\
&= -\left(u_{2,j}u_{3,2} - u_{3,2}u_{2,j}\right)e^{-u_j-u_2} \\
&= 0
\end{aligned}$$

- Case where $i \coloneqq 2, 3$. We have:

$$\begin{aligned}
\mathbb{R}^i{}_{0i1} &= -\left(u_{i,0}\left(u_{i,1} - u_{0,1}\right) - u_{i,1}u_{1,0}\right)e^{-u_0-u_1} \\
&= -\left(\dot{b}\left(r^{-1} - u'\right) - r^{-1}\dot{v}\right)e^{-u-v} \\
\mathbb{R}^i{}_{1i0} &= -\left(u_{i,1}\left(u_{i,0} - u_{1,0}\right) - u_{i,0}u_{0,1}\right)e^{-u_1-u_0} \\
&= -\left(r^{-1}\left(\dot{b} - \dot{v}\right) - u'\dot{b}\right)e^{-u-v}
\end{aligned}$$

These are the only non-zero terms of the form $\mathbb{R}^i{}_{jil}$.

1.4 Tensor of Ricci

1.4.1 Generalities

The curvature tensor $\mathscr{R}$ is of type $(1,3)$, so we can contract the upper index with the second index.

Definition 1.4.1.1: The Ricci tensor

The **Ricci tensor** is defined by:

$$\mathscr{R}\mathrm{ic} = [\mathscr{R}]_2^1$$

That is, for any vector fields Y and Z:

$$\mathscr{R}\mathrm{ic}(Y, Z) = [\mathscr{R}]_2^1(Y, Z).$$

1.4.2 Computation of the components in $\mathscr{E}$ and $\mathscr{E}^*$

Definition 1.4.2.1: The Ricci tensor

The **components of the Ricci tensor in** $\mathscr{E}$ **and** $\mathscr{E}^*$ are:

$$\mathbf{R}_{jl} \coloneqq \mathscr{R}\text{ic}(e_j, e_l).$$

Therefore, we have the decomposition:

$$\mathscr{R}\text{ic} = \mathbf{R}_{jl}\ e_j \otimes e_l.$$

Next, we demonstrate the usual properties of the Ricci tensor. For part (ii), note that for any vector fields Y and Z, we have a linear mapping from TM (at each point p of M, we have a linear mapping from T_pM):

$$X \longmapsto \mathscr{R}(X,Y)Z.$$

Thus, we can consider the trace of this linear mapping.

Proposition 1.4.2.2: Usual properties

Let $j,k,l,n \in \{0,1,2,3\}$ and two vector fields Y and Z.

(i) (**Symmetry**) We have:

$$\mathbf{R}_{jl} = \mathbf{R}_{lj}.$$

(ii) (a) We have:

$$\mathscr{R}\text{ic}(Y,Z) = \text{tr}\left(X \longmapsto \mathscr{R}(X,Y)Z\right).$$

(b) We have:

$$\mathbf{R}_{jl} = \mathbf{R}^i{}_{jil}.$$

(iii) (**Contracted Bianchi identity**) We have:

$$\nabla_n \mathbf{R}_{jl} - \nabla_l \mathbf{R}_{jn} = -\nabla_k \mathbf{R}^k{}_{jln}.$$

Proof. (i) We use the first three points of Proposition 1.3.2.3.

- By point (i), we have:

$$\mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk} = 0$$

- By point (iii), we have:

$$\begin{aligned} g^{ki}\mathbf{R}_{iklj} &= g^{ik}\mathbf{R}_{iklj} \\ &= -g^{ik}\mathbf{R}_{kilj} \\ &= -g^{ki}\mathbf{R}_{ikjl} \end{aligned}$$

  Thus, we have:

$$g^{ki}\mathbf{R}_{iklj} = 0.$$

- By point (ii), we have:

$$\begin{aligned} 0 &= g^{ki}\left(\mathbf{R}_{ijkl} + \mathbf{R}_{iklj} + \mathbf{R}_{iljk}\right) \\ &= \mathbf{R}^k{}_{jkl} + 0 + \mathbf{R}^k{}_{ljk} \\ &= \mathbf{R}_{jl} - \mathbf{R}^k{}_{lkj} \\ &= \mathbf{R}_{jl} - \mathbf{R}_{lj} \end{aligned}$$

Thus, we have:

$$\mathbf{R}_{jl} = \mathbf{R}_{lj}.$$

(ii) We prove both points simultaneously. Recall that for any endomorphism ϕ of TM, the **trace of** ϕ is defined by:

$$\mathrm{tr}(\phi) := e^i\left(\phi(e_i)\right).$$

The trace is independent of the chosen basis.

Since:

$$\mathscr{R} = \mathbf{R}^i{}_{mkn}\; e_i \otimes e^m \otimes e^k \otimes e^n$$

we have:

$$\begin{aligned}
[\mathscr{R}]_2^1 &= \mathbf{R}^i{}_{mkn}\; e^k(e_i) e^m \otimes e^n \\
&= \mathbf{R}^i{}_{mkn}\; \delta_i^k e^m \otimes e^n \\
&= \mathbf{R}^i{}_{min}\; e^m \otimes e^n
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\mathscr{R}\mathrm{ic}(e_j, e_l) &= [\mathscr{R}]_2^1\,(e_j, e_l) \\
&= \left(\mathbf{R}^i{}_{min}\; e^m \otimes e^n\right)(e_j, e_l) \\
&= \mathbf{R}^i{}_{mkn}\; e^m(e_j) e^n(e_l) \\
&= \mathbf{R}^i{}_{min}\; \delta_j^m \delta_l^n \\
&= \mathbf{R}^i{}_{jil}
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
\mathrm{tr}\left(X \longmapsto \mathscr{R}(X, e_j) e_l\right) &= e^p\left(\mathscr{R}(e_p, e_p) e_l\right) \\
&= e^p\left(\mathbf{R}^i{}_{mkn}\; e^m(e_l) e^k(e_p) e^n(e_j) e_i\right) \\
&= \mathbf{R}^i{}_{mkn}\; \delta_l^m \delta_p^k \delta_j^n e^p(e_i) \\
&= \mathbf{R}^i{}_{lpj}\; \delta_i^p \\
&= \mathbf{R}^i{}_{lij} \\
&= \mathbf{R}^i{}_{jil}
\end{aligned}$$

Hence, the result follows from the bilinearity of $\mathscr{R}\mathrm{ic}$ and $(Y, Z) \longmapsto \mathrm{tr}\left(X \longmapsto \mathscr{R}(X, Y) Z\right)$.

(iii) We use points (ii) and (iv) of Proposition 1.3.2.3.

- By point (iv), we have:

$$\nabla_n \mathbf{R}_{ijkl} + \nabla_k \mathbf{R}_{ijln} + \nabla_l \mathbf{R}_{ijnk} = 0.$$

$$\nabla_n \mathbf{R}_{ijkl} + \nabla_j \mathbf{R}_{iknl} + \nabla_k \mathbf{R}_{injl} = 0.$$

- Thus, with point (ii), we have:

$$\begin{aligned}
0 &= g^{ki}\left(\nabla_n \mathbf{R}_{ijkl} + \nabla_k \mathbf{R}_{ijln} + \nabla_l \mathbf{R}_{ijnk}\right) \\
&= \nabla_n \mathbf{R}^k{}_{jkl} + \nabla_k \mathbf{R}^k{}_{jln} + \nabla_l \mathbf{R}^k{}_{jnk} \\
&= \nabla_n \mathbf{R}_{jl} + \nabla_k \mathbf{R}^k{}_{jln} - \nabla_l \mathbf{R}^k{}_{jkn}
\end{aligned}$$

Thus, we have:

$$\nabla_n \mathbf{R}_{jl} - \nabla_l \mathbf{R}_{jn} = -\nabla_k \mathbf{R}^k{}_{jln}.$$

□

Example 1.4.2.3: Components of the Ricci tensor in bases $\mathscr{C}$ and $\mathscr{C}_\perp$

(i) The **components of the Ricci tensor in $\mathscr{C}$ and $\mathscr{C}^*$** are given by:

$$\mathrm{R}_{ij} := \mathrm{R}^k{}_{ikj}.$$

Thus, we have the decomposition:

$$\mathscr{R}\mathrm{ic} = \mathrm{R}_{jl}\ \mathrm{d}x^j \otimes \mathrm{d}x^l.$$

(ii) The **components of the Ricci tensor in $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$** are given by:

$$\mathbb{R}_{ij} := \mathbb{R}^k{}_{ikj}.$$

Thus, we have the decomposition:

$$\mathscr{R}\mathrm{ic} = \mathbb{R}_{jl}\ \theta^j \otimes \theta^l.$$

We also have:

$$\mathbb{R}_{ij} = e^{-u_i-u_j}\mathrm{R}_{ij}.$$

1.4.3 Calculation of components in $\mathscr{C}$ and $\mathscr{C}^*$

Lemma 1.4.3.1: Usual properties

We have:

$$\mathrm{R}_{jl} = \partial_i\Gamma^i_{jl} - \partial_l\Gamma^i_{ij} + \Gamma^p_{jl}\Gamma^i_{ip} - \Gamma^p_{ij}\Gamma^i_{lp}.$$

Proof. By Proposition 1.3.3.1, we have:

$$\begin{aligned}\mathrm{R}_{jl} &= \mathrm{R}^i{}_{jil}\\ &= \partial_i\Gamma^i_{jl} - \partial_l\Gamma^i_{ij} + \Gamma^p_{jl}\Gamma^i_{ip} - \Gamma^p_{ij}\Gamma^i_{lp}\end{aligned}$$

□

Notation 1.4.3.2: Hat notation

We denote:

$$\hat{u}_{i,i} := \sum_{k\neq i} u_{k,i} \qquad \hat{u}_{i,jj} := \sum_{k\neq i} u_{k,jj} \qquad \hat{u}^2_{i,i} := \sum_{k\neq i} u^2_{k,i} \qquad \hat{u}_{ij,j} := \sum_{k\neq i,j} u_{k,j} \qquad \hat{u}_{ij,jj} := \sum_{k\neq i,j} u_{k,jj}$$

Theorem 1.4.3.3: Exact values of R_{jl}

Let $j,l \in \{0,1,2,3\}$ be distinct. We have:

$$\begin{aligned}\mathrm{R}_{jl} &= -\left[\sum_{p\neq j,l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l}\right]\\ \mathrm{R}_{jj} &= -\left(\hat{u}_{j,jj} + \hat{u}^2_{j,j} - u_{j,j}\hat{u}_{j,j}\right) - \eta_j\sum_{p\neq j}\eta_p\left[u_{j,pp} + u_{j,p}\left(\hat{u}_{p,p} - u_{p,p}\right)\right]e^{2u_j-2u_p}\end{aligned}$$

Proof. We have two cases.

- We have:

$$\begin{aligned}
\mathrm{R}_{jl} =& \partial_i \Gamma^i_{jl} - \partial_j \Gamma^i_{li} + \Gamma^i_{ip}\Gamma^p_{jl} - \Gamma^i_{jp}\Gamma^p_{il} \\
=& \partial_j \Gamma^j_{jl} + \partial_l \Gamma^l_{jl} - \partial_j \Gamma^i_{li} + \Gamma^i_{ij}\Gamma^j_{jl} + \Gamma^i_{il}\Gamma^l_{jl} - \Gamma^i_{jp}\Gamma^p_{il} \\
=& -\left[\sum_{p\neq j,l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l}\right]
\end{aligned}$$

- We have:

$$\begin{aligned}
\mathrm{R}_{jj} =& \partial_i \Gamma^i_{jj} - \partial_j \Gamma^i_{ji} + \Gamma^i_{ip}\Gamma^p_{jj} - \Gamma^i_{ji}\Gamma^i_{ij} - \Gamma^i_{jj}\Gamma^j_{ij} \\
=& \partial_i\left(-\eta_j\eta_i u_{j,i} e^{2u_j-2u_i}\right) - \partial_j u_{i,j} + u_{i,p}\left(-\eta_j\eta_p u_{j,p} e^{2u_j-2u_p}\right) - u^2_{i,j} - \left(-\eta_j\eta_i u_{j,i} e^{2u_j-2u_i}\right) u_{j,i} \\
=& -\eta_j\eta_i\left(u_{j,ii} + 2u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right) e^{2u_j-2u_i} - u_{i,jj} - \eta_j\eta_p u_{i,p}u_{j,p} e^{2u_j-2u_p} - u^2_{i,j} + \eta_j\eta_i u^2_{j,i} e^{2u_j-2u_i} \\
=& -\left(\hat{u}_{j,jj} + \hat{u}^2_{j,j} - u_{j,j}\hat{u}_{j,j}\right) - \eta_j \sum_{p\neq j} \eta_p\left[u_{j,pp} + u_{j,p}\left(\hat{u}_{p,p} - u_{p,p}\right)\right] e^{-2u_p+2u_j}
\end{aligned}$$

□

We will calculate the values of the components of the Ricci tensor in the following subsection.

1.4.4 Calculation of components in $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$

Lemma 1.4.4.1: Usual properties

We have:

$$\mathbb{R}_{jl} = \mathrm{d}\omega^i{}_j\left(\theta_i, \theta_l\right) + \omega^i{}_k \wedge \omega^k{}_j\left(\theta_i, \theta_l\right).$$

Proof. We have directly:

$$\begin{aligned}
\mathbb{R}_{jl} &= \Omega^i_j\left(\theta_i, \theta_l\right) \\
&= \mathrm{d}\omega^i{}_j\left(\theta_i, \theta_l\right) + \omega^i{}_k \wedge \omega^k{}_j\left(\theta_i, \theta_l\right)
\end{aligned}$$

□

We deduce the values of the components of the Ricci tensor.

Theorem 1.4.4.2: Exact values of $\mathbb{R}_{jl}$

Let i, j, k, l such that $\{0,1,2,3\} = \{i,j,k,l\}$ and $p \neq j, l$. We have:

$$\begin{aligned}
\mathbb{R}_{jl} &= -\left[\sum_{p\neq j,l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l}\right] e^{-u_j-u_l} \\
\mathbb{R}_{jj} &= -\left(\hat{u}_{j,jj} + \hat{u}^2_{j,j} - u_{j,j}\hat{u}_{j,j}\right) e^{-2u_j} - \eta_j \sum_{p\neq j} \eta_p\left[u_{j,pp} + u_{j,p}\left(\hat{u}_{p,p} - u_{p,p}\right)\right] e^{-2u_p}
\end{aligned}$$

Proof. We have two cases.

- Since:

$$\mathbb{R}^j{}_{jjl} = 0 \quad , \quad \mathbb{R}^l{}_{jll}$$

we have:

$$\begin{aligned}\mathbb{R}_{jl} =& \mathbb{R}^p{}_{jpl}\\ =& \sum_{p\neq j,l} \mathbb{R}^p{}_{jpl}\\ =& -\left[\sum_{p\neq j,l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l}\right] e^{-u_j-u_l}\end{aligned}$$

- Since:

$$\mathbb{R}^j{}_{jjj} = 0$$

we have:

$$\begin{aligned}\mathbb{R}_{jj} =& \mathbb{R}^p{}_{jpj}\\ =& \mathbb{R}^i{}_{jij} + \mathbb{R}^k{}_{jkj} + \mathbb{R}^l{}_{jlj}\\ =& -\left(u_{i,jj} + u_{i,j}\left(u_{i,j} - u_{j,j}\right)\right) e^{-2u_j} - \eta_i\eta_j\left(u_{j,ii} + u_{j,i}\left(u_{j,i} - u_{i,i}\right)\right) e^{-2u_i}\\ &- \eta_k\eta_j u_{i,k}u_{j,k}e^{-2u_k} - \eta_l\eta_j u_{i,l}u_{j,l}e^{-2u_l}\\ &- \left(u_{k,jj} + u_{k,j}\left(u_{k,j} - u_{j,j}\right)\right) e^{-2u_j} - \eta_k\eta_j\left(u_{j,kk} + u_{j,k}\left(u_{j,k} - u_{k,k}\right)\right) e^{-2u_k}\\ &- \eta_i\eta_j u_{k,i}u_{j,i}e^{-2u_i} - \eta_l\eta_j u_{k,l}u_{j,l}e^{-2u_l}\\ &- \left(u_{l,jj} + u_{l,j}\left(u_{l,j} - u_{j,j}\right)\right) e^{-2u_j} - \eta_l\eta_j\left(u_{j,ll} + u_{j,l}\left(u_{j,l} - u_{l,l}\right)\right) e^{-2u_l}\\ &- \eta_k\eta_j u_{l,k}u_{j,k}e^{-2u_k} - \eta_i\eta_j u_{l,i}u_{j,i}e^{-2u_i}\\ =& -\sum_{p\neq j}\left(u_{p,jj} + u_{p,j}\left(u_{p,j} - u_{j,j}\right)\right) e^{-2u_j}\\ &- \eta_l\eta_j\left[\left(u_{j,ll} + u_{j,l}\left(u_{j,l} + u_{k,l} + u_{i,l} - u_{l,l}\right)\right)\right] e^{-2u_l}\\ =& -\left(\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - u_{j,j}\hat{u}_{j,j}\right) e^{-2u_j} - \eta_j\sum_{p\neq j}\eta_p\left[u_{j,pp} + u_{j,p}\left(\hat{u}_{p,p} - u_{p,p}\right)\right] e^{-2u_p}\end{aligned}$$

□

We deduce the values of the components for the metric h.

Example 1.4.4.3: Example 7 for the metric h

We revisit the example with the metric h (see 1.1.4.1, 1.1.5.1, 1.2.5.5, 1.2.6.3, 1.2.6.5, 1.3.4.5, and 1.3.5.4) with:

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

(1) **First method**. We use Theorem 1.4.3.3. We deal with the off-diagonal elements first, and then the diagonal elements.

- **Off-diagonal zero terms**. We have:

$$\mathbb{R}_{3l} = -\left[\sum_{p\neq 3,l} -u_{3,l}u_{p,3} - u_{l,3}u_{p,l} + u_{p,3}u_{p,l}\right] e^{-u_3-u_l} = 0$$

$$\mathbb{R}_{j3} = -\left[\sum_{p\neq j,3} -u_{j,3}u_{p,j} - u_{l,j}u_{p,3} + u_{p,j}u_{p,3}\right] e^{-u_j-u_3} = 0.$$

$$\mathbb{R}_{12} = -\left[\sum_{p\neq 1,2} -u_{1,2}u_{p,1} - u_{2,1}u_{p,2} + u_{p,1}u_{p,2}\right] e^{-u_1-u_2} = -\left[-u_{2,1}u_{3,2} + u_{3,1}u_{3,2}\right] e^{-u_1-u_2} = 0$$

$$\mathbb{R}_{2l} = -\left[\sum_{p\neq 2,l} -u_{2,l}u_{p,2} - u_{l,2}u_{p,l} + u_{p,2}u_{p,l}\right]e^{-u_2-u_l} = -\left[-u_{2,l}u_{3,2} + u_{3,2}u_{3,l}\right]e^{-u_2-u_l} = 0$$

- **Non-zero terms**. We have :

$$\begin{aligned}
\mathbb{R}_{10} &= -\left[\sum_{p\neq 1,0} u_{p,10} - u_{1,0}u_{p,1} - u_{0,1}u_{p,0} + u_{p,1}u_{p,0}\right]e^{-u_1-u_0}\\
&= -\left[-u_{1,0}u_{2,1} - u_{0,1}u_{2,0} + u_{2,1}u_{2,0} - u_{1,0}u_{3,1} - u_{0,1}u_{3,0} + u_{3,1}u_{3,0}\right]e^{-u_1-u_0}\\
&= -2\left[-u_{1,0}u_{2,1} - u_{0,1}u_{2,0} + u_{2,1}u_{2,0}\right]e^{-u_1-u_0}\\
&= -2\left[-\dot{v}r^{-1} - \dot{b}u' + \dot{b}r^{-1}\right]e^{-u-v}\\
&=2\left[\dot{v}r^{-1} + \dot{b}\left(u' - r^{-1}\right)\right]e^{-u-v}\\
\mathbb{R}_{00} &= -\left[\hat{u}_{0,00} + \hat{u}_{0,0}^2 - u_{0,0}\hat{u}_{0,0}\right]e^{-2u_0} - \eta_0\eta_1\left[u_{0,11} + u_{0,1}\left(\hat{u}_{1,1} - u_{1,1}\right)\right]e^{-2u_1}\\
&\quad - \eta_0\eta_2\left[u_{0,22} + u_{0,2}\left(\hat{u}_{2,2} - u_{2,2}\right)\right]e^{-2u_2} - \eta_0\eta_3\left[u_{0,33} + u_{0,3}\left(\hat{u}_{3,3} - u_{3,3}\right)\right]e^{-2u_3}\\
&= -\left[\hat{u}_{0,00} + \hat{u}_{0,0}^2 - u_{0,0}\hat{u}_{0,0}\right]e^{-2u_0} - \eta_0\eta_1\left[u_{0,11} + u_{0,1}\left(\hat{u}_{1,1} - u_{1,1}\right)\right]e^{-2u_1}\\
&= -\left[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2\left(\dot{b}\right)^2 - \dot{u}\left(\dot{v} + 2\dot{b}\right)\right]e^{-2u} + \left[u'' + u'\left(u' + 2r^{-1} - v'\right)\right]e^{-2v}\\
\mathbb{R}_{11} &= -\left[\hat{u}_{1,11} + \hat{u}_{1,1}^2 - u_{1,1}\hat{u}_{1,1}\right]e^{-2u_1} - \eta_1\eta_0\left[u_{1,00} + u_{1,0}\left(\hat{u}_{0,0} - u_{0,0}\right)\right]e^{-2u_0}\\
&\quad - \eta_1\eta_2\left[u_{1,22} + u_{1,2}\left(\hat{u}_{2,2} - u_{2,2}\right)\right]e^{-2u_2} - \eta_1\eta_3\left[u_{1,33} + u_{1,3}\left(\hat{u}_{3,3} - u_{3,3}\right)\right]e^{-2u_3}\\
&= -\left[\hat{u}_{1,11} + \hat{u}_{1,1}^2 - u_{1,1}\hat{u}_{1,1}\right]e^{-2u_1} - \eta_1\eta_0\left[u_{1,00} + u_{1,0}\left(\hat{u}_{0,0} - u_{0,0}\right)\right]e^{-2u_0}\\
&= -\left[u'' + \left(u'\right)^2 - v'\left(u' + 2r^{-1}\right)\right]e^{-2v} + \left[\ddot{v} + \dot{v}\left(\dot{v} + 2\dot{b} - \dot{u}\right)\right]e^{-2u}\\
\mathbb{R}_{22} &= -\left(\hat{u}_{2,22} + \hat{u}_{2,2}^2 - u_{2,2}\hat{u}_{2,2}\right)e^{-2u_2} - \eta_2\eta_0\left[u_{2,00} + u_{2,0}\left(\hat{u}_{0,0} - u_{0,0}\right)\right]e^{-2u_0}\\
&\quad - \eta_2\eta_1\left[u_{2,11} + u_{2,1}\left(\hat{u}_{1,1} - u_{1,1}\right)\right]e^{-2u_1} - \eta_2\eta_3\left[u_{2,33} + u_{2,3}\left(\hat{u}_{3,3} - u_{3,3}\right)\right]e^{-2u_3}\\
&= -\left(\hat{u}_{2,22} + \hat{u}_{2,2}^2\right)e^{-2u_2} - \eta_2\eta_0\left[u_{2,00} + u_{2,0}\left(\hat{u}_{0,0} - u_{0,0}\right)\right]e^{-2u_0}\\
&\quad - \eta_2\eta_1\left[u_{2,11} + u_{2,1}\left(\hat{u}_{1,1} - u_{1,1}\right)\right]e^{-2u_1}\\
&=r^{-2}e^{-2b} + \left[\ddot{b} + \dot{b}\left(\dot{v} + 2\dot{b} - \dot{u}\right)\right]e^{-2u} - r^{-1}\left(u' - v' + r^{-1}\right)e^{-2v}\\
\mathbb{R}_{33} &= -\left(\hat{u}_{3,33} + \hat{u}_{3,3}^2 - u_{3,3}\hat{u}_{3,3}\right)e^{-2u_3} - \eta_3\eta_0\left[u_{3,00} + u_{3,0}\left(\hat{u}_{0,0} - u_{0,0}\right)\right]e^{-2u_0}\\
&\quad - \eta_3\eta_1\left[u_{3,11} + u_{3,1}\left(\hat{u}_{1,1} - u_{1,1}\right)\right]e^{-2u_1} - \eta_3\eta_2\left[u_{3,22} + u_{3,2}\left(\hat{u}_{2,2} - u_{2,2}\right)\right]e^{-2u_2}\\
&= -\eta_3\eta_0\left[u_{3,00} + u_{3,0}\left(\hat{u}_{0,0} - u_{0,0}\right)\right]e^{-2u_0} - \eta_3\eta_1\left[u_{3,11} + u_{3,1}\left(\hat{u}_{1,1} - u_{1,1}\right)\right]e^{-2u_1}\\
&\quad - \eta_3\eta_2\left[u_{3,22} + u_{3,2}\left(\hat{u}_{2,2} - u_{2,2}\right)\right]e^{-2u_2}\\
&=r^{-2}e^{-2b} + \left[\ddot{b} + \dot{b}\left(\dot{v} + 2\dot{b} - \dot{u}\right)\right]e^{-2u} - r^{-1}\left(u' - v' + r^{-1}\right)e^{-2v}
\end{aligned}$$

(2) **Second method**. We use definition 1.4.1.1.

- Case $j := 0$. We have:

$$\begin{aligned}
\mathbb{R}_{00} &=\mathbb{R}^0_{000} + \mathbb{R}^1_{010} + \mathbb{R}^2_{020} + \mathbb{R}^3_{030}\\
&=\mathbb{R}^1_{010} + 2\mathbb{R}^2_{020}\\
&= \left(u'' + u'(u' - v')\right)e^{-2v} - \left(\ddot{v} + \dot{v}(\dot{v} - \dot{u})\right)e^{-2u} - 2\left(\ddot{b} + \dot{b}(\dot{b} - \dot{u})\right)e^{-2u} + 2r^{-1}u'e^{-2v}\\
&= -\left[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{u}(\dot{v} + 2\dot{b})\right]e^{-2u} + \left[u'' + u'(u' + 2r^{-1} - v')\right]e^{-2v}\\
\mathbb{R}_{01} &=\mathbb{R}^0_{001} + \mathbb{R}^1_{011} + \mathbb{R}^2_{021} + \mathbb{R}^3_{031}\\
&=2\mathbb{R}^2_{021}\\
&=2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v})e^{-u-v}\\
\mathbb{R}02 &=\mathbb{R}^0_{002} + \mathbb{R}^1_{012} + \mathbb{R}^2_{022} + \mathbb{R}^3_{032} = 0
\end{aligned}$$

$$\mathbb{R}_{03} = \mathbb{R}^0_{003} + \mathbb{R}^1_{013} + \mathbb{R}^2_{023} + \mathbb{R}^3_{033} = 0$$

- Case $j \coloneqq 1$. We have:

$$\begin{aligned}
\mathbb{R}_{10} =& \mathbb{R}^0_{100} + \mathbb{R}^1_{110} + \mathbb{R}^2_{120} + \mathbb{R}^3_{130} \\
=& 2\mathbb{R}^2_{120} \\
=& 2\mathbb{R}^2_{021} \\
=& \mathbb{R}_{01} \\
\mathbb{R}_{11} =& \mathbb{R}^0_{101} + \mathbb{R}^1_{111} + \mathbb{R}^2_{121} + \mathbb{R}^3_{131} \\
=& \mathbb{R}^0_{101} + 2\mathbb{R}^2_{121} \\
=& -\left(u'' + u'(u' - v')\right) e^{-2v} + \left(\ddot{v} + \dot{v}(\dot{v} - \dot{u})\right) e^{-2u} \\
&+ 2r^{-1} v' e^{-2v} + 2\dot{v}\dot{b} e^{-2u} \\
=& -\left[u'' + (u')^2 - v'(u' + 2r^{-1})\right] e^{-2v} + \left[\ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{u})\right] e^{-2u} \\
\mathbb{R}_{12} =& \mathbb{R}^0_{102} + \mathbb{R}^1_{112} + \mathbb{R}^2_{122} + \mathbb{R}^3_{132} = 0 \\
\mathbb{R}_{13} =& \mathbb{R}^0 103 + \mathbb{R}^1_{113} + \mathbb{R}^2_{123} + \mathbb{R}^3_{133} = 0
\end{aligned}$$

- Case $j \coloneqq 2$. We have:

$$\begin{aligned}
\mathbb{R}_{20} =& \mathbb{R}^0_{200} + \mathbb{R}^1_{210} + \mathbb{R}^2_{220} + \mathbb{R}^3_{230} = 0 \\
\mathbb{R}_{21} =& \mathbb{R}^0_{201} + \mathbb{R}^1_{211} + \mathbb{R}^2_{221} + \mathbb{R}^3_{231} = 0 \\
\mathbb{R}_{22} =& \mathbb{R}^0_{202} + \mathbb{R}^1_{212} + \mathbb{R}^2_{222} + \mathbb{R}^3_{232} \\
=& \mathbb{R}^0_{202} + \mathbb{R}^1_{212} + \mathbb{R}^3_{232} \\
=& (\ddot{b} + \dot{b}(\dot{b} - \dot{u}))e^{-2u} - u' r^{-1} e^{-2v} \\
&+ r^{-1} v' e^{-2v} + \dot{v}\dot{b} e^{-2u} \\
&+ r^{-2} + (\dot{b})^2 e^{-2u} - r^{-2} e^{-2v} \\
=& r^{-2} + [\ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u} - r^{-1}(u' - v' + r^{-1})e^{-2v} \\
\mathbb{R}_{23} =& \mathbb{R}^0_{203} + \mathbb{R}^1_{213} + \mathbb{R}^2_{223} + \mathbb{R}^3_{233} = 0
\end{aligned}$$

- Case $j \coloneqq 3$. We have:

$$\begin{aligned}
\mathbb{R}_{30} =& \mathbb{R}^0_{300} + \mathbb{R}^1_{310} + \mathbb{R}^2_{320} + \mathbb{R}^3_{330} = 0 \\
\mathbb{R}_{31} =& \mathbb{R}^0_{301} + \mathbb{R}^1_{311} + \mathbb{R}^2_{321} + \mathbb{R}^3_{331} = 0 \\
\mathbb{R}_{32} =& \mathbb{R}^0_{302} + \mathbb{R}^1_{312} + \mathbb{R}^2_{322} + \mathbb{R}^3_{332} = 0 \\
\mathbb{R}_{33} =& \mathbb{R}^0_{303} + \mathbb{R}^1_{313} + \mathbb{R}^2_{323} + \mathbb{R}^3_{333} \\
=& \mathbb{R}^0_{303} + \mathbb{R}^1_{313} + \mathbb{R}^2_{323} \\
=& \mathbb{R}^0_{202} + \mathbb{R}^1_{212} + \mathbb{R}^3_{232} \\
=& \mathbb{R}_{22}
\end{aligned}$$

Therefore, the matrices $\mathbb{R}_{jl}$ and R_{jl} have the form :

$$\mathbb{R}_{jl} = \begin{pmatrix} \mathbb{R}_{00} & \mathbb{R}_{01} & 0 & 0 \\ \mathbb{R}_{10} & \mathbb{R}_{11} & 0 & 0 \\ 0 & 0 & \mathbb{R}_{22} & 0 \\ 0 & 0 & 0 & \mathbb{R}_{33} \end{pmatrix} \quad , \quad \mathrm{R}_{jl} = \begin{pmatrix} \mathrm{R}_{00} & \mathrm{R}_{01} & 0 & 0 \\ \mathrm{R}_{10} & \mathrm{R}_{11} & 0 & 0 \\ 0 & 0 & \mathrm{R}_{22} & 0 \\ 0 & 0 & 0 & \mathrm{R}_{33} \end{pmatrix}$$

with:

$$\begin{aligned}
\mathbb{R}_{00} =& -\left[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{u}(\dot{v} + 2\dot{b})\right] e^{-2u} + \left[u'' + u'(u' + 2r^{-1} - v')\right] e^{-2v} \\
\mathrm{R}_{00} =& -\left[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2(\dot{b})^2 - \dot{u}(\dot{v} + 2\dot{b})\right] + \left[u'' + u'(u' + 2r^{-1} - v')\right] e^{2u-2v} \\
\mathbb{R}_{01} =& 2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v})e^{-u-v} \\
\mathrm{R}_{01} =& 2(\dot{b}(u' - r^{-1}) + r^{-1}\dot{v}) \\
\mathbb{R}_{11} =& -\left[u'' + (u')^2 - v'(u' + 2r^{-1})\right] e^{-2v} + \left[\ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{u})\right] e^{-2u} \\
\mathrm{R}_{11} =& -\left[u'' + (u')^2 - v'(u' + 2r^{-1})\right] + \left[\ddot{v} + \dot{v}(\dot{v} + 2\dot{b} - \dot{u})\right] e^{-2u+2v} \\
\mathbb{R}_{22} =& \mathbb{R}_{33} = r^{-2} + \left[\ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u})\right] e^{-2u} - r^{-1}(u' - v' + r^{-1})e^{-2v} \\
\mathrm{R}_{22} =& 1 + r^2[\ddot{b} + \dot{b}(\dot{v} + 2\dot{b} - \dot{u})]e^{-2u} - r(u' - v' + r^{-1})e^{-2v} \\
\mathrm{R}_{33} =& \sin^2\vartheta \mathrm{R}_{22}
\end{aligned}$$

1.5 Scalar Curvature

Definition 1.5.0.1: Scalar Curvature

The **scalar curvature** is defined by:

$$\mathscr{S} := \left[[\mathscr{R}ic]^1\right]_1^1.$$

Since the operations $[\bullet]_1^1$ and $[\bullet]^1$ are independent of the chosen basis, $\mathscr{S}$ is also independent of the chosen basis.

Proposition 1.5.0.2: Usual Properties

(i) (a) We have:

$$\mathscr{S} = \mathrm{tr}_g \mathscr{R}ic.$$

(b) We have:

$$\mathscr{S} = \mathbf{g}^{ik}\mathbf{R}_{ik}.$$

(ii) (**Twice Contracted Bianchi Identity**)

(a) We have:

$$\begin{aligned}
\nabla^l \mathbf{R}_{lj} &= \frac{1}{2}\nabla_j \mathscr{S} \\
\nabla_k \mathbf{R}^k{}_j &= \frac{1}{2}\nabla_j \mathscr{S} \\
\mathbf{g}^{kl}\nabla_k \mathbf{R}_{lj} &= \frac{1}{2}\nabla_j \mathscr{S}
\end{aligned}$$

(b) We have:

$$\begin{aligned}
\left[[\mathscr{R}ic]^1\right]_2^1 &= \frac{1}{2}\mathrm{d}\mathscr{S} \\
\mathrm{div}[\mathscr{R}ic]^1 &= \frac{1}{2}\mathrm{d}\mathscr{S} \\
\mathrm{div}_g \mathscr{R}ic &= \frac{1}{2}\mathrm{d}\mathscr{S}
\end{aligned}$$

Proof. (i) Since:

$$
\begin{aligned}
[\mathscr{R}ic]^1 : (\alpha, X) \longmapsto g^* (\alpha, \mathscr{R}ic(\bullet, X)) \\
\mathscr{R}ic = \mathbf{R}_{kl}\ e^k \otimes e^l
\end{aligned}
$$

we have:

$$
\begin{aligned}
[\mathscr{R}ic]^1(e^i, e_j) &= g^* \left(e^i, \mathscr{R}ic(\bullet, e_j)\right) \\
&= g^* \left(e^i, \mathbf{R}_{kl}\ e^l(e_j) e^k\right) \\
&= g^* \left(e^i, \mathbf{R}_{kl}\ \delta^l_j e^k\right) \\
&= g^* \left(e^i, \mathbf{R}_{kj}\ e^k\right) \\
&= \mathbf{R}_{kj} g^* \left(e^i, e^k\right) \\
&= \mathbf{R}_{kj} \mathbf{g}^{ik}
\end{aligned}
$$

and we have:

$$
\begin{aligned}
\mathscr{S} &= \operatorname{tr}\left([\mathscr{R}ic]^1\right) \\
&= e^i \left([\mathscr{R}ic]^1(\bullet, e_i)\right) \\
&= e^i \left(\mathbf{R}^j{}_l\ e^l(e_i) e_j\right) \\
&= \mathbf{R}^j{}_l\ \delta^l_i e^i\left(e_j\right) \\
&= \mathbf{R}^j{}_i\ \delta^i_j \\
&= \mathbf{R}^i{}_i \\
&= \mathbf{g}^{ik} \mathbf{R}_{ki}
\end{aligned}
$$

(ii) (a) By the twice contracted Bianchi identity (see proposition 1.4.2.2), we have:

$$
\nabla_n \mathbf{R}_{jl} - \nabla_l \mathbf{R}_{jn} + \nabla_k \mathbf{R}^k{}_{jln} = 0.
$$

Since:

$$
\begin{aligned}
\mathbf{g}^{nl} \nabla_k \mathbf{R}^k{}_{njl} &= \mathbf{g}^{nl} \mathbf{g}^{kp} \nabla_k \mathbf{R}_{pnjl} \\
&= \mathbf{g}^{nl} \nabla^p \mathbf{R}_{pnjl} \\
&= \mathbf{g}^{nl} \nabla^p \mathbf{R}_{nplj} \\
&= \nabla^p \mathbf{R}^l{}_{plj} \\
&= \nabla^p \mathbf{R}_{pj} \\
&= \nabla^p \mathbf{R}_{jp}
\end{aligned}
$$

we then contract:

$$
\begin{aligned}
0 &= \mathbf{g}^{nl} \left(\nabla_n \mathbf{R}_{jl} - \nabla_j \mathbf{R}_{nl} + \nabla_k \mathbf{R}^k{}_{njl}\right) \\
&= \mathbf{g}^{nl} \nabla_n \mathbf{R}_{jl} - \mathbf{g}^{nl} \nabla_j \mathbf{R}_{nl} + \mathbf{g}^{nl} \nabla_k \mathbf{R}^k{}_{njl} \\
&= \nabla^l \mathbf{R}_{jl} - \nabla_j \mathscr{S} + \nabla^p \mathbf{R}_{jp}
\end{aligned}
$$

i.e., we have:

$$
\nabla^l \mathbf{R}_{jl} = \frac{1}{2} \nabla_j \mathscr{S}.
$$

For the other two equalities, we just need to notice that:

$$
\begin{aligned}
\nabla^l \mathbf{R}_{jl} &= \mathbf{g}^{kl} \nabla_k \mathbf{R}_{lj} \\
&= \frac{1}{2} \nabla_j \mathscr{S}
\end{aligned}
$$

(b) We consider the base $\mathscr{C}$. Since:

$$\mathrm{div}_g \mathscr{R}\mathrm{ic} = \left[\nabla \left[\mathscr{R}\mathrm{ic}\right]^1\right]_2^1,$$

we have:

$$\begin{aligned}\mathrm{div}_g \mathscr{R}\mathrm{ic}(\partial_j) &= \left[\nabla \left[\mathscr{R}\mathrm{ic}\right]^1\right]_2^1 (\partial_j)\\ &= \nabla_k \mathbf{R}^k{}_j\\ &= \frac{1}{2}\nabla_j \mathscr{S}\\ &= \frac{1}{2}\mathrm{d}\mathscr{S}(\partial_j)\end{aligned}$$

Since this is true for all j, we have:

$$\mathrm{div}_g \mathscr{R}\mathrm{ic} = \frac{1}{2}\mathrm{d}\mathscr{S}.$$

□

Example 1.5.0.3: Examples of calculations in bases $\mathscr{C}$ and $\mathscr{C}_\perp$

Therefore, we have:

$$\mathscr{S} = \mathrm{g}^{jj}\mathrm{R}_{jj} = \eta^{jj}\mathbb{R}_{jj}.$$

We can directly show the last equality without using the invariance under change of basis. Since g_{jl} and η_{jl} are diagonal, we have:

$$\begin{aligned}\mathscr{S} &= \mathrm{g}^{jl}\mathrm{R}_{jl}\\ &= \mathrm{g}^{jj}\mathrm{R}_{jj}\\ &= \mathrm{g}^{jj}e^{2u_j}\mathbb{R}_{jj}\\ &= \mathrm{g}^{jj}\eta^j \mathrm{g}_{jj}\mathbb{R}_{jj}\\ &= \eta^j\mathbb{R}_{jj}\end{aligned}$$

1.5.1 Practical calculations in $\mathscr{C}$ and $\mathscr{C}^*$

Proposition 1.5.1.1: Value in terms of Christoffel symbols

We have:

$$\mathscr{S} = \mathrm{g}^{kl}\left(\partial_i\Gamma^i_{kl} - \partial_k\Gamma^i_{li} + \Gamma^i_{ip}\Gamma^p_{kl} - \Gamma^i_{kp}\Gamma^p_{il}\right).$$

Proof. Using lemma 1.5.0.2, we have:

$$\mathscr{S} = \mathrm{g}^{kl}\mathrm{R}_{kl}.$$

Since:

$$\mathrm{R}_{kl} = \partial_i\Gamma^i_{kl} - \partial_k\Gamma^i_{li} + \Gamma^i_{ip}\Gamma^p_{kl} - \Gamma^i_{kp}\Gamma^p_{il},$$

we can conclude that:

$$\mathscr{S} = \mathrm{g}^{kl}\left(\partial_i\Gamma^i_{kl} - \partial_k\Gamma^i_{li} + \Gamma^i_{ip}\Gamma^p_{kl} - \Gamma^i_{kp}\Gamma^p_{il}\right).$$

□

Corollary 1.5.1.2: Exact value

We have:

$$\mathscr{S} = -\sum_k \eta_k \left(2\hat{u}_{k,kk} + \hat{u}^2_{k,k} - 2u_{k,k}\hat{u}_{k,k} + \left(\hat{u}_{k,k}\right)^2\right)e^{-2u_k}.$$

Proof. We have:

$$\begin{aligned}
\mathscr{S} =& g^{kl}\left(\partial_i\Gamma^i_{kl} - \partial_k\Gamma^i_{li} + \Gamma^i_{ip}\Gamma^p_{kl} - \Gamma^i_{kp}\Gamma^p_{il}\right)\\
=& g^{kk}\left(\partial_i\Gamma^i_{kk} - \partial_k\Gamma^i_{ki} + \Gamma^i_{ip}\Gamma^p_{kk} - \Gamma^i_{ki}\Gamma^i_{ik} - \Gamma^i_{kk}\Gamma^k_{ik}\right)\\
=& \eta_k\left(\partial_i\left(-\eta_k\eta_i u_{k,i}e^{2u_k-2u_i}\right) - \partial_k u_{i,k} + u_{i,p}\left(-\eta_k\eta_p u_{k,p}e^{2u_k-2u_p}\right) - u_{i,k}^2 - \left(-\eta_k\eta_i u_{k,i}e^{2u_k-2u_i}\right)u_{k,i}\right)e^{-2u_k}\\
=& \eta_k\left(-\eta_k\eta_i\left(u_{k,ii} + 2u_{k,i}\left(u_{k,i} - u_{i,i}\right)\right)e^{2u_k-2u_i} - u_{i,kk} - \eta_k\eta_p u_{i,p}u_{k,p}e^{2u_k-2u_p} - u_{i,k}^2 + \eta_k\eta_i u_{k,i}^2 e^{2u_k-2u_i}\right)e^{-2u_k}\\
=& -\eta_i\left(u_{k,ii} + 2u_{k,i}\left(u_{k,i} - u_{i,i}\right)\right)e^{-2u_i} - \eta_k u_{i,kk}e^{-2u_k} - \eta_p u_{i,p}u_{k,p}e^{-2u_p} - \eta_k u_{i,k}^2 e^{-2u_k} + \eta_i u_{k,i}^2 e^{-2u_i}\\
=& -\eta_k\left(u_{i,kk} + 2u_{i,k}\left(u_{i,k} - u_{k,k}\right)\right)e^{-2u_k} - \eta_k u_{i,kk}e^{-2u_k} - \eta_k u_{i,k}u_{p,k}e^{-2u_k} - \eta_k u_{i,k}^2 e^{-2u_k} + \eta_k u_{i,k}^2 e^{-2u_k}\\
=& -\eta_k\left(u_{i,kk} + 2u_{i,k}\left(u_{i,k} - u_{k,k}\right) - u_{i,kk} - u_{i,k}u_{p,k} + u_{i,k}^2 + u_{i,k}^2\right)e^{-2u_k}\\
=& -\eta_k\left(2u_{i,k}\left(u_{i,k} - u_{k,k}\right) - u_{i,k}u_{p,k} + u_{i,k}^2\right)e^{-2u_k}\\
=& -\eta_k\left(2(u_{i,k}^2 - u_{i,k}u_{k,k}) - u_{i,k}u_{p,k}\right)e^{-2u_k}\\
=& -\sum_k \eta_k\left(2\hat{u}k,kk + \hat{u}^2k,k - 2u_{k,k}\hat{u}k,k + (\hat{u}k,k)^2\right)e^{-2u_k}
\end{aligned}$$

□

We will calculate the value of $\mathscr{S}$ using the metric h in the next subsection.

1.5.2 Practical calculations in $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$

Using the formula:

$$\mathscr{S} = \eta_j\mathbb{R}_{jj},$$

we can determine the value of the scalar curvature.

Theorem 1.5.2.1: Value of the scalar tensor

We have:

$$\mathscr{S} = -\sum_k \eta_k\left(2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k}\hat{u}_{k,k} + \left(\hat{u}_{k,k}\right)^2\right)e^{-2u_k}.$$

Proof. We have:

$$\begin{aligned}
\mathscr{S} =& \eta_j\mathbb{R}_{jj}\\
=& -\sum_j \eta_j e^{-2u_j}\sum_{k\neq j}\left(u_{k,jj} + u_{k,j}\left(u_k - u_j\right)_{,j}\right) - \sum_j\sum_{k\neq j}\eta_k\left(u_{j,kk} + u_{j,k}\left(-u_k + \sum_{i\neq k}u_i\right)_{,k}\right)e^{-2u_k}\\
=& -\sum_k\sum_{j\neq k}\eta_k\left(u_{j,kk} + u_{j,k}\left(-u_k + \sum_{i\neq k}u_i\right)_{,k}\right)e^{-2u_k} - \sum_k\sum_{j\neq k}\eta_k e^{-2u_k}\left(u_{j,kk} + u_{j,k}\left(u_j - u_k\right)_{,k}\right)\\
=& -\sum_k \eta_k e^{-2u_k}\sum_{j\neq k}\left(2u_{j,kk} + u_{j,k}\left(2u_j - 2u_k + \sum_{i\neq j,k}u_i\right)_{,k}\right)\\
=& -\sum_k \eta_k\left(2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k}\hat{u}_{k,k} + \left(\hat{u}_{k,k}\right)^2\right)e^{-2u_k}
\end{aligned}$$

□

Example 1.5.2.2: Example 8 of the metric h

We revisit the example with the metric h (see 1.1.4.1, 1.1.5.1, 1.2.5.5, 1.2.6.3 , 1.2.6.5, 1.3.4.5 , 1.3.5.4, and 1.4.4.3) with:

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

We calculate Scal in two different ways.

(1) Using Theorem 1.5.2.1, we have:

$$\begin{aligned}
\mathscr{S} =& -\eta_0\left(2\hat{u}_{0,00} + \hat{u}_{0,0}^2 - 2u_{0,0}\hat{u}_{0,0} + \left(\hat{u}_{0,0}\right)^2\right)e^{-2u_0}\\
&-\eta_1\left(2\hat{u}_{1,11} + \hat{u}_{1,1}^2 - 2u_{1,1}\hat{u}_{1,1} + \left(\hat{u}_{1,1}\right)^2\right)e^{-2u_1}\\
&-\eta_2\left(2\hat{u}_{2,22} + \hat{u}_{2,2}^2 - 2u_{2,2}\hat{u}_{2,2} + \left(\hat{u}_{2,2}\right)^2\right)e^{-2u_2}\\
=& -\left(2\left(\ddot{v} + 2\ddot{b}\right) + (\dot{v})^2 + 2\left(\dot{b}\right)^2 - 2\dot{u}\left(\dot{v} + 2\dot{b}\right) + \left(\dot{v} + 2\dot{b}\right)^2\right)e^{-2u}\\
&+\left(2\left(u'' - 2r^{-2}\right) + \left(u'\right)^2 + 2r^{-2} - 2v'\left(u' + 2r^{-1}\right) + \left(u' + 2r^{-1}\right)^2\right)e^{-2v}\\
&-2r^{-2}e^{-2b}\\
=& -2\left[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 3\left(\dot{b}\right)^2 - \dot{u}\dot{v} - 2\dot{u}\dot{b} + 2\dot{v}\dot{b}\right]e^{-2u}\\
&+2\left[u'' - 2r^{-2} + \left(u'\right)^2 + 3r^{-2} - u'v' + 2r^{-1}(u' - v')u'\right]e^{-2v} - 2r^{-2}e^{-2b}
\end{aligned}$$

(2) Using Definition 1.5.0.1, we have:

$$\begin{aligned}
\mathscr{S} =& \eta_j\mathbb{R}_{jj}\\
=& \eta_0\mathbb{R}_{00} + \eta_1\mathbb{R}_{11} + \eta_2\mathbb{R}_{22} + \eta_3\mathbb{R}_{33}\\
=& \mathbb{R}_{00} - \mathbb{R}_{11} - 2\mathbb{R}_{22}\\
=& -\left[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 2\left(\dot{b}\right)^2 - \dot{u}\left(\dot{v} + 2\dot{b}\right)\right]e^{-2u} + \left[u'' + u'\left(u' + 2r^{-1} - v'\right)\right]e^{-2v}\\
&+\left[u'' + \left(u'\right)^2 - v'\left(u' + 2r^{-1}\right)\right]e^{-2v} - \left[\ddot{v} + \dot{v}\left(\dot{v} + 2\dot{b} - \dot{u}\right)\right]e^{-2u}\\
&-2r^{-2} - 2\left[\ddot{b} + \dot{b}\left(\dot{v} + 2\dot{b} - \dot{u}\right)\right]e^{-2u} + 2r^{-1}\left(u' - v' + r^{-1}\right)e^{-2v}\\
=& -2\left[\ddot{v} + 2\ddot{b} + (\dot{v})^2 + 3\left(\dot{b}\right)^2 - \dot{u}\dot{v} - 2\dot{u}\dot{b} + 2\dot{v}\dot{b}\right]e^{-2u}\\
&+2\left[u'' - 2r^{-2} + \left(u'\right)^2 + 3r^{-2} - u'v' + 2r^{-1}(u' - v')u'\right]e^{-2v} - 2r^{-2}e^{-2b}
\end{aligned}$$

We see the efficiency of the first method, which directly gives a very good factorization.

1.6 Tensor of Einstein

Definition 1.6.0.1: The tensor of Einstein

The **tensor of Einstein** is defined by:

$$\mathscr{G} := \mathscr{R} - \frac{1}{2}\mathscr{S}g.$$

That is, for all vector fields X and Y, we have:

$$\mathscr{G}(X,Y) = \mathscr{R}(X,Y) - \frac{1}{2}\mathscr{S}g(X,Y).$$

Next, we define the components of $\mathscr{G}$ in the $\mathscr{E}$ basis.

Definition 1.6.0.2: The components of the tensor of Einstein in $\mathscr{E}$ and $\mathscr{E}^*$

The **components of the Ricci tensor in $\mathscr{E}$ and $\mathscr{E}^*$** are:

$$\mathbf{G}_{jl} := \mathscr{G}(e_j, e_l).$$

Proposition 1.6.0.3: Simple properties of the tensor of Einstein in $\mathscr{E}$ and $\mathscr{E}^*$

(i) (**Symmetry**)

(a) We have:
$$\mathscr{G}(X,Y) = \mathscr{G}(Y,X).$$

(b) We have:
$$\mathbf{G}_{jl} = \mathbf{G}_{lj}.$$

(ii) (**Twice contracted Bianchi identity**)

(a) We have:

$$\begin{aligned}\nabla^l \mathbf{G}_{lj} &= 0\\ \nabla_k \mathbf{G}^k{}_j &= 0\\ \mathbf{g}^{kl}\nabla_k \mathbf{G}_{lj} &= 0\end{aligned}$$

(b) We have:

$$\begin{aligned}\left[[\mathscr{G}]^1\right]^1_2 &= 0\\ \operatorname{div}[\mathscr{G}]^1 &= 0\\ \operatorname{div}_g \mathscr{G} &= 0\end{aligned}$$

Proof. (i) (Symmetry)

(a) Since $\mathscr{R}$ and g are symmetric, we have:

$$\begin{aligned}\mathscr{G}(X,Y) &= \mathscr{R}(X,Y) - \frac{1}{2}\mathscr{S}g(X,Y)\\ &= \mathscr{R}(Y,X) - \frac{1}{2}\mathscr{S}g(Y,X)\\ &= \mathscr{G}(Y,X)\end{aligned}$$

(b) We have:

$$\begin{aligned}\mathbf{G}_{jl} &= \mathscr{G}(e_j, e_l)\\ &= \mathscr{G}(e_l, e_j)\\ &= \mathbf{G}_{lj}\end{aligned}$$

(ii) By proposition 1.5.0.2, we have:

$$\nabla^n \mathbf{R}_{nl} = \frac{1}{2}\nabla_l \mathscr{S}$$

Hence, we have:

$$\nabla^n \mathbf{R}_{nl} = \frac{1}{2}\mathbf{g}_{nl}\nabla^n \mathscr{S}.$$

Therefore, by linearity of ∇, we obtain:

$$\begin{aligned}\nabla^n \mathbf{G}_{nl} &= \nabla^n \left(\mathbf{R}_{nl} - \frac{1}{2}\mathbf{g}_{nl}\mathscr{S}\right)\\ &= 0\end{aligned}$$

□

Example 1.6.0.4: Components of the Einstein tensor in the bases $\mathscr{C}$ and $\mathscr{C}_\perp$

(i) The **components of the Einstein tensor in $\mathscr{C}$ and $\mathscr{C}^*$** are:

$$\mathrm{G}_{jl} := \mathrm{R}_{jl} - \frac{1}{2}\mathrm{g}_{jl}\mathscr{S}.$$

We have the decomposition:

$$\mathscr{G} = \mathrm{G}_{jl}\ \mathrm{d}x^j \otimes \mathrm{d}x^l.$$

(ii) The **components of the Einstein tensor in $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$** are:

$$\mathbb{G}_{jl} := \mathbb{R}_{jl} - \frac{1}{2}\eta_{jl}\mathscr{S}.$$

We have the decomposition:

$$\mathscr{G} = \mathbb{G}_{jl}\ \theta^j \otimes \theta^l.$$

We have:

$$\mathbb{G}_{ij} = e^{-u_i - u_j}\mathrm{G}_{ij}.$$

Theorem 1.6.0.5: Value of the Einstein tensor

Let $j, l \in \{0,1,2,3\}$ **distinct**.

(i) We have:

$$\mathbb{G}_{jl} = -\left[\sum_{p\neq j,l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l}\right] e^{-u_j - u_l}$$

$$\mathrm{G}_{jl} = -\left[\sum_{p\neq j,l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l}\right]$$

(ii) We have:

$$\mathbb{G}_{jj} = \frac{1}{2}\left[(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2\right] e^{-2u_j} + \eta_j \sum_{k\neq j}\eta_k \left[\hat{u}_{jk,kk} + \frac{1}{2}\hat{u}_{k,k}^2 + \frac{1}{2}\hat{u}_{k,k}\left(\hat{u}_{jk,k} - 2u_{k,k} - u_{j,k}\right) + u_{j,k}u_{k,k}\right] e^{-2u_k}$$

$$\mathrm{G}_{jj} = \frac{1}{2}\left[(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2\right] + \eta_j \sum_{k\neq j} \eta_k \left[\hat{u}_{jk,kk} + \frac{1}{2}\hat{u}_{k,k}^2 + \frac{1}{2}\hat{u}_{k,k}\left(\hat{u}_{jk,k} - 2u_{k,k} - u_{j,k}\right) + u_{j,k}u_{k,k}\right] e^{2u_j - 2u_k}$$

Proof. (i) We have:

$$\mathbb{G}_{jl} = \mathbb{R}_{jl} = -\left[\sum_{p\neq j,l} u_{p,jl} - u_{j,l}u_{p,j} - u_{l,j}u_{p,l} + u_{p,j}u_{p,l}\right] e^{-u_j - u_l}.$$

(ii) We have:

$$\begin{aligned}
\mathbb{G}_{jj} =& \mathbb{R}_{jj} - \frac{1}{2}\eta_j \mathbb{R} \\
=& -\left(\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - \hat{u}_{j,j}u_{j,j}\right) e^{-2u_j} - \eta_j \sum_{k\neq j} \eta_k \left(u_{j,kk} + u_{j,k}\left(\hat{u}_{k,k} - u_{k,k}\right)\right) e^{-2u_k} \\
&+ \frac{1}{2}\eta_j \sum_{k} \eta_k \left(2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k}\hat{u}_{k,k} + \left(\hat{u}_{k,k}\right)^2\right) e^{-2u_k} \\
=& -\left(\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - \hat{u}_{j,j}u_{j,j}\right) e^{-2u_j} - \eta_j \sum_{k\neq j} \eta_k \left(u_{j,kk} + u_{j,k}\left(\hat{u}_{k,k} - u_{k,k}\right)\right) e^{-2u_k} \\
&+ \frac{1}{2}\eta_j \sum_{k\neq j} \eta_k \left(2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k}\hat{u}_{k,k} + \left(\hat{u}_{k,k}\right)^2\right) e^{-2u_k} \\
&+ \frac{1}{2}\left(2\hat{u}_{j,jj} + \hat{u}_{j,j}^2 - 2u_{j,j}\hat{u}_{j,j} + (\hat{u}_{j,j})^2\right) e^{-2u_j} \\
=& \frac{1}{2}\left[(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2\right] e^{-2u_j} + \frac{1}{2}\eta_j \sum_{k\neq j} \eta_k \left(2\hat{u}_{k,kk} + \hat{u}_{k,k}^2 - 2u_{k,k}\hat{u}_{k,k} + \left(\hat{u}_{k,k}\right)^2 - 2u_{j,kk} - 2u_{j,k}\left(\hat{u}_{k,k} - u_{k,k}\right)\right) e^{-2u_k} \\
=& \frac{1}{2}\left[(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2\right] e^{-2u_j} + \eta_j \sum_{k\neq j} \eta_k \left[\hat{u}_{k,kk} - u_{j,kk} + \frac{1}{2}\hat{u}_{k,k}^2 + \frac{1}{2}\hat{u}_{k,k}\left(\hat{u}_{k,k} - 2u_{k,k} - u_{j,k}\right) + u_{j,k}u_{k,k}\right] e^{-2u_k} \\
=& \frac{1}{2}\left[(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2\right] e^{-2u_j} + \eta_j \sum_{k\neq j} \eta_k \left[\hat{u}_{jk,kk} + \frac{1}{2}\hat{u}_{k,k}^2 + \frac{1}{2}\hat{u}_{k,k}\left(\hat{u}_{jk,k} - 2u_{k,k} - u_{j,k}\right) + u_{j,k}u_{k,k}\right] e^{-2u_k}
\end{aligned}$$

□

Example 1.6.0.6: Sequence 9 of the metric h

We revisit the example with the metric h (see 1.1.4.1, 1.1.5.1, 1.2.5.5, 1.2.6.3, 1.2.6.5, 1.3.4.5, 1.3.5.4, 1.4.4.3, and 1.5.2.2) with :

$$u_0(t,r) := u(t,r) \qquad u_1(t,r) := v(t,r)$$
$$u_2(t,r) := b(t) + \ln(r) \qquad u_3(t,r,\vartheta) := b(t) + \ln(r) + \ln\sin(\vartheta)$$

We have:

$$\mathbb{G}_{jj} = \frac{1}{2}\left[(\hat{u}_{j,j})^2 - \hat{u}_{j,j}^2\right] e^{-2u_j} + \eta_j \sum_{k\neq j} \eta_k \left[\hat{u}_{jk,kk} + \frac{1}{2}\hat{u}_{k,k}^2 + \frac{1}{2}\hat{u}_{k,k}\left(\hat{u}_{jk,k} - 2u_{k,k} - u_{j,k}\right) + u_{j,k}u_{k,k}\right] e^{-2u_k}$$

- **Case** $j := 0$. We have :

$$\begin{aligned}
\mathbb{G}_{00} =& \frac{1}{2}\left[(\hat{u}_{0,0})^2 - \hat{u}_{0,0}^2\right] e^{-2u_0} \\
&+ \eta_0\eta_1 \left[\hat{u}_{01,11} + \frac{1}{2}\hat{u}_{1,1}^2 + \frac{1}{2}\hat{u}_{1,1}\left(\hat{u}_{01,1} - 2u_{1,1} - u_{0,1}\right) + u_{0,1}u_{1,1}\right] e^{-2u_1} \\
&+ \eta_0\eta_2 \left[\hat{u}_{02,22} + \frac{1}{2}\hat{u}_{2,2}^2 + \frac{1}{2}\hat{u}_{2,2}\left(\hat{u}_{02,2} - 2u_{2,2} - u_{0,2}\right) + u_{0,2}u_{2,2}\right] e^{-2u_2} \\
&+ \eta_0\eta_3 \left[\hat{u}_{03,33} + \frac{1}{2}\hat{u}_{3,3}^2 + \frac{1}{2}\hat{u}_{3,3}\left(\hat{u}_{03,3} - 2u_{3,3} - u_{0,3}\right) + u_{0,3}u_{3,3}\right] e^{-2u_3}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}\left[\left(\dot{v}+2\dot{b}\right)^2-(\dot{v})^2-2\left(\dot{b}\right)^2\right]e^{-2u}\\
&-\left[-2r^{-2}+\frac{1}{2}\left(\left(u'\right)^2+2r^{-2}\right)+\frac{1}{2}\left(u'+2r^{-1}\right)\left(2r^{-1}-2v'-u'\right)+u'v'\right]e^{-2v}\\
&+r^{-2}e^{-2b}\\
=&\dot{b}\left(2\dot{v}+\dot{b}\right)e^{-2u}-r^{-1}\left(r^{-1}-2v'\right)e^{-2v}+r^{-2}e^{-2b}
\end{aligned}
$$

- **Case** $j := 1$. We have :

$$
\begin{aligned}
\mathbb{G}_{11}=&\frac{1}{2}\left[(\hat{u}_{1,1})^2-\hat{u}_{1,1}^2\right]e^{-2u_1}\\
&+\eta_1\eta_0\left[\hat{u}_{10,00}+\frac{1}{2}\hat{u}_{0,0}^2+\frac{1}{2}\hat{u}_{0,0}\left(\hat{u}_{10,0}-2u_{0,0}-u_{1,0}\right)+u_{1,0}u_{0,0}\right]e^{-2u_0}\\
&+\eta_1\eta_2\left[\hat{u}_{12,22}+\frac{1}{2}\hat{u}_{2,2}^2+\frac{1}{2}\hat{u}_{2,2}\left(\hat{u}_{12,2}-2u_{2,2}-u_{1,2}\right)+u_{1,2}u_{2,2}\right]e^{-2u_2}\\
&+\eta_1\eta_3\left[\hat{u}_{13,33}+\frac{1}{2}\hat{u}_{3,3}^2+\frac{1}{2}\hat{u}_{3,3}\left(\hat{u}_{13,3}-2u_{3,3}-u_{1,3}\right)+u_{1,3}u_{3,3}\right]e^{-2u_3}\\
=&\frac{1}{2}\left[\left(u'+2r^{-1}\right)^2-\left(u'\right)^2-2\left(u'\right)^2\right]e^{-2v}\\
&-\left[2\ddot{b}+\frac{1}{2}\left((\dot{v})^2+2\left(\dot{b}\right)^2\right)+\frac{1}{2}\left(\dot{v}+2\dot{b}\right)\left(2\dot{b}-2\dot{u}-\dot{v}\right)+\dot{u}\dot{v}\right]e^{-2u}\\
&-r^{-2}e^{-2b}\\
=&\left[\dot{b}\left(2\dot{u}-3\dot{b}\right)-2\ddot{b}\right]e^{-2u}+r^{-1}\left(2u'+r^{-1}\right)e^{-2v}-r^{-2}e^{-2b}
\end{aligned}
$$

- **Case** $j := 2$. We have :

$$
\begin{aligned}
\mathbb{G}_{22}=&\frac{1}{2}\left[(\hat{u}_{2,2})^2-\hat{u}_{2,2}^2\right]e^{-2u_2}\\
&+\eta_2\eta_0\left[\hat{u}_{20,00}+\frac{1}{2}\hat{u}_{0,0}^2+\frac{1}{2}\hat{u}_{0,0}\left(\hat{u}_{20,0}-2u_{0,0}-u_{2,0}\right)+u_{2,0}u_{0,0}\right]e^{-2u_0}\\
&+\eta_2\eta_1\left[\hat{u}_{21,11}+\frac{1}{2}\hat{u}_{1,1}^2+\frac{1}{2}\hat{u}_{1,1}\left(\hat{u}_{21,1}-2u_{1,1}-u_{2,1}\right)+u_{2,1}u_{1,1}\right]e^{-2u_1}\\
&+\eta_2\eta_3\left[\hat{u}_{23,33}+\frac{1}{2}\hat{u}_{3,3}^2+\frac{1}{2}\hat{u}_{3,3}\left(\hat{u}_{23,3}-2u_{3,3}-u_{2,3}\right)+u_{2,3}u_{3,3}\right]e^{-2u_3}\\
=&-\left[\ddot{v}+\ddot{b}+\frac{1}{2}\left((\dot{v})^2+2\left(\dot{b}\right)^2\right)+\frac{1}{2}\left(\dot{v}+2\dot{b}\right)\left(\dot{v}+\dot{b}-2\dot{u}-\dot{b}\right)+\dot{u}\dot{b}\right]e^{-2u}\\
&+\left[u''-r^{-2}+\frac{1}{2}\left(\left(u'\right)^2+2r^{-2}\right)+\frac{1}{2}\left(u'+2r^{-1}\right)\left(u'-2v'\right)+r^{-1}v'\right]e^{-2v}\\
=&\left[-\ddot{v}-\ddot{b}+(\dot{u}-\dot{v})\left(\dot{v}+\dot{b}\right)-\left(\dot{b}\right)^2\right]e^{-2u}+\left[u''+u'\left(u'-v'+r^{-1}\right)-r^{-1}v'\right]e^{-2v}
\end{aligned}
$$

- **Case** $j := 3$. Since $\mathbb{R}_{33}=\mathbb{R}_{22}$, we have:

$$\mathbb{G}_{33}=\mathbb{G}_{22}.$$

- We have:

$$\mathbb{G}_{01}=\mathbb{G}_{10}=\mathbb{R}_{01}=2\left(\dot{b}\left(u'-r^{-1}\right)+r^{-1}\dot{v}\right)e^{-u-v}.$$

Therefore, the matrices $\mathbb{G}_{jl}$ and G_{jl} are of the form:

$$
\mathbb{G}_{jl}=\begin{pmatrix}\mathbb{G}_{00}&\mathbb{G}_{01}&0&0\\\mathbb{G}_{10}&\mathbb{G}_{11}&0&0\\0&0&\mathbb{G}_{22}&0\\0&0&0&\mathbb{G}_{33}\end{pmatrix}\quad,\quad G_{jl}=\begin{pmatrix}G_{00}&G_{01}&0&0\\G_{10}&G_{11}&0&0\\0&0&G_{22}&0\\0&0&0&G_{33}\end{pmatrix}
$$

with:

$$
\begin{aligned}
\mathbb{G}_{00} =& \dot{b}\left(2\dot{v}+\dot{b}\right)e^{-2u} - r^{-1}\left(r^{-1}-2v'\right)e^{-2v} + r^{-2}e^{-2b} \\
G_{00} =& \dot{b}\left(2\dot{v}+\dot{b}\right) - r^{-1}\left(r^{-1}-2v'\right)e^{2u-2v} + r^{-2}e^{-2b+2u} \\
\mathbb{G}_{01} =& 2\left(\dot{b}\left(u'-r^{-1}\right)+r^{-1}\dot{v}\right)e^{-u-v} \\
G_{01} =& 2\left(\dot{b}\left(u'-r^{-1}\right)+r^{-1}\dot{v}\right) \\
\mathbb{G}_{11} =& \left[\dot{b}\left(2\dot{u}-3\dot{b}\right)-2\ddot{b}\right]e^{-2u} + r^{-1}\left(2u'+r^{-1}\right)e^{-2v} - r^{-2}e^{-2b} \\
G_{11} =& \left[\dot{b}\left(2\dot{u}-3\dot{b}\right)-2\ddot{b}\right]e^{-2u+2v} + r^{-1}\left(2u'+r^{-1}\right) - r^{-2}e^{-2b+2v} \\
\mathbb{G}_{22} =& \mathbb{G}_{33} = \left[-\ddot{v}-\ddot{b}+\left(\dot{u}-\dot{v}\right)\left(\dot{v}+\dot{b}\right)-\left(\dot{b}\right)^2\right]e^{-2u} + \left[u''+u'\left(u'-v'+r^{-1}\right)-r^{-1}v'\right]e^{-2v} \\
G_{22} =& r^2\left[-\ddot{v}-\ddot{b}+\left(\dot{u}-\dot{v}\right)\left(\dot{v}+\dot{b}\right)-\left(\dot{b}\right)^2\right]e^{-2u} + r^2\left[u''+u'\left(u'-v'+r^{-1}\right)-r^{-1}v'\right]e^{-2v} \\
G_{33} =& \sin^2\vartheta G_{22}
\end{aligned}
$$

Chapter 2

Practical Resolution in the Case of Spherically Symmetric Metrics

We study the metric h discussed in the examples of the previous chapter, see:

1.1.4.1, 1.1.5.1, 1.2.5.5, 1.2.6.3 , 1.2.6.5, 1.3.4.5 , 1.3.5.4, 1.4.4.3, 1.5.2.2, 1.6.0.6

$$\begin{aligned}
h &= e^{2u(t,r)}\mathrm{d}t \otimes \mathrm{d}t - e^{2v(t,r)}\mathrm{d}r \otimes \mathrm{d}r - r^2 e^{2b(t)}\left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right) \\
&= e^{2u(t,r)}\mathrm{d}t \otimes \mathrm{d}t - e^{2v(t,r)}\mathrm{d}r \otimes \mathrm{d}r - r^2 e^{2b(t)} h_\Omega
\end{aligned}$$

where:

$$h_\Omega := \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi$$

is the **metric on the unit 2-sphere in spherical coordinates** (see point (2) of example 1.1.3.3).

We will denote, as desired:

$$(0,1,2,3) := (t, r, \vartheta, \phi)$$

Following Einstein's article [3], several solutions to the general relativity equation have been given in specific cases. Here are the references for those discussed in this chapter:

- Schwarzschild metric: [1], [18], [19], [20]
- Reissner-Nordström metric: [7], [12], [13]
- FLRW metric: [4], [5], [10], [14], [15], [16], [20], [21],

2.1 General Solution

2.1.1 Objects Associated with Curvature

For the calculations, we will need the Christoffel symbols and the components of the Einstein tensor.

According to example 1.2.5.5, the Christoffel symbols are given by:

$$\Gamma^0 = \begin{pmatrix} \dot u & u' & 0 & 0 \\ u' & \dot v e^{-2u+2v} & 0 & 0 \\ 0 & 0 & r^2 \dot b e^{2b-2u} & 0 \\ 0 & 0 & 0 & r^2 \dot b \sin^2\vartheta e^{2b-2u} \end{pmatrix} \qquad \Gamma^1 = \begin{pmatrix} u' e^{2u-2v} & \dot v & 0 & 0 \\ \dot v & v' & 0 & 0 \\ 0 & 0 & -r e^{2b-2v} & 0 \\ 0 & 0 & 0 & -r\sin^2\vartheta e^{2b-2v} \end{pmatrix}$$

$$\Gamma^2 = \begin{pmatrix} 0 & 0 & \dot b & 0 \\ 0 & 0 & r^{-1} & 0 \\ \dot b & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & -\cos\vartheta\sin\vartheta \end{pmatrix} \qquad \Gamma^3 = \begin{pmatrix} 0 & 0 & 0 & \dot b \\ 0 & 0 & 0 & r^{-1} \\ 0 & 0 & 0 & \cot\vartheta \\ \dot b & r^{-1} & \cot\vartheta & 0 \end{pmatrix}$$

According to example 1.6.0.6, the non-zero components of the Einstein tensor are given by:

In the bases $\mathscr{C}_\perp$ **and** $\mathscr{C}_\perp^*$	$$\begin{aligned} \mathbb{G}_{00} &= \dot{b}\left(2\dot{v}+\dot{b}\right)e^{-2u} - r^{-1}\left(r^{-1}-2v'\right)e^{-2v} + r^{-2}e^{-2b} \\ \mathbb{G}_{01} &= 2\left(\dot{b}\left(u'-r^{-1}\right)+r^{-1}\dot{v}\right)e^{-u-v} \\ \mathbb{G}_{11} &= \left[\dot{b}\left(2\dot{u}-3\dot{b}\right)-2\ddot{b}\right]e^{-2u} + r^{-1}\left(2u'+r^{-1}\right)e^{-2v} - r^{-2}e^{-2b} \\ \mathbb{G}_{22} &= \left[-\ddot{v}-\ddot{b}+\left(\dot{u}-\dot{v}\right)\left(\dot{v}+\dot{b}\right)-\left(\dot{b}\right)^2\right]e^{-2u} + \left[u''+u'\left(u'-v'+r^{-1}\right)-r^{-1}v'\right]e^{-2v} \\ \mathbb{G}_{33} &= \mathbb{G}_{22} \end{aligned}$$
In the bases $\mathscr{C}$ **and** $\mathscr{C}^*$	$$\begin{aligned} \mathrm{G}_{00} &= e^{2u}\mathbb{G}_{00} \\ &= \dot{b}\left(2\dot{v}+\dot{b}\right) - r^{-1}\left(r^{-1}-2v'\right)e^{2u-2v} + r^{-2}e^{-2b+2u} \\ \mathrm{G}_{01} &= e^{u+v}\mathbb{G}_{01} \\ &= 2\left(\dot{b}\left(u'-r^{-1}\right)+r^{-1}\dot{v}\right) \\ \mathrm{G}_{11} &= e^{2v}\mathbb{G}_{11} \\ &= \left[\dot{b}\left(2\dot{u}-3\dot{b}\right)-2\ddot{b}\right]e^{-2u+2v} + r^{-1}\left(2u'+r^{-1}\right) - r^{-2}e^{-2b+2v} \\ \mathrm{G}_{22} &= r^2\mathbb{G}_{22} \\ &= r^2\left[-\ddot{v}-\ddot{b}+\left(\dot{u}-\dot{v}\right)\left(\dot{v}+\dot{b}\right)-\left(\dot{b}\right)^2\right]e^{-2u} + r^2\left[u''+u'\left(u'-v'+r^{-1}\right)-r^{-1}v'\right]e^{-2v} \\ \mathrm{G}_{33} &= r^2\sin^2\vartheta\,\mathbb{G}_{33} \\ &= -r^2\sin^2\vartheta\left[2\ddot{b}+3\left(\dot{b}\right)^2\right]e^{2b-2u} + r^2\sin^2\vartheta\left[u''+\left(u'+r^{-1}\right)\left(u'-v'\right)\right]e^{-2v} \end{aligned}$$

2.1.2 Form of the tensor field S_{jl}

Depending on the utility of the tensors, it is useful to raise or lower indices (see subsections 1.1.8 and 1.1.9). The obtained tensors are related by simple formulas. We begin with an example.

Example 2.1.2.1: Forms of the energy-momentum tensor in the case of a perfect fluid

In the literature, an usual example of the energy-momentum tensor is the energy-momentum tensor for a perfect fluid in thermodynamic equilibrium. It can be expressed in six different equivalent forms:

$$\mathbb{T}_{jl} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \qquad \mathrm{T}_{jl} = \begin{pmatrix} \rho c^2 e^{2u} & 0 & 0 & 0 \\ 0 & Pe^{2v} & 0 & 0 \\ 0 & 0 & r^2P & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta P \end{pmatrix}$$

$$\mathbb{T}^j_{\ l} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix} \qquad \mathrm{T}^j_{\ l} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}$$

$$\mathbb{T}^{jl} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \qquad \mathrm{T}^{jl} = \begin{pmatrix} \rho c^2 e^{-2u} & 0 & 0 & 0 \\ 0 & Pe^{-2v} & 0 & 0 \\ 0 & 0 & r^{-2}P & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\vartheta P \end{pmatrix}$$

where ρ is the **mass density of the fluid** and P is the **hydrostatic pressure**. As indicated by the notation T or $\mathbb{T}$ and the indices, these six tensors represent the case of a perfect fluid with or without tetrads and of type $(2,0)$ or $(1,1)$ or $(0,2)$.
These six tensors are related by the standard formulas (see the results in subsections 1.1.8 and 1.1.9):

$$\mathrm{T}^j_{\ j} = \mathrm{h}^{jj}\mathrm{T}_{jj} \qquad \mathrm{T}^{jj} = \mathrm{h}^{jj}\mathrm{T}^j_{\ j}$$

$$\mathbb{T}^j_j = \eta^{jj}\mathbb{T}_{jj} \qquad \mathbb{T}^{jj} = \eta^{jj}\mathbb{T}^j_j$$
$$\mathrm{T}_{jj} = e^{2u_j}\mathbb{T}_{jj} \qquad \mathrm{T}^j_j = \mathbb{T}^j_j \qquad \mathrm{T}^{jj} = e^{-2u_j}\mathbb{T}^{jj}$$

It can be observed that in the case where the tensor T_{jl} is diagonal, we have:

$$\mathbb{T}^j_j = \mathrm{T}^j_j.$$

This is no longer true if T_{jl} is non-diagonal.
We will prefer the tensors T_{jl} and $\mathbb{T}_{jl}$ when using the Einstein equation, and the tensors T^j_l and $\mathbb{T}^j_l$ when using the Bianchi identity.

The Einstein equation is written as:

In the $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$ bases	**In the $\mathscr{C}$ and $\mathscr{C}^*$ bases**
$\mathbb{G}_{jl} = \mathbb{S}_{jl}$	$\mathrm{G}_{jl} = \mathrm{S}_{jl}$

.

with a right-hand side that, in the case with a cosmological constant, is given by:

In the $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$ bases	**In the $\mathscr{C}$ and $\mathscr{C}^*$ bases**
$\mathbb{S}_{jl} := -\Lambda\eta_{jl} + \kappa\mathbb{T}_{jl}$	$\mathrm{S}_{jl} := -\Lambda g_{jl} + \kappa\mathrm{S}_{jl}$

We have defined:

$$\kappa := \frac{8\pi G}{c^4}.$$

In this section, we will consider tensors $\mathbb{S}_{jl}$ and S_{jl} of the following form, as $\mathbb{G}_{jl}$ and G_{jl} only contain two non-zero off-diagonal terms that are equal in symmetric positions $(j,l) := (0,1)$ and $(j,l) := (1,0)$:

$$\mathrm{S}_{jl} := \begin{pmatrix} \mathrm{S}_{00} & \mathrm{S}_{01} & 0 & 0 \\ \mathrm{S}_{10} & \mathrm{S}_{11} & 0 & 0 \\ 0 & 0 & \mathrm{S}_{22} & 0 \\ 0 & 0 & 0 & \mathrm{S}_{33} \end{pmatrix} := \begin{pmatrix} \alpha e^{2u} & \xi e^{u+v} & 0 & 0 \\ \xi e^{u+v} & \beta e^{2v} & 0 & 0 \\ 0 & 0 & r^2\gamma & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta\gamma \end{pmatrix}$$
$$\mathbb{S}_{jl} = \begin{pmatrix} \mathbb{S}_{00} & \mathbb{S}_{01} & 0 & 0 \\ \mathbb{S}_{10} & \mathbb{S}_{11} & 0 & 0 \\ 0 & 0 & \mathbb{S}_{22} & 0 \\ 0 & 0 & 0 & \mathbb{S}_{33} \end{pmatrix} := \begin{pmatrix} \alpha & \xi & 0 & 0 \\ \xi & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$$

where $\alpha(t,r), \beta(t,r), \gamma(t,r), \xi(t,r)$ are functions of the variables t and r. Thus, we have:

$$\mathrm{S}^j_l = \mathrm{g}^{ll}\mathrm{S}_{jl} = \begin{pmatrix} \mathrm{S}^0_0 & \mathrm{S}^0_1 & 0 & 0 \\ \mathrm{S}^1_0 & \mathrm{S}^1_1 & 0 & 0 \\ 0 & 0 & \mathrm{S}^2_2 & 0 \\ 0 & 0 & 0 & \mathrm{S}^3_3 \end{pmatrix} = \begin{pmatrix} \alpha & \xi e^{-u+v} & 0 & 0 \\ -\xi e^{u-v} & -\beta & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{pmatrix}$$
$$\mathbb{S}^j_l = \eta^{ll}\mathrm{S}_{jl} = \begin{pmatrix} \mathbb{S}^0_0 & \mathbb{S}^0_1 & 0 & 0 \\ \mathbb{S}^1_0 & \mathbb{S}^1_1 & 0 & 0 \\ 0 & 0 & \mathbb{S}^2_2 & 0 \\ 0 & 0 & 0 & \mathbb{S}^3_3 \end{pmatrix} = \begin{pmatrix} \alpha & \xi & 0 & 0 \\ -\xi & -\beta & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{pmatrix}$$

Matrix representations are given by:

In the $\mathscr{C}_\perp$ and $\mathscr{C}_\perp^*$ bases	**In the $\mathscr{C}$ and $\mathscr{C}^*$ bases**
$\begin{pmatrix} \mathbb{G}_{00} & \mathbb{G}_{01} & 0 & 0 \\ \mathbb{G}_{10} & \mathbb{G}_{11} & 0 & 0 \\ 0 & 0 & \mathbb{G}_{22} & 0 \\ 0 & 0 & 0 & \mathbb{G}_{33} \end{pmatrix} = \begin{pmatrix} \mathbb{S}_{00} & \mathbb{S}_{01} & 0 & 0 \\ \mathbb{S}_{10} & \mathbb{S}_{11} & 0 & 0 \\ 0 & 0 & \mathbb{S}_{22} & 0 \\ 0 & 0 & 0 & \mathbb{S}_{33} \end{pmatrix}$	$\begin{pmatrix} \mathrm{G}_{00} & \mathrm{G}_{01} & 0 & 0 \\ \mathrm{G}_{10} & \mathrm{G}_{11} & 0 & 0 \\ 0 & 0 & \mathrm{G}_{22} & 0 \\ 0 & 0 & 0 & \mathrm{G}_{33} \end{pmatrix} = \begin{pmatrix} \mathrm{S}_{00} & \mathrm{S}_{01} & 0 & 0 \\ \mathrm{S}_{10} & \mathrm{S}_{11} & 0 & 0 \\ 0 & 0 & \mathrm{S}_{22} & 0 \\ 0 & 0 & 0 & \mathrm{S}_{33} \end{pmatrix}$.

In both cases, we have the following four fundamental equations:

$$\begin{aligned}
&(E_0 \ : \ \mathbb{G}_{00} = \mathbb{S}_{00}) \ : \ \dot{b}\left(2\dot{v} + \dot{b}\right)e^{-2u} - r^{-1}\left(r^{-1} - 2v'\right)e^{-2v} + r^{-2}e^{-2b} = \alpha \\
&(E_{01} \ : \ \mathbb{G}_{01} = \mathbb{S}_{01}) \ : \ \left(\dot{b}\left(u' - r^{-1}\right) + r^{-1}\dot{v}\right)e^{-u-v} = \xi \\
&(E_1 \ : \ \mathbb{G}_{11} = \mathbb{S}_{11}) \ : \ \left[\dot{b}\left(2\dot{u} - 3\dot{b}\right) - 2\ddot{b}\right]e^{-2u} + r^{-1}\left(2u' + r^{-1}\right)e^{-2v} - r^{-2}e^{-2b} = \beta \\
&(E_2 \ : \ \mathbb{G}_{22} = \mathbb{S}_{22}) \ : \ \left[-\ddot{v} - \ddot{b} + (\dot{u} - \dot{v})\left(\dot{v} + \dot{b}\right) - \left(\dot{b}\right)^2\right]e^{-2u} + \left[u'' + u'\left(u' - v' + r^{-1}\right) - r^{-1}v'\right]e^{-2v} = \gamma
\end{aligned}$$

2.1.3 Using the Bianchi Identity

According to Proposition 1.6.0.3, the twice-contracted Bianchi identity gives four equations for $l \in \{0, 1, 2, 3\}$:

$$\nabla_j \mathrm{G}^j_l = 0.$$

Using Proposition 1.2.5.2, we have:

$$\nabla_j \mathrm{G}^j_l = \partial_j \mathrm{G}^j_l + \Gamma^j{}_{jk}\mathrm{G}^k_l - \Gamma^k{}_{lj}\mathrm{G}^j_k.$$

Therefore, we have:

$$\nabla_j \mathrm{G}^j_l = \partial_j \mathrm{G}^j_l + \Gamma^j{}_{jk}\mathrm{G}^k_l - \Gamma^k{}_{lj}\mathrm{G}^j_k = 0.$$

Using the equality:

$$\mathrm{G}^j_l = \mathrm{S}^j_l$$

and the linearity of ∇_j, we obtain the following four equalities for $l \in \{0, 1, 2, 3\}$:

$$0 = \nabla_j \mathrm{S}^j_l = \partial_j \mathrm{S}^j_l + \Gamma^j{}_{jk}\mathrm{S}^k_l - \Gamma^k{}_{lj}\mathrm{S}^j_k$$

The following proposition gives the two nontrivial equalities derived from these four equations.

Proposition 2.1.3.1: Application of the Bianchi Identity

The identity:

$$\nabla_j \mathrm{G}^j_l = 0$$

is equivalent to the two equations:

$$\begin{aligned}
&(B_0 \ : \ \nabla_j \mathrm{S}^j_0 = 0) \ : \ \dot{\alpha} - \left[\xi' + 2\left(u' + r^{-1}\right)\xi\right]e^{u-v} + 2\dot{b}\left(\alpha + \gamma\right) - \left(u'\alpha - v'\beta + 2r^{-1}\gamma\right) + \dot{v}\beta = 0 \\
&(B_1 \ : \ \nabla_j \mathrm{S}^j_1 = 0) \ : \ \left[\dot{\xi} + \left(\dot{v} - 2\dot{u} - 2\dot{b}\right)\xi\right]e^{-u+v} - \beta' - (\alpha + \beta)u' + 2r^{-1}(\gamma - \beta) = 0
\end{aligned}$$

Proof. We have:

$$0 = \nabla_j \mathrm{S}^j_l = \partial_j \mathrm{S}^j_l + \Gamma^j{}_{jk}\mathrm{S}^k_l - \Gamma^k{}_{lj}\mathrm{S}^j_k$$

There are four cases.

- Case $l := 0$. We have:

$$\begin{aligned}
0 = \nabla_j \mathrm{S}_0^j &= \partial_j \mathrm{S}_0^j + {\Gamma^j}_{jk} \mathrm{S}_0^k - {\Gamma^k}_{0j} \mathrm{S}_k^j \\
&= \partial_0 \mathrm{S}_0^0 + \partial_1 \mathrm{S}_0^1 + {\Gamma^j}_{j0} \mathrm{S}_0^0 + {\Gamma^j}_{j1} \mathrm{S}_0^1 - {\Gamma^0}_{01} \mathrm{S}_0^1 - {\Gamma^j}_{0j} \mathrm{S}_j^j \\
&= \dot\alpha - \left[\xi' + \left(u' - v'\right)\xi\right] e^{u-v} + \left(\dot u + \dot v + 2\dot b\right)\alpha - \left(u' + v' + 2r^{-1}\right)\xi e^{u-v} - \dot v \alpha - \left(\dot u \alpha - \dot v \beta - 2\dot b \gamma\right) \\
&= \dot\alpha - \left[\xi' + 2\left(u' + r^{-1}\right)\xi\right] e^{u-v} + 2\dot b\left(\alpha + \gamma\right) - \left(u'\alpha - v'\beta + 2r^{-1}\gamma\right) + \dot v \beta
\end{aligned}$$

- Case $l := 1$. We have:

$$\begin{aligned}
0 = \nabla_j \mathrm{S}_1^j &= \partial_j \mathrm{S}_1^j + {\Gamma^j}_{jk} \mathrm{S}_1^k - {\Gamma^k}_{1j} \mathrm{S}_k^j \\
&= \partial_0 \mathrm{S}_1^0 + \partial_1 \mathrm{S}_1^1 + {\Gamma^j}_{j0} \mathrm{S}_1^0 + {\Gamma^j}_{j1} \mathrm{S}_1^1 - {\Gamma^1}_{10} \mathrm{S}_1^0 - {\Gamma^j}_{1j} \mathrm{S}_j^j \\
&= \left[\dot\xi + \left(\dot v - \dot u\right)\xi\right] e^{-u+v} - \beta' - \left(\dot u + \dot v + 2\dot b\right)\xi e^{-u+v} - \left(u' + v' + 2r^{-1}\right)\beta - \dot v \xi e^{-u+v} - \left(u'\alpha - v'\beta - 2r^{-1}\gamma\right) \\
&= \left[\dot\xi + \left(\dot v - 2\dot u - 2\dot b\right)\xi\right] e^{-u+v} - \beta' - \left(\alpha + \beta\right)u' + 2r^{-1}\left(\gamma - \beta\right)
\end{aligned}$$

- Case $l := 2$. We have:

$$\begin{aligned}
0 = \nabla_j \mathrm{S}_2^j &= \partial_j \mathrm{S}_2^j + {\Gamma^j}_{jk} \mathrm{S}_2^k - {\Gamma^k}_{2j} \mathrm{S}_k^j \\
&= {\Gamma^j}_{j2} \mathrm{S}_2^2 - {\Gamma^j}_{2j} \mathrm{S}_j^j \\
&= \cot\vartheta \mathrm{S}_2^2 - \cot\vartheta \mathrm{S}_3^3 \\
&= 0
\end{aligned}$$

- Case $l := 3$. We have:

$$\begin{aligned}
0 = \nabla_j \mathrm{S}_3^j &= \partial_j \mathrm{S}_3^j + {\Gamma^j}_{jk} \mathrm{S}_3^k - {\Gamma^k}_{3j} \mathrm{S}_k^j \\
&= {\Gamma^j}_{j3} \mathrm{S}_3^3 - {\Gamma^j}_{3j} \mathrm{S}_j^j \\
&= \cot\vartheta \mathrm{S}_2^2 - \cot\vartheta \mathrm{S}_3^3 \\
&= 0
\end{aligned}$$

The last two cases are trivial.

□

2.1.4 The six fundamental equations

The six fundamental equations that will be used throughout this section are:

$$\begin{aligned}
&(E_0 \;:\; \mathbb{G}_{00} = \mathbb{S}_{00}) &:\;& \dot b\left(2\dot v + \dot b\right) e^{-2u} - r^{-1}\left(r^{-1} - 2v'\right) e^{-2v} + r^{-2} e^{-2b} = \alpha \\
&(E_{01} \;:\; \mathbb{G}_{01} = \mathbb{S}_{01}) &:\;& \left(\dot b\left(u' - r^{-1}\right) + r^{-1}\dot v\right) e^{-u-v} = \xi \\
&(E_1 \;:\; \mathbb{G}_{11} = \mathbb{S}_{11}) &:\;& \left[\dot b\left(2\dot u - 3\dot b\right) - 2\ddot b\right] e^{-2u} + r^{-1}\left(2u' + r^{-1}\right) e^{-2v} - r^{-2} e^{-2b} = \beta \\
&(E_2 \;:\; \mathbb{G}_{22} = \mathbb{S}_{22}) &:\;& \left[-\ddot v - \ddot b + \left(\dot u - \dot v\right)\left(\dot v + \dot b\right) - \left(\dot b\right)^2\right] e^{-2u} + \left[u'' + u'\left(u' - v' + r^{-1}\right) - r^{-1} v'\right] e^{-2v} = \gamma \\
&(B_0 \;:\; \nabla_j \mathrm{S}_0^j = 0) &:\;& \dot\alpha - \left[\xi' + 2\left(u' + r^{-1}\right)\xi\right] e^{u-v} + 2\dot b\left(\alpha + \gamma\right) - \left(u'\alpha - v'\beta + 2r^{-1}\gamma\right) + \dot v \beta = 0 \\
&(B_1 \;:\; \nabla_j \mathrm{S}_1^j = 0) &:\;& \left[\dot\xi + \left(\dot v - 2\dot u - 2\dot b\right)\xi\right] e^{-u+v} - \beta' - \left(\alpha + \beta\right)u' + 2r^{-1}\left(\gamma - \beta\right) = 0
\end{aligned}$$

The first four equations are derived from the general relativity equation, and the last two are due to the Bianchi identities.

The two Bianchi identities are derived from the general relativity equation, but due to their importance in the calculations, they are considered as two additional fundamental equations.

2.1.5 Case where G and S are diagonal

From equation (E_{01}), we have the following equivalences:

(i) G_{jl} and S_{jl} are diagonal;

(ii) $\xi = 0$;

(iii) $r\dot{b}u' + \dot{v} - \dot{b} = 0$.

The last equation is a criterion that is generally difficult to exploit. However, we have the following simple result.

Proposition 2.1.5.1: Case where $\xi = 0$

Consider the case:

$$\xi = 0.$$

(i) The following conditions are equivalent:

  (a) $u' = 0$, *i.e.*, u is independent of r;

  (b) $\dot{v} = \dot{b}$;

  (c) The function v can be decomposed as $v(t, r) = b(t) + v_r(r)$.

(ii) Suppose that:

$$\dot{b} = 0.$$

Then $\dot{v} = 0$, *i.e.*, v is independent of t.

Proof. (i) From point (iii) of the equivalence mentioned before the proposition and since $\xi = 0$, we have:

$$r\dot{b}u' + \dot{v} - \dot{b} = 0.$$

This leads to the equivalences between:

- $u' = 0$;
- $\dot{v} - \dot{b} = 0$;
- $\dot{v} = \dot{b}$.

The last point is equivalent, by integration, to the existence of a function v_r depending only on r such that:

$$v(t, r) = b(t) + v_r(r).$$

(ii) Suppose that:

$$\dot{b} = 0.$$

Then, according to point (iii) of the equivalence mentioned before the proposition:

$$\dot{v} = 0$$

i.e., v is independent of t.

□

The two cases studied in this proposition will be treated separately, the first one in the following subsection, and the second one, which is more extensive, will be addressed in the next section.

2.1.6 Case where the functions u' and ξ are zero

We consider the case where:

$$u' = \xi = 0.$$

Hence, there exists a constant c such that:

$$e^u = c.$$

By point (i) of Proposition 2.1.5.1, the function v has separated variables. We assume that v can be written as:

$$v(t,r) := b(t) + v(r).$$

Thus, the studied metric is of the form:

$$h = c^2 \mathrm{d}t \otimes \mathrm{d}t - e^{2b(t)} \left(e^{2v(r)} \mathrm{d}r \otimes \mathrm{d}r + r^2 \left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2 \vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi \right) \right)$$

Throughout this subsection, we consider an energy-momentum tensor that satisfies the six fundamental equations of the form:

$$\mathbb{S}_{jl} := \begin{pmatrix} \alpha(t,r) & 0 & 0 & 0 \\ 0 & \beta(t,r) & 0 & 0 \\ 0 & 0 & \beta(t,r) & 0 \\ 0 & 0 & 0 & \beta(t,r) \end{pmatrix}$$

where α, β are functions of the variables t and r.

We have:

$$(B_1 \ : \ \nabla_j S^j_1 = 0) \quad : \quad -\beta' = 0$$

Thus, β does not depend on r, so we have:

$$\beta(t,r) = \beta(t).$$

The equation (E_{01}) is trivial, and the five fundamental equations become:

$$\begin{aligned}
(E_0 \ : \ \mathbb{G}_{00} = \mathbb{S}_{00}) \quad &: \quad 3c^{-2} \left(\dot{b}\right)^2 + r^{-1} e^{-2b} \left[\left(2v' - r^{-1}\right) e^{-2v} + r^{-1} \right] = \alpha \\
(E_1 \ : \ \mathbb{G}_{11} = \mathbb{S}_{11}) \quad &: \quad -c^{-2} \left[2\ddot{b} + 3 \left(\dot{b}\right)^2 \right] + r^{-2} e^{-2b} \left(e^{-2v} - 1 \right) = \beta \\
(E_2 \ : \ \mathbb{G}_{22} = \mathbb{S}_{22}) \quad &: \quad -c^{-2} \left[2\ddot{b} + 3 \left(\dot{b}\right)^2 \right] - r^{-1} v' e^{-2b-2v} = \beta \\
(B_0 \ : \ \nabla_j S^j_0 = 0) \quad &: \quad \dot{\alpha} + \dot{b} \left(2\alpha + 3\beta\right) + \left(v' - 2r^{-1}\right) \beta = 0 \\
(B_1 \ : \ \nabla_j S^j_1 = 0) \quad &: \quad -\beta' = 0
\end{aligned}$$

Theorem 2.1.6.1: Generalized metric

There exists $k \in \mathbb{R}$ such that:

$$h = c^2 \mathrm{d}t \otimes \mathrm{d}t - e^{2b(t)} \left(\frac{1}{1 - kr^2} \mathrm{d}r \otimes \mathrm{d}r + r^2 h_\Omega \right)$$

where:

$$\begin{aligned}
\left(\dot{b}\right)^2 + kc^2 e^{-2b} &= \frac{\alpha c^2}{3} \\
\ddot{b} + \left(\dot{b}\right)^2 &= -\frac{c^2}{6} \left(3\beta + \alpha\right)
\end{aligned}$$

Proof. We prove the result in three steps:

$$\begin{aligned}
&(1) \quad e^{-2v} = 1 - kr^2 \\
&(2) \quad \left(\dot{b}\right)^2 + kc^2 e^{-2b} = \frac{\alpha c^2}{3} \\
&(3) \quad \ddot{b} + \left(\dot{b}\right)^2 = -\frac{c^2}{6}\left(3\beta + \alpha\right)
\end{aligned}$$

(1) We provide two methods.

(a) We have

$$\begin{aligned}
0 &= \beta - \beta \\
&= \mathbb{G}_{11} - \mathbb{G}_{22} \\
&= -c^{-2}\left[2\ddot{b} + 3\left(\dot{b}\right)^2\right] + r^{-2}e^{-2b}\left(e^{-2v} - 1\right) + c^{-2}\left[2\ddot{b} + 3\left(\dot{b}\right)^2\right] + r^{-1}v'e^{-2b-2v} \\
&= r^{-1}e^{-2b}\left(r^{-1}e^{-2v} + v'e^{-2v} - r^{-1}\right)
\end{aligned}$$

So, by setting $g := e^{-2v}$, we have:

$$\begin{aligned}
0 &= r^{-1}e^{-2v} + v'e^{-2v} - r^{-1} \\
&= r^{-1}g - \frac{1}{2}g' - r^{-1}
\end{aligned}$$

Thus, g is a solution of the differential equation:

$$(E) \ : \ y' - 2r^{-1}y = -2r^{-1}.$$

The homogeneous solutions are generated by the function:

$$r \longmapsto \exp\left(2\int_1^r \mathfrak{r}^{-1}\mathrm{d}\mathfrak{r}\right) = e^{2\ln(r)} = r^2.$$

We have the constant particular solution:

$$r \longmapsto 1.$$

Therefore, there exists a real constant k such that:

$$g = 1 - kr^2.$$

Thus, we have:

$$e^{2v} = \frac{1}{g} = \frac{1}{1 - kr^2}.$$

(b) Using (E_1) and separating the variables r and t, we have:

$$e^{2b}\left(\beta + c^{-2}\left[2\ddot{b} + 3\left(\dot{b}\right)^2\right]\right) = r^{-2}\left(e^{-2v} - 1\right)$$

Hence, there exists a constant k such that:

$$r^{-2}\left(e^{-2v} - 1\right) = -k$$

In other words, we have:

$$e^{2v} = \frac{1}{1 - kr^2}.$$

We also obtain another equation that appears in point (3):

$$e^{2b}\left(\beta + c^{-2}\left[2\ddot{b} + 3\left(\dot{b}\right)^2\right]\right) = -k$$

In other words, we have:

$$2\ddot{b} + 3\left(\dot{b}\right)^2 + kc^2e^{-2b} = -\beta c^2.$$

The advantage of the second method is that it only uses equation (E_1).

(2) Since:

$$g' = -2kr$$

we have:

$$\begin{aligned}
\alpha &= \mathbb{G}_{00} \\
&= 3c^{-2}\left(\dot{b}\right)^2 + r^{-1}\left(2v' - r^{-1}\right)e^{-2b-2v} + r^{-2}e^{-2b} \\
&= 3c^{-2}\left(\dot{b}\right)^2 + r^{-1}e^{-2b}\left(2v'e^{-2v} - r^{-1}e^{-2v} + r^{-1}\right) \\
&= 3c^{-2}\left(\dot{b}\right)^2 + r^{-1}e^{-2b}\left(-g' - r^{-1}g + r^{-1}\right) \\
&= 3c^{-2}\left(\dot{b}\right)^2 + r^{-1}e^{-2b}\left(2kr - r^{-1} + kr + r^{-1}\right) \\
&= 3c^{-2}\left(\dot{b}\right)^2 + 3ke^{-2b}
\end{aligned}$$

Therefore, we have:

$$\dot{b} + kc^2e^{-2b} = \frac{\alpha c^2}{3}.$$

(3) We have:

$$\begin{aligned}
\beta &= \mathbb{G}_{22} \\
&= -c^{-2}\left(2\ddot{b} + 3\left(\dot{b}\right)^2\right) - r^{-1}v'e^{-2b-2v} \\
&= -c^{-2}\left(2\ddot{b} + 3\left(\dot{b}\right)^2\right) - \frac{1}{2r}g'e^{-2b} \\
&= -c^{-2}\left(2\ddot{b} + 3\left(\dot{b}\right)^2\right) - ke^{-2b}
\end{aligned}$$

Thus, we have the two equations (the second one is from point (ii)):

$$\begin{aligned}
2\ddot{b} + 3\left(\dot{b}\right)^2 + kc^2e^{-2b} &= -\beta c^2 \\
\left(\dot{b}\right)^2 + kc^2e^{-2b} &= \frac{\alpha c^2}{3}
\end{aligned}$$

By subtracting these two equations, we obtain:

$$\ddot{b} + \left(\dot{b}\right)^2 = -\frac{c^2}{6}\left(3\beta + \alpha\right)$$

□

In the following corollary, we give the relations with the usual notations used in the case of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric.

Corollary 2.1.6.2: Other versions

(i) The **scale factor** is defined as:

$$a := e^b.$$

Then there exists $k \in \mathbb{R}$ such that:

$$h = c^2 \mathrm{d}t \otimes \mathrm{d}t - a(t)^2\left(\frac{1}{1-kr^2}\mathrm{d}r \otimes \mathrm{d}r + r^2 h_\Omega\right)$$

where:

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= \frac{\alpha c^2}{3} \\ \frac{\ddot{a}}{a} &= -\frac{c^2}{6}\left(3\beta + \alpha\right)\end{aligned}$$

(ii) The **Hubble parameter** is defined as:

$$H := \dot{b} = \frac{\dot{a}}{a}.$$

The **spatial curvature** is defined as:

$$K := ke^{-b} = \frac{k}{a}.$$

Then we have:

$$\begin{aligned}\left(\dot{H}\right)^2 + \frac{Kc^2}{a} &= \frac{\alpha c^2}{3} \\ \dot{H} + H^2 &= -\frac{c^2}{6}\left(3\beta + \alpha\right)\end{aligned}$$

Example 2.1.6.3: Usual example

We take the notations from Example 2.1.2.1. We consider the case where:

$$\alpha := \frac{8\pi G}{c^2}\rho - \Lambda \quad , \quad \beta := \frac{8\pi G}{c^4}P + \Lambda.$$

Therefore, we have the Friedmann-Lemaître equations:

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} &= \frac{8\pi G}{3}\rho - \frac{\Lambda c^2}{3} \\ \frac{\ddot{a}}{a} &= -\frac{8\pi G}{c^2}\left(P + \rho c^2\right) - \frac{\Lambda}{3}\end{aligned}$$

2.2 Case where the functions $\dot{b}$ and ξ are zero.

We consider the case where:

$$\dot{b} = 0 \quad \wedge \ \xi = 0.$$

By point (ii) of Proposition 2.1.5.1, the function v is independent of the variable t, *i.e.*, v can be written as:

$$v(t,r) := v(r).$$

Thus, the studied metric is of the type:

$$\begin{aligned}h &= e^{2u(t,r)}\mathrm{d}t \otimes \mathrm{d}t - e^{2v(r)}\mathrm{d}r \otimes \mathrm{d}r - r^2\left(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi\right) \\ &= e^{2u(t,r)}\mathrm{d}t \otimes \mathrm{d}t - e^{2v(r)}\mathrm{d}r \otimes \mathrm{d}r - r^2 h_\Omega\end{aligned}$$

In this subsection, we consider an energy-momentum tensor of the general form:

$$\mathbb{S}_{jl} := \begin{pmatrix}\alpha(t,r) & 0 & 0 & 0 \\ 0 & \beta(t,r) & 0 & 0 \\ 0 & 0 & \gamma(t,r) & 0 \\ 0 & 0 & 0 & \gamma(t,r)\end{pmatrix}$$

where α, β, γ are functions of the variables t and r.

From the equation:

$$(E_0 \;:\; \mathbb{G}_{00} = \mathbb{S}_{00}) \quad : \quad r^{-1}\left[\left(2v' - r^{-1}\right)e^{-2v} + r^{-1}\right] = \alpha$$

and since the left-hand side only depends on r, we deduce that α is independent of t, *i.e.*,

$$\alpha(t, r) \coloneqq \alpha(r).$$

The equation (E_{01}) is trivial, and the five fundamental equations become:

$$\begin{aligned}
(E_0 \;:\; \mathbb{G}_{00} = \mathbb{S}_{00}) \quad &: \quad r^{-1}\left[\left(2v' - r^{-1}\right)e^{-2v} + r^{-1}\right] = \alpha \\
(E_1 \;:\; \mathbb{G}_{11} = \mathbb{S}_{11}) \quad &: \quad r^{-1}\left[\left(2u' + r^{-1}\right)e^{-2v} - r^{-1}\right] = \beta \\
(E_2 \;:\; \mathbb{G}_{22} = \mathbb{S}_{22}) \quad &: \quad \left[u'' + \left(u' + r^{-1}\right)\left(u' - v'\right)\right]e^{-2v} = \gamma \\
(B_0 \;:\; \nabla_j \mathbb{S}_0^j = 0) \quad &: \quad u'\alpha - v'\beta + 2r^{-1}\gamma = 0 \\
(B_1 \;:\; \nabla_j \mathbb{S}_1^j = 0) \quad &: \quad -\beta' - (\alpha + \beta)\,u' + 2r^{-1}(\gamma - \beta) = 0
\end{aligned}$$

We define:

$$A(r) \coloneqq \int_0^r \alpha(\mathfrak{r})\mathfrak{r}^2 \mathrm{d}\mathfrak{r}.$$

Example 2.2.0.1: Three simple examples

Throughout this section, we will consider three usual cases as examples (see Example 2.1.2.1 for the notations). We assume that the functions $\rho(r)$ and $P(r)$ are independent of t.

(1) **Case of a perfect fluid.** We are in the case where (also see Example 2.1.2.1):

$$\mathbb{S}_{jl} \coloneqq \begin{pmatrix} \kappa\rho c^2 & 0 & 0 & 0 \\ 0 & \kappa P & 0 & 0 \\ 0 & 0 & \kappa P & 0 \\ 0 & 0 & 0 & \kappa P \end{pmatrix} \qquad \mathrm{S}_{jl} \coloneqq \begin{pmatrix} \kappa\rho c^2 e^{2u} & 0 & 0 & 0 \\ 0 & \kappa P e^{2v} & 0 & 0 \\ 0 & 0 & \kappa r^2 P & 0 \\ 0 & 0 & 0 & \kappa r^2 \sin^2\vartheta P \end{pmatrix}$$

We are thus in the case where:

$$\alpha \coloneqq \kappa\rho c^2 \quad , \quad \beta \coloneqq \kappa P.$$

(2) **Case with a cosmological constant.** We are in the case where:

$$\mathbb{S}_{jl} \coloneqq \begin{pmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{pmatrix} \qquad \mathrm{S}_{jl} \coloneqq \kappa \begin{pmatrix} -\Lambda e^{2u} & 0 & 0 & 0 \\ 0 & \Lambda e^{2v} & 0 & 0 \\ 0 & 0 & r^2\Lambda & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta\Lambda \end{pmatrix}$$

We are thus in the case where:

$$\alpha \coloneqq -\Lambda \quad , \quad \beta \coloneqq \Lambda.$$

(3) **Case of a perfect fluid with a cosmological constant.** We are in the case where:

$$\mathbb{S}_{jl} \coloneqq \begin{pmatrix} \kappa\rho - \Lambda & 0 & 0 & 0 \\ 0 & \kappa P + \Lambda & 0 & 0 \\ 0 & 0 & \kappa P + \Lambda & 0 \\ 0 & 0 & 0 & \kappa P + \Lambda \end{pmatrix}$$

$$\mathrm{S}_{jl} \coloneqq \begin{pmatrix} (\kappa\rho - \Lambda)\,e^{2u} & 0 & 0 & 0 \\ 0 & (\kappa P + \Lambda)\,e^{2v} & 0 & 0 \\ 0 & 0 & r^2(\kappa P + \Lambda) & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta\,(\kappa P + \Lambda) \end{pmatrix}$$

We are thus in the case where:

$$\alpha := \kappa\rho - \Lambda \quad , \quad \beta := \kappa P + \Lambda.$$

At the end of the section, we will consider a more complicated example with an electric charge.

2.2.1 Section plan

We will consider different cases in this section. Here is a brief outline:

- **General solution**. In this subsection, we make no additional assumptions.
- **Exterior metric**. We assume the existence of a real number $R > 0$ such that for all $r > R$:

  $$\alpha(r) = -\beta(r)$$

  and the metric is asymptotically flat, *i.e.*, we have:

  $$\lim_{r\to\infty} \mathrm{g}_{ii} = \eta_i.$$

  We examine what happens outside a "ball" of radius R.
- **Interior metric**. We retain the assumptions of the exterior metric. Additionally, we assume that α is constant on $[0, R]$, $\gamma = \beta$, β is independent of t, and β is continuous at $r := R$.

  We examine what happens inside a "ball" of radius R.
- **Case where $\beta(r) = 0$ for $r > R$**. We assume the hypotheses from the previous point, with the additional assumption that β (and thus α) is zero for $r > R$.
- **Example of the Reissner–Nordström metric**. We consider a longer example involving an electric charge.

2.2.2 General solution

We start with general results on the functions appearing in the equation of general relativity.

Theorem 2.2.2.1: Usual properties

(i) We have:

$$e^{2v} = \left(1 - \frac{A}{r}\right)^{-1}.$$

(ii) We have:

$$u' = \frac{r^3\beta + A}{2r(r - A)}.$$

(iii) We have:

$$\beta' = -\frac{r^3\beta + A}{2r(r - A)}(\alpha + \beta) + 2\frac{\gamma - \beta}{r}.$$

Proof. (i) We have:

$$\begin{aligned}\mathbb{G}_{00} &= r^{-1}e^{-2v}\left(2v' - r^{-1}\right) + r^{-2}\\ &= r^{-2}\left(rv'e^{-2v} + \left(1 - e^{-2v}\right)\right)\\ &= r^{-2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\left(1 - e^{-2v}\right)\right)\end{aligned}$$

By equation (E_0), we have:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\left(1 - e^{-2v}\right)\right) = \alpha r^2$$

Upon integration, there exists a constant C such that:

$$r\left(1 - e^{-2v}\right) = \int_0^r \alpha(\mathfrak{r})\mathfrak{r}^2\mathrm{d}\mathfrak{r} + C = A + C$$

Thus, we have:

$$e^{-2v} = 1 - \frac{1}{r}\left(A + C\right)$$

and hence:

$$e^{2v} = \left(1 - \frac{A + C}{r}\right)^{-1}.$$

To determine the value of C, we consider the neighborhood of $r := 0_+$. We can assume that α is a constant with value α_0 in this neighborhood of 0_+. Therefore, in this neighborhood of 0_+, we have:

$$A(r) \approx \int_0^r \alpha_0\mathfrak{r}^2\mathrm{d}\mathfrak{r} \approx \alpha_0\frac{r^3}{3}$$

which leads to:

$$e^{-2v} \approx 1 - \frac{\alpha_0 r^3/3 + C}{r} \approx 1 - \frac{\alpha_0 r^2}{3} + \frac{C}{r}.$$

Thus, to avoid a singularity in the metric at 0_+, we set:

$$C := 0.$$

Consequently, we have:

$$e^{2v} = \left(1 - \frac{A}{r}\right)^{-1}.$$

(ii) Since:

$$e^{2v} = \left(1 - \frac{A}{r}\right)^{-1}$$

we have from (E_1):

$$\begin{aligned}\beta &= \mathbb{G}_{11}\\ &= r^{-1}e^{-2v}\left(2u' + r^{-1}\right) - r^{-2}\\ &= r^{-1}\left(1 - \frac{A}{r}\right)\left(2u' + r^{-1}\right) - r^{-2}\end{aligned}$$

which gives:

$$u' = \frac{1}{2}\frac{r\beta + r^{-1}}{1 - \frac{A}{r}} - \frac{1}{2}r^{-1} = \frac{r^3\beta + A}{2r(r - b)}.$$

(iii) Two proofs are provided.

(1) Using (B_1), we have:

$$\begin{aligned}\beta' &= -(\alpha+\beta)\,u' + 2r^{-1}(\gamma-\beta)\\ &= -\frac{r^3\beta + A}{2r(r-A)}(\alpha+\beta) + 2\frac{\gamma-\beta}{r}\end{aligned}$$

(2) Another proof is given without using the Bianchi identity. By differentiating equation

$$(E_1) \quad : \quad \mathbb{G}_{11} = \beta$$

we obtain:

$$\begin{aligned}\beta' &= \frac{\mathrm{d}}{\mathrm{d}r}\mathbb{G}_{11}\\ &= \frac{\mathrm{d}}{\mathrm{d}r}\left(e^{-2v}\left(2r^{-1}u' + r^{-2}\right) - r^{-2}\right)\\ &= e^{-2v}\left(-4r^{-1}v'u' - 2r^{-2}v' - 2r^{-2}u' + 2r^{-1}u'' - 2r^{-3}\right) + 2r^{-3}\\ &= 2r^{-1}e^{-2v}\left(-2v'u' - r^{-1}v' - r^{-1}u' + u'' - r^{-2}\right) + 2r^{-3}\\ &= 2r^{-1}\left[-e^{-2v}\left(-u'' + u'v' - (u')^2 + r^{-1}(u'+v') + r^{-2}\right)\right.\\ &\left.\left[-u'e^{-2v}\left(v'+u'\right)\right] + 2r^{-3}\right.\\ &= 2r^{-1}\left[e^{-2v}\left(u'' - u'v' + (u')^2 + r^{-1}(u'-v') - r^{-2}e^{2v} + r^{-2}e^{2v}\right) - u'e^{-2v}\left(v'+u'\right)\right] + 2r^{-3}\\ &= 2r^{-1}\left[e^{-2v}\left(e^{2v}\mathbb{G}_{11} - e^{2v}\mathbb{G}_{22} - r^{-2}e^{2v}\right) - u'e^{-2v}\left(v'+u'\right)\right] + 2r^{-3}\\ &= 2r^{-1}\left[\beta - \gamma - r^{-2} - u'e^{-2v}\left(v'+u'\right)\right] + 2r^{-3}\\ &= -2r^{-1}u'\left(v'+u'\right)e^{-2v} + 2\frac{\gamma-\beta}{r}\\ &= -\left(\mathbb{G}_{00} + \mathbb{G}_{11}\right)u'e^{-2v} + 2\frac{\gamma-\beta}{r}\\ &= -(\alpha+\beta)\,u' + 2\frac{\gamma-\beta}{r}\\ &= -\frac{r^3\beta + A}{2r(r-A)}(\alpha+\beta) + 2\frac{\gamma-\beta}{r}\end{aligned}$$

□

So the metric is of the form:

$$h = e^{2u}dt\otimes dt - \left(1 - \frac{A}{r}\right)^{-1} dr\otimes dr - r^2 h_\Omega$$

where:

$$u' = \frac{r^3\beta + A}{2r(r-A)}.$$

Point (iii) is a generalization of the Tolman-Oppenheimer-Volkoff equation (known as TOV).

Example 2.2.2.2: Suite 1 - Two Simple Examples

Let's consider the three examples from 2.2.0.1.

(1) **Case of perfect fluid.** We have:

$$\alpha := \kappa\rho c^2, \quad \beta := \kappa P.$$

Let's define:

$$m(r) := 4\pi\int_0^r \rho(\mathfrak{r})\mathfrak{r}^2\mathrm{d}\mathfrak{r}.$$

Then we have:

$$\begin{aligned} A(r) &= \int_0^r \frac{8\pi G}{c^2}\rho(\mathfrak{r})\mathfrak{r}^2 \mathrm{d}\mathfrak{r} \\ &= \frac{2G}{c^2} m(r) \end{aligned}$$

Therefore, by theorem 2.2.2.1, we have:

(i) We have:

$$e^{2v} = \left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}.$$

(ii) We have:

$$\begin{aligned} u' &= \frac{r^3\kappa P + 2Gm(r)/c^2}{2r(r - 2Gm(r)/c^2)} \\ &= \frac{8\pi G r^3 P + 2Gc^2 m(r)}{2c^2 r(c^2 r - 2Gm(r))} \end{aligned}$$

(iii) We have:

$$P' = -\frac{8\pi G r^3 P + 2Gc^2 m(r)}{2c^2 r(c^2 r - 2Gm(r))}(\rho c^2 + P).$$

(iv) The metric is of the form:

$$h = e^{2u} dt \otimes dt - \left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1} dr \otimes dr - r^2 h_\Omega$$

(2) **Case with cosmological constant.** We have:

$$\alpha := -\Lambda, \quad \beta := \Lambda.$$

Therefore, we have:

$$A(r) = -\int_0^r \Lambda \mathfrak{r}^2 \mathrm{d}\mathfrak{r} = -\frac{\Lambda r^3}{3}.$$

By theorem 2.2.2.1, we have:

(i) We have:

$$e^{2v} = \left(1 + \frac{\Lambda r^2}{3}\right)^{-1}.$$

(ii) We have:

$$\begin{aligned} u' &= \frac{r^3\Lambda - \Lambda r^3/3}{2r(r - \Lambda r^3/3)} \\ &= \frac{\Lambda r}{3 - \Lambda r^2} \end{aligned}$$

Thus, there exists a constant k such that:

$$u = -\frac{1}{2}\ln|3 - \Lambda r^2| + k$$

Therefore, we have:

$$e^{2u} = \frac{e^{2k}}{3}\left|1 - \frac{\Lambda r^2}{3}\right|^{-1}.$$

(iii) The metric is of the form:

$$h = \frac{e^{2k}}{3}\left|1 - \frac{\Lambda r^2}{3}\right|^{-1} dt \otimes dt - \left(1 + \frac{\Lambda r^2}{3}\right)^{-1} dr \otimes dr - r^2 h_\Omega$$

(3) **Case of perfect fluid with cosmological constant.** We have:

$$\alpha := \kappa\rho - \Lambda, \quad \beta := \kappa P + \Lambda.$$

Therefore, we have:

$$A(r) = \int_0^r \left(\kappa\rho c^2 - \Lambda\right)\mathfrak{r}^2 \mathrm{d}\mathfrak{r} = \frac{2Gm(r)}{c^2} - \frac{\Lambda r^3}{3}.$$

By theorem 2.2.2.1, we have:

(i) We have:

$$e^{2v} = \left(1 - \frac{2Gm(r)}{c^2 r} + \frac{\Lambda r^2}{3}\right)^{-1}.$$

(ii) We have:

$$\begin{aligned} u' &= \frac{r^3(\kappa P + \Lambda) + 2Gm(r)/c^2 - \Lambda r^3/3}{2r(r - 2Gm(r)/c^2 + \Lambda r^3/3)} \\ &= \frac{24\pi G r^3 P + 6Gc^2 m(r) - \Lambda c^4 r^3}{2c^2 r(3c^2 r - 6Gm(r) + \Lambda c^2 r^3)} \end{aligned}$$

(iii) We have:

$$P' = -\frac{24\pi G r^3 P + 6Gc^2 m(r) - \Lambda c^4 r^3}{2c^2 r(3c^2 r - 6Gm(r) + \Lambda c^2 r^3)}(\rho c^2 + P).$$

(iv) The metric is of the form:

$$h = e^{2u} dt \otimes dt - \left(1 - \frac{2Gm(r)}{c^2 r} + \frac{\Lambda r^2}{3}\right)^{-1} dr \otimes dr - r^2 h_\Omega$$

2.2.3 Exterior Metric

We make two assumptions:

- There exists a real number $R > 0$ such that:

$$\forall r > R,\ \alpha(r) = -\beta(r).$$

- The metric is asymptotically flat, meaning:

$$\lim_{r\to\infty} \mathrm{g}_{ii} = \eta_i.$$

Theorem 2.2.3.1: Form of the exterior metric

For $r > R$, we have:

$$h = c^2\left(1 - \frac{A}{r}\right)\mathrm{d}t \otimes \mathrm{d}t - \left(1 - \frac{A}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r - r^2 h_\Omega.$$

Proof. We have:

$$\begin{aligned}
0 &= \alpha(r) - \alpha(r) \\
&= \alpha(r) + \beta(r) \\
&= \mathbb{G}_{00} + \mathbb{G}_{11} \\
&= r^{-1}e^{-2v}\left(2v' - r^{-1}\right) + r^{-2} + r^{-1}e^{-2v}\left(2u' + r^{-1}\right) - r^{-2} \\
&= r^{-1}e^{-2v}\left(u' + v'\right)
\end{aligned}$$

Thus, we have $u' + v' = 0$. Therefore, there exists a constant C such that:

$$u + v = C$$

Since the spacetime is asymptotically flat, we have:

$$\lim_{r\to\infty} e^{2u} = \mathrm{g}_{00} = c^2\eta_{00} = c^2 \quad , \quad \lim_{r\to\infty} -e^{2v} = \mathrm{g}_{11} = \eta_{11} = -1$$

This implies:

$$\lim_{r\to\infty} u = \ln c \quad , \quad \lim_{r\to\infty} v = 0.$$

Hence, we have:

$$C = \lim_{r\to\infty}(u + v) = \ln c.$$

Thus, we have $u = \ln c - v$ and:

$$\begin{aligned}
e^{2u} &= e^{2\ln c - 2v} \\
&= c^2 e^{-2v} \\
&= c^2\left(1 - \frac{A}{r}\right)
\end{aligned}$$

□

Example 2.2.3.2: Sequence 2 - Two Simple Examples

Let's consider examples (1) and (3) from 2.2.2.2 of the **perfect fluid** with or without a cosmological constant (also see 2.1.2.1 and 2.2.0.1). For both examples, we assume that there exists a constant $R > 0$ such that:

$$\rho(r) := \begin{cases} \rho(r) & \text{if } r \le R \\ 0 & \text{if } r > R. \end{cases} \quad , \quad P(r) := \begin{cases} P(r) & \text{if } r \le R \\ 0 & \text{if } r > R. \end{cases}$$

(a) The **mass of the central object** is defined by:

$$M := m(R) = 4\pi \int_0^R \rho(\mathfrak{r})\mathfrak{r}^2 \mathrm{d}\mathfrak{r}.$$

Thus, for all $r \ge R$, we have:

$$m(r) = m(R) = M.$$

(b) The **Schwarzschild radius** is defined by:

$$R_s := A(R) = \frac{2G}{c^2}m(R) = \frac{2GM}{c^2}.$$

Therefore, we have:

$$A(r) = \frac{2G}{c^2}m(r) = \frac{R_s}{M}m(r).$$

Let's analyze both examples.

(1) We consider the case where:
$$\alpha := \kappa\rho c^2 \quad , \quad \beta := \kappa P.$$

Using example (1) from 2.2.3.2, the metric has the form:

$$h = \left(1 - \frac{R_s}{r}\right) \mathrm{d}t \otimes \mathrm{d}t - \left(1 - \frac{R_s}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r - r^2 h_\Omega.$$

(3) We consider the case where:
$$\alpha := \kappa\rho c^2 - \Lambda \quad , \quad \beta := \kappa P + \Lambda.$$

Using example (3) from 2.2.3.2, the metric has the form:

$$h = \left(1 - \frac{R_s}{r} + \frac{\Lambda r^2}{3}\right) \mathrm{d}t \otimes \mathrm{d}t - \left(1 - \frac{R_s}{r} + \frac{\Lambda r^2}{3}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r - r^2 h_\Omega.$$

Corollary 2.2.3.3: Static Metric

For $r > R$, the metric h is static.

Proof. Since v is independent of time and:
$$u = \ln(c) - v$$
Thus, u is independent of t. Therefore, the metric h is static for $r > R$. □

By (E_1) and (E_2), we also see that β and γ are independent of t for $r > R$.

2.2.4 Interior Metric

We assume that:

- α is constant on $[0, R]$, *i.e.*, there exists a constant α_0 such that $\alpha = \alpha_0$ on $[0, R]$. We assume that:
$$\alpha = \alpha_0 < \frac{3}{R^2}$$
The condition on α_0 is due to the square root $\sqrt{3 - \alpha_0 r^2}$ appearing in the results of this subsection.
- $\gamma := \beta$, β is independent of t, and β is continuous at $r = R$. We define:
$$\beta_R := \beta(R_-) = \beta(R_+).$$
- for all $r > R$, we have:
$$\alpha(r) = -\beta(r).$$
This condition will allow us to match the interior metric with the exterior metric.

Throughout this subsection, we have:

$$r \leq R \quad : \quad \mathbb{S}_{jl} := \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \beta(r) & 0 & 0 \\ 0 & 0 & \beta(r) & 0 \\ 0 & 0 & 0 & \beta(r) \end{pmatrix}$$

$$r > R \quad : \quad \mathbb{S}_{jl} := \begin{pmatrix} \alpha(r) & 0 & 0 & 0 \\ 0 & -\alpha(r) & 0 & 0 \\ 0 & 0 & -\alpha(r) & 0 \\ 0 & 0 & 0 & -\alpha(r) \end{pmatrix}$$

Therefore, on $[0, R]$ we have:

$$\begin{aligned} A(r) &= \int_0^r \alpha_0 \mathfrak{r}^2 \mathrm{d}\mathfrak{r} \\ &= \frac{\alpha_0 r^3}{3} \end{aligned}$$

Theorem 2.2.4.1: Interior Metric

For $r \leq R$, we have:

$$h = \frac{c^2}{4\sqrt{3}\alpha_0^2}\left(3\left(\alpha_0+\beta_R\right)\sqrt{3-\alpha_0 R^2} - \left(\alpha_0+3\beta_R\right)\sqrt{3-\alpha_0 r^2}\right)^2 \mathrm{d}t \otimes \mathrm{d}t - \frac{3}{3-\alpha_0 r^2}\mathrm{d}r \otimes \mathrm{d}r - r^2 h_\Omega.$$

Proof. By (B_1) and since $\gamma = \beta$, we have:

$$u' = -\frac{\beta'}{\alpha_0 + \beta}.$$

Thus, there exists a constant C_1 such that:

$$u = -\ln\left|\alpha_0 + \beta\right| + C_1.$$

We have:

$$\begin{aligned} \alpha_0 + \beta &= \mathbb{G}_{00} + \mathbb{G}_{11} \\ &= 2r^{-1}e^{-2v}\left(v' + u'\right) \\ &= 2r^{-1}\left(1 - \frac{A}{r}\right)\left(v' + u'\right) \\ &= 2r^{-1}\left(1 - \frac{\alpha_0 r^2}{3}\right)\left(v' + u'\right) \end{aligned}$$

and:

$$\begin{aligned} re^{C_1} &= r\exp\left(u + \ln\left|\alpha_0 + \beta\right|\right) \\ &= re^u\left|\alpha_0 + \beta\right| \\ &= re^u\left(2r^{-1}e^{-2v}\left|v' + u'\right|\right) \\ &= 2e^u e^{-2v}\left|v' + u'\right| > 0 \end{aligned}$$

Thus, by continuity of $v' + u'$, $\varepsilon \in \{\pm 1\}$ such that $|v' + u'| = \varepsilon(v' + u')$. With $U := e^u$, we have:

$$\begin{aligned} re^{C_1} &= 2e^u e^{-2v}\left|v' + u'\right| \\ &= 2\varepsilon e^u e^{-2v}\left(v' + u'\right) \\ &= 2\varepsilon u' e^u e^{-2v} - \varepsilon e^u \frac{\mathrm{d}}{\mathrm{d}r}\left(e^{-2v}\right) \\ &= 2\varepsilon u' e^u\left(1 - \frac{\alpha_0 r^2}{3}\right) + \frac{2\varepsilon\alpha_0 r}{3}e^u \\ &= 2\varepsilon\left(1 - \frac{\alpha_0 r^2}{3}\right)U' + \frac{2\varepsilon\alpha_0 r}{3}U. \end{aligned}$$

Thus, U is a solution of the differential equation:

$$(E)\ :\ 2\varepsilon\left(1 - \frac{\alpha_0 r^2}{3}\right)y' + \frac{2\varepsilon\alpha_0 r}{3}y = re^{C_1}$$

We have:

$$\alpha_0 < \frac{3}{R^2} \leq \frac{3}{r^2}$$

which means $1 - \alpha_0/r^2 \geq 0$. Thus, the homogeneous solutions are generated by the function:

$$r \mapsto \exp\left(-\int_0^r \frac{2\varepsilon\alpha_0\mathfrak{r}/3}{2\varepsilon\left(1-\alpha_0\mathfrak{r}^2/3\right)}\mathrm{d}\mathfrak{r}\right) = \exp\left(\frac{1}{2}\ln\left(1-\alpha_0 r^2/3\right)\right) = \sqrt{1-\frac{\alpha_0 r^2}{3}}.$$

We have the constant particular solution:

$$r \mapsto \frac{3\varepsilon e^{C_1}}{2\alpha_0}.$$

Thus, there exists a constant C_2 such that:

$$U = \frac{3\varepsilon e^{C_1}}{2\alpha_0} + C_2\sqrt{1-\frac{\alpha_0 r^2}{3}}.$$

Let's calculate the two constants C_1 and C_2.

- **Using** $\beta(R_-) = \beta_R = \beta(R_+)$. We have:

$$re^{C_1} = re^u\left|\alpha_0 + \beta\right| = \varepsilon r\left(\alpha_0 + \beta\right)U$$

which means:

$$e^{-C_1}\left(\alpha_0 + \beta\right)U = \varepsilon$$

and thus at R_-, we have:

$$\begin{aligned}
\varepsilon &= e^{-C_1}\left(\alpha_0 + \beta(R_-)\right)U(R) \\
&= e^{-C_1}\left(\alpha_0 + \beta_R\right)U(R) \\
&= e^{-C_1}\left(\alpha_0 + \beta_R\right)\left(\frac{3\varepsilon e^{C_1}}{2\alpha_0} + C_2\sqrt{1-\frac{\alpha_0 R^2}{3}}\right) \\
&= \frac{3\varepsilon\left(\alpha_0 + \beta_R\right)}{2\alpha_0} + C_2\left(\alpha_0 + \beta_R\right)e^{-C_1}\sqrt{1-\frac{\alpha_0 R^2}{3}}
\end{aligned}$$

which means:

$$\frac{\varepsilon e^{C_1}}{2\alpha_0} = -C_2\frac{\left(\alpha_0 + \beta_R\right)}{\left(\alpha_0 + 3\beta_R\right)}\sqrt{1-\frac{\alpha_0 R^2}{3}}.$$

- **Using the continuity of the metric at** R. Since:

$$g_{00}(R_-) = g_{00}(R_+)$$

we have:

$$\begin{aligned}
c^2\left(1-\frac{\alpha_0 R^2}{3}\right) &= c^2\left(1-\frac{A(R)}{R}\right) \\
&= g_{00}(R_+) \\
&= g_{00}(R_-) \\
&= e^{2u(R_-)} \\
&= U(R_-)^2 \\
&= \left(\frac{3\varepsilon e^{C_1}}{2\alpha_0} + C_2\sqrt{1-\frac{\alpha_0 R^2}{3}}\right)^2 \\
&= \left(-3C_2\frac{\left(\alpha_0 + \beta_R\right)}{\left(\alpha_0 + 3\beta_R\right)}\sqrt{1-\frac{\alpha_0 R^2}{3}} + C_2\sqrt{1-\frac{\alpha_0 R^2}{3}}\right)^2 \\
&= 4C_2^2\frac{\alpha_0^2}{\left(\alpha_0 + 3\beta_R\right)^2}\left(1-\frac{\alpha_0 R^2}{3}\right)
\end{aligned}$$

which means:

$$C_2 = \pm \frac{c(\alpha_0 + 3\beta_R)}{2\alpha_0}$$

and thus:

$$\frac{\varepsilon e^{C_1}}{2\alpha_0} = \mp \frac{c(\alpha_0 + \beta_R)}{2\alpha_0} \sqrt{1 - \frac{\alpha_0 R^2}{3}}$$

Thus, we have:

$$e^u = U = \mp \frac{3c(\alpha_0 + \beta_R)}{2\alpha_0} \sqrt{1 - \frac{\alpha_0 R^2}{3}} \pm \frac{c(\alpha_0 + 3\beta_R)}{2\alpha_0} \sqrt{1 - \frac{\alpha_0 r^2}{3}}.$$

Therefore, we have:

$$\begin{aligned} e^{2u} &= \frac{c^2}{4\alpha_0^2} \left(3(\alpha_0 + \beta_R) \sqrt{1 - \frac{\alpha_0 R^2}{3}} - (\alpha_0 + 3\beta_R) \sqrt{1 - \frac{\alpha_0 r^2}{3}} \right)^2 \\ &= \frac{c^2}{4\sqrt{3}\alpha_0^2} \left(3(\alpha_0 + \beta_R) \sqrt{3 - \alpha_0 R^2} - (\alpha_0 + 3\beta_R) \sqrt{3 - \alpha_0 r^2} \right)^2 \end{aligned}$$

Hence the result. □

Corollary 2.2.4.2: Integral form and central value of β

(i) We have:

$$\beta = \alpha_0 \frac{(\alpha_0 + 3\beta_R)\sqrt{3 - \alpha_0 r^2} - (\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2}}{3(\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2} - (\alpha_0 + 3\beta_R)\sqrt{3 - \alpha_0 r^2}}.$$

(ii) We have:

$$\beta_c := \lim_{r \to 0_+} \beta = \alpha_0 \frac{(\alpha_0 + 3\beta_R)\sqrt{3} - (\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2}}{3(\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2} - (\alpha_0 + 3\beta_R)\sqrt{3}}.$$

Proof. We have:

$$\alpha_0 + \beta = \varepsilon e^{C_1} e^{-u}$$

$$e^u = \mp \frac{3c(\alpha_0 + \beta_R)}{2\alpha_0} \sqrt{1 - \frac{\alpha_0 R^2}{3}} \pm \frac{c(\alpha_0 + 3\beta_R)}{2\alpha_0} \sqrt{1 - \frac{\alpha_0 r^2}{3}}.$$

$$\varepsilon e^{C_1} = \mp c(\alpha_0 + \beta_R) \sqrt{1 - \frac{\alpha_0 R^2}{3}}$$

Therefore, we have:

$$\begin{aligned} \beta &= \varepsilon e^{C_1} e^{-u} - \alpha_0 \\ &= \frac{\mp 2c\alpha_0(\alpha_0 + \beta_R)\sqrt{1 - \alpha_0 R^2/3}}{\mp 3c(\alpha_0 + \beta_R)\sqrt{1 - \alpha_0 R^2/3} \pm c(\alpha_0 + 3\beta_R)\sqrt{1 - \alpha_0 r^2/3}} - \alpha_0 \\ &= \frac{2\alpha_0(\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2}}{3(\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2} - (\alpha_0 + 3\beta_R)\sqrt{3 - \alpha_0 r^2}} - \alpha_0 \\ &= \alpha_0 \frac{(\alpha_0 + 3\beta_R)\sqrt{3 - \alpha_0 r^2} - (\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2}}{3(\alpha_0 + \beta_R)\sqrt{3 - \alpha_0 R^2} - (\alpha_0 + 3\beta_R)\sqrt{3 - \alpha_0 r^2}} \end{aligned}$$

Hence the result for β_c by taking the limit $r \to 0$. □

Example 2.2.4.3: Sequence 3 - Two simple examples

We revisit examples (1) and (3) from 2.2.3.2 of the **perfect fluid** with or without a cosmological constant (also see 2.1.2.1, 2.2.0.1, and 2.2.2.2). For both examples, we assume the existence of a constant $R > 0$ such that:

$$\rho(r) := \begin{cases} \rho_0 & \text{if } r \leq R \\ 0 & \text{if } r > R. \end{cases} \quad , \quad P(r) := \begin{cases} P(r) & \text{if } r \leq R \\ 0 & \text{if } r > R. \end{cases}$$

Thus, we have:

$$P_R = P(R_-) = P(R_+) = 0$$

We will consider the two examples.

(1) We are in the case where:

$$\alpha := \kappa\rho c^2 \quad , \quad \beta := \kappa P.$$

We observe that:

$$\frac{\alpha_0 R^2}{3} = \frac{A(R)}{R} = \frac{R_s}{R}$$
$$\frac{\alpha_0 r^2}{3} = \frac{\alpha_0 R^2}{3}\frac{r^2}{R^2} = \frac{R_s r^2}{R^3}$$

(i) The metric takes the form:

$$h = \frac{c^2}{4}\left(3\sqrt{1-\frac{R_s}{R}} - \sqrt{1-\frac{r^2 R_s}{R^3}}\right)^2 \mathrm{d}t\otimes\mathrm{d}t - \left(1-\frac{r^2R_s}{R^3}\right)^{-1}\mathrm{d}r\otimes\mathrm{d}r - r^2 h_\Omega$$

(ii) We have:

$$P = \rho_0 c^2 \frac{\sqrt{1-R_s r^2/R^3} - \sqrt{1-R_s/R}}{3\sqrt{1-R_s/R} - \sqrt{1-R_s r^2/R^3}}.$$

(iii) We have:

$$P_c := \lim_{r\to 0_+} P = \rho_0 c^2 \frac{1-\sqrt{1-R_s/R}}{3\sqrt{1-R_s/R}-1}.$$

(3) We are in the case where:

$$\alpha := \kappa\rho c^2 - \Lambda \quad , \quad \beta := \kappa P + \Lambda.$$

We observe that:

$$\frac{\alpha_0 R^2}{3} = \frac{R_s}{R} - \frac{\Lambda R^2}{3}$$
$$\frac{\alpha_0 r^2}{3} = \frac{\alpha_0 R^2}{3}\frac{r^2}{R^2} = \frac{R_s r^2}{R^3} - \frac{\Lambda r^2}{3}$$

Thus, we have:

$$\beta_R = \beta(R_-) = \beta(R_+) = \Lambda.$$

(i) The metric takes the form:

$$h = \frac{c^2}{4}\left(3\sqrt{1-\frac{R_s}{R}+\frac{\Lambda R^2}{3}} - \sqrt{1-\frac{r^2R_s}{R^3}+\frac{\Lambda r^2}{3}}\right)^2 \mathrm{d}t\otimes\mathrm{d}t - \left(1-\frac{r^2R_s}{R^3}+\frac{\Lambda r^2}{3}\right)^{-1}\mathrm{d}r\otimes\mathrm{d}r - r^2h_\Omega$$

(ii) We have:

$$P = \rho_0 c^2 \frac{\sqrt{1-R_s r^2/R^3+\Lambda r^2/3} - \sqrt{1-R_s/R+\Lambda R^2/3}}{3\sqrt{1-R_s/R+\Lambda R^2/3} - \sqrt{1-R_s r^2/R^3+\Lambda r^2/3}}.$$

(iii) We have:

$$P_c := \lim_{r\to 0_+} P = \rho_0 c^2 \frac{1-\sqrt{1-R_s/R+\Lambda R^2/3}}{3\sqrt{1-R_s/R+\Lambda R^2/3}-1}.$$

2.2.5 Case where $\beta(r) = 0$ for $r > R$

We consider the conditions from the previous subsection. Additionally, we assume the existence of two tensors:

$$r \leq R \quad : \quad \mathbb{S}_{jl}^{(1)} := \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \beta(r) & 0 & 0 \\ 0 & 0 & \beta(r) & 0 \\ 0 & 0 & 0 & \beta(r) \end{pmatrix} \quad , \quad \mathbb{S}_{jl}^{(2)} := \begin{pmatrix} \lambda\alpha_0 & 0 & 0 & 0 \\ 0 & \mu\beta(r) + \delta(r) & 0 & 0 \\ 0 & 0 & \mu\beta(r) + \delta(r) & 0 \\ 0 & 0 & 0 & \mu\beta(r) + \delta(r) \end{pmatrix}$$

$$r > R \quad : \quad \mathbb{S}_{jl}^{(1)} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbb{S}_{jl}^{(2)} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where α_0, λ, and μ are real constants, and:

$$\delta(r) := \begin{cases} \delta(r) & \text{if } r \leq R \\ 0 & \text{if } r > R. \end{cases}$$

We assume that $\mathbb{S}_{jl}^{(1)}$ and $\mathbb{S}_{jl}^{(2)}$ are solutions of the five fundamental equations discussed earlier.

Next, we will study the Newtonian limits of the obtained formulas. Let us recall the classical definition of the Newtonian limit.

Example 2.2.5.1: Newtonian limit

In the usual Newtonian limit, we require:

(1) The velocity v of the fluid is much smaller than the speed of light, *i.e.*, $v << c$. This implies:

$$P \equiv \frac{1}{2}\rho v^2 << \rho c^2.$$

(2) The Schwarzschild radius is much smaller than the position r and the radius R, *i.e.*,

$$R_s = \frac{2GM}{c^2} << r, R.$$

In the case $G :=: c = 1$, we have:

$$P << \rho$$
$$m(r), M, R_s << r, R$$

Definition 2.2.5.2: Definition of the Newtonian limit

Let's consider a tensor:

$$\mathbb{S}_{jl} := \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \beta(r) & 0 & 0 \\ 0 & 0 & \beta(r) & 0 \\ 0 & 0 & 0 & \beta(r) \end{pmatrix}$$

We say that a tensor $\mathbb{S}_{jl}$ satisfies the **Newtonian condition** if it fulfills the following conditions:

(i) We have:

$$\alpha_0 r^2, \alpha_0 R^2 << 1.$$

(ii) We have:

$$\beta << \alpha_0.$$

Proposition 2.2.5.3: Newtonian approximation of β

Let's consider a tensor:

$$\mathbb{S}_{jl} := \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \beta(r) & 0 & 0 \\ 0 & 0 & \beta(r) & 0 \\ 0 & 0 & 0 & \beta(r) \end{pmatrix}$$

Suppose that $\mathbb{S}_{jl}$ satisfies the five fundamental equations and the Newtonian condition.

(i) Then we have:

$$\beta \approx \frac{\alpha_0^2}{12}\left(R^2 - r^2\right).$$

(ii) Then we have:

$$\beta_c \approx \frac{\alpha_0^2}{12} R^2.$$

Proof. (i) Since $\beta(r) = 0$ for $r > R$ (and thus $\beta_R = 0$), we have:

$$\begin{aligned} \beta &= \alpha_0 \frac{\sqrt{3 - \alpha_0 r^2} - \sqrt{3 - \alpha_0 R^2}}{3\sqrt{3 - \alpha_0 R^2} - \sqrt{3 - \alpha_0 r^2}} \\ &= \alpha_0 \frac{\sqrt{1 - \alpha_0 r^2/3} - \sqrt{1 - \alpha_0 R^2/3}}{3\sqrt{1 - \alpha_0 R^2/3} - \sqrt{1 - \alpha_0 r^2/3}} \\ &= \alpha_0 \frac{\left(1 - \alpha_0 r^2/6\right) - \left(1 - \alpha_0 R^2/6\right)}{2} \\ &\approx \frac{\alpha_0^2}{12}\left(R^2 - r^2\right) \end{aligned}$$

(ii) We have:

$$\beta_c = \lim_{r \to 0^+} \beta(r) \approx \frac{\alpha_0^2}{12} R^2.$$

□

Example 2.2.5.4: Example 4 - Two simple examples

We revisit the example of 2.2.4.3 of a **perfect fluid** without a cosmological constant (also see 2.1.2.1, 2.2.0.1, 2.2.2.2, and 2.2.3.2). Suppose that:

$$\alpha_0 := \frac{8\pi G}{c^2}\rho_0 \quad , \quad \beta := \frac{8\pi G}{c^4} P.$$

In the Newtonian approximation, we recover the standard approximations:

$$\begin{aligned} P &\approx \frac{2\pi G}{3}\rho_0^2\left(R^2 - r^2\right) \\ P_c &\approx \frac{2\pi G}{3}\rho_0^2 R^2 \end{aligned}$$

Proposition 2.2.5.5: Conditions for having two equal TOVs

Suppose that $\mathbb{S}_{jl}^{(1)}$ and $\mathbb{S}_{jl}^{(2)}$ are solutions to the five fundamental equations.

(i) We have:

$$\delta = \alpha_0 \frac{(3-\mu)F(R^2)F(\lambda r^2) + (1-3\mu)F(r^2)F(\lambda R^2) + 3(\mu-1)\left[F(R^2)F(\lambda R^2) + F(r^2)F(\lambda r^2)\right]}{\left(3F(\lambda R^2) - F(\lambda r^2)\right)\left(3F(R^2) - F(r^2)\right)}$$

where:

$$F(x) := \sqrt{(3-\alpha_0 x)}.$$

(ii) In the Newtonian approximation, we have:

$$\delta \approx \frac{\alpha_0^2}{24}\left[(3-\mu)\left(R^2 + \lambda r^2\right) + (1-3\mu)\left(r^2 + \lambda R^2\right) + 3(\mu-1)(1+\lambda)\left(r^2 + R^2\right)\right].$$

Proof. (i) By Corollary 2.2.4.2, we have two integral forms for β and $\mu\beta + \delta$:

$$(F_1) \ : \ \beta = \alpha_0 \frac{\sqrt{3-\alpha_0 r^2} - \sqrt{3-\alpha_0 R^2}}{3\sqrt{3-\alpha_0 R^2} - \sqrt{3-\alpha_0 r^2}}$$

$$(F_2) \ : \ \mu\beta + \delta = \alpha_0 \frac{\sqrt{3-\lambda\alpha_0 r^2} - \sqrt{3-\lambda\alpha_0 R^2}}{3\sqrt{3-\lambda\alpha_0 R^2} - \sqrt{3-\lambda\alpha_0 r^2}}$$

By subtracting $\mu(F_1)$ from (F_2), we obtain:

$$\begin{aligned}\delta &= \alpha_0 \frac{\sqrt{3-\lambda\alpha_0 r^2} - \sqrt{3-\lambda\alpha_0 R^2}}{3\sqrt{3-\lambda\alpha_0 R^2} - \sqrt{3-\lambda\alpha_0 r^2}} - \mu\alpha_0 \frac{\sqrt{3-\alpha_0 r^2} - \sqrt{3-\alpha_0 R^2}}{3\sqrt{3-\alpha_0 R^2} - \sqrt{3-\alpha_0 r^2}} \\ &= \alpha_0 \frac{(3-\mu)F(R^2)F(\lambda r^2) + (1-3\mu)F(r^2)F(\lambda R^2) + 3(\mu-1)\left[F(R^2)F(\lambda R^2) + F(r^2)F(\lambda r^2)\right]}{\left(3F(\lambda R^2) - F(\lambda r^2)\right)\left(3F(R^2) - F(r^2)\right)}\end{aligned}$$

(ii) We have:

$$\alpha_0 r^2, \alpha_0 R^2 << 1.$$

Therefore, for $x := r^2, R^2, \lambda r^2, \lambda R^2$, we have:

$$F(x) \approx \sqrt{3}\left(1 - \frac{\alpha_0 x}{6}\right).$$

Thus, we have:

$$\delta \approx \frac{\alpha_0^2}{24}\left[(3-\mu)\left(R^2 + \lambda r^2\right) + (1-3\mu)\left(r^2 + \lambda R^2\right) + 3(\mu-1)(1+\lambda)\left(r^2 + R^2\right)\right].$$

□

Corollary 2.2.5.6: Case $\lambda := -1$ and $\mu := 1$

We consider the case where $\lambda := -1$ and $\mu := 1$.

(i) We have:

$$\delta = 2\alpha_0 \frac{F(R^2)F(-r^2) - F(r^2)F(-R^2)}{\left(3F(-R^2) - F(-r^2)\right)\left(3F(R^2) - F(r^2)\right)}.$$

(ii) In the Newtonian approximation, we have:

(a) We have:

$$\delta \approx \frac{\alpha_0^2}{6}\left(R^2 - r^2\right) \approx 2\beta.$$

(b) We have:

$$\mathbb{S}^{(1)}_{jl} \approx \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbb{S}^{(2)}_{jl} \approx \begin{pmatrix} -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 2.2.5.7: Example 6 - Perfect Fluid

We revisit the example from 2.2.5.4 of a **perfect fluid** without a cosmological constant (also see 2.1.2.1, 2.2.0.1, 2.2.2.2, 2.2.3.2, and 2.2.4.3). We assume that:

$$\alpha_0 := \frac{8\pi G}{c^2}\rho_0 \quad , \quad \beta := \frac{8\pi G}{c^4}P.$$

We consider the case where $\lambda := -1$ and $\mu := 1$.

(i) We have:

$$\delta = 2\frac{8\pi G}{c^2}\rho_0 \frac{F(R^2)F(-r^2) - F(r^2)F(-R^2)}{(3F(-R^2) - F(-r^2))\,(3F(R^2) - F(r^2))}$$

with:

$$F(x) := \sqrt{3 - 8\pi G\rho x/c^2}.$$

(ii) In the Newtonian approximation, we have:

(a) We have:

$$\delta \approx \frac{2\pi G}{3}\rho_0^2\left(R^2 - r^2\right) \approx 2P.$$

(b) We have:

$$\mathbb{S}^{(1)}_{jl} \approx \begin{pmatrix} \frac{8\pi G}{c^2}\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathbb{S}^{(2)}_{jl} \approx \begin{pmatrix} -\frac{8\pi G}{c^2}\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.2.6 Example of the Reissner-Nordström Metric

In this subsection, we study the Reissner-Nordström metric. It corresponds to the gravitational field of a non-rotating, spherically symmetric, charged body with mass M and charge q.

We start with a proposition.

Proposition 2.2.6.1: Simple expression of α and β

Suppose that for all $r > R$:

$$\alpha(r) = -\beta(r) = \nu_0 + \frac{\nu_1}{r^4}.$$

Let:

$$A_s := A(R) - \frac{\nu_0 R^3}{3} + \frac{\nu_1}{R}.$$

Then, for $r > R$:

$$h = c^2\left(1 - \frac{A_s}{r} - \frac{\nu_0 r^2}{3} + \frac{\nu_1}{r^2}\right)\mathrm{d}t \otimes \mathrm{d}t - \left(1 - \frac{A_s}{r} - \frac{\nu_0 r^2}{3} + \frac{\nu_1}{r^2}\right)^{-1}\mathrm{d}r \otimes \mathrm{d}r - r^2 h_\Omega.$$

Proof. For $r > R$, we have:

$$\begin{aligned}
A(r) &= \int_0^r \alpha(\mathfrak{r})\mathfrak{r}^2 \mathrm{d}\mathfrak{r} \\
&= \int_0^R \alpha(\mathfrak{r})\mathfrak{r}^2 \mathrm{d}\mathfrak{r} + \int_R^r \alpha(\mathfrak{r})\mathfrak{r}^2 \mathrm{d}\mathfrak{r} \\
&= A(R) + \int_R^r \nu_0 \mathfrak{r}^2 \mathrm{d}\mathfrak{r} + \int_R^r \frac{\nu_1}{\mathfrak{r}^2} \mathrm{d}\mathfrak{r} \\
&= A(R) - \frac{\nu_0 R^3}{3} + \frac{\nu_1}{R} + \frac{\nu_0 r^3}{3} - \frac{\nu_1}{r} \\
&= A_s + \frac{\nu_0 r^3}{3} - \frac{\nu_1}{r}
\end{aligned}$$

Thus, we have:

$$1 - \frac{A}{r} = 1 - \frac{A_s}{r} - \frac{\nu_0 r^2}{3} + \frac{\nu_1}{r^2}.$$

□

The energy-momentum tensor is derived from the electromagnetic tensor. We assume that there is no magnetic field, and the electric field has only a radial component. We provide the definition of the energy-momentum tensor.

Definition 2.2.6.2: Definition of the energy-momentum tensor

In spherical polar coordinates, the energy-momentum tensor is defined as:

$$\mathbb{T}_{jl} := \frac{1}{\mu_0}\left(\frac{1}{4}\eta_{jl}F_{ik}F^{ik} - \eta_{lk}F_{ji}F^{ki}\right)$$

where:

$$E_r := \frac{Q}{4\pi\varepsilon_0 r^2} \quad , \quad F_{jl} := \begin{pmatrix} 0 & E_r/c & 0 & 0 \\ -E_r/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad F^{jl} := \begin{pmatrix} 0 & -E_r/c & 0 & 0 \\ E_r/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The energy-momentum tensor has a simple form.

Lemma 2.2.6.3: Value of the components of the energy-momentum tensor

We have:

$$\mathbb{T}_{jl} = \frac{\Phi}{r^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where:

$$\Phi := -\frac{GQ^2}{4\pi\varepsilon_0 c^4 r^4}.$$

Proof. Since:

$$F_{01}F^{01} = F_{10}F^{10}$$

we have:

$$\begin{aligned}
\mathbb{T}_{jl} &= \frac{1}{\mu_0}\left(\frac{1}{4}\eta_{jl}F_{ik}F^{ik} - \eta_{lk}F_{ji}F^{ki}\right) \\
&= \frac{1}{\mu_0}\left(\frac{1}{4}\eta_{jl}\left(F_{01}F^{01} + F_{10}F^{10}\right) - \eta_{lk}F_{j0}F^{k0} - \eta_{lk}F_{j1}F^{k1}\right) \\
&= \frac{1}{\mu_0}\left(\frac{1}{2}\eta_{jl}F_{01}F^{01} - \eta_{l1}\delta_{j,1}F_{01}F^{01} - \eta_{l0}\delta_{j,0}F_{01}F^{01}\right) \\
&= \frac{F_{01}F^{01}}{\mu_0}\left(\frac{1}{2}\eta_{jl} - \eta_{l1}\delta_{j,1} - \eta_{l0}\delta_{j,0}\right)
\end{aligned}$$

We have two cases.

(1) **Case** $j \neq l$. We have:

$$\begin{aligned}
\mathbb{T}_{jl} &= \frac{F_{01}F^{01}}{\mu_0}\left(\frac{1}{2}\eta_{jl} - \eta_{l1}\delta_{j,1} - \eta_{l0}\delta_{j,0}\right) \\
&= 0
\end{aligned}$$

(2) **Case** $j = l$. We have:

$$\begin{aligned}
\mathbb{T}_{00} &= \frac{F_{01}F^{01}}{\mu_0}\left(\frac{1}{2}\eta_{00} - \eta_{01}\delta_{0,1} - \eta_{00}\delta_{j,0}\right) \\
&= -\frac{F_{01}F^{01}}{2\mu_0} \\
\mathbb{T}_{11} &= \frac{F_{01}F^{01}}{\mu_0}\left(\frac{1}{2}\eta_{11} - \eta_{11}\delta_{1,1} - \eta_{10}\delta_{1,0}\right) \\
&= \frac{F_{01}F^{01}}{2\mu_0} \\
\mathbb{T}_{22} &= \frac{F_{01}F^{01}}{\mu_0}\left(\frac{1}{2}\eta_{22} - \eta_{21}\delta_{2,1} - \eta_{20}\delta_{2,0}\right) \\
&= -\frac{F_{01}F^{01}}{2\mu_0} \\
\mathbb{T}_{33} &= \frac{F_{01}F^{01}}{\mu_0}\left(\frac{1}{2}\eta_{33} - \eta_{31}\delta_{3,1} - \eta_{30}\delta_{3,0}\right) \\
&= -\frac{F_{01}F^{01}}{2\mu_0}
\end{aligned}$$

And since $c^2\varepsilon_0\mu_0 = 1$, we have:

$$\begin{aligned}
\frac{F_{01}F^{01}}{2\mu_0} &= -\frac{E_r}{2\mu_0 c^2} \\
&= -\frac{GQ^2}{4\pi\varepsilon_0 c^4 r^4} \\
&= \frac{\Phi}{r^4}
\end{aligned}$$

□

We use the notations from the examples in this chapter (see, for example, 2.1.2.1) with:

$$\alpha(r) := \begin{cases} \kappa c^2 \rho(r) - \Lambda + \Phi(r) & \text{if } r \leq R \\ -\Lambda + \frac{\Phi_0}{r^4} & \text{if } r > R. \end{cases}$$
$$\beta(r) := \begin{cases} \kappa P(r) + \Lambda - \Phi(r) & \text{if } r \leq R \\ \Lambda - \frac{\Phi_0}{r^4} & \text{if } r > R. \end{cases}$$
$$\gamma(r) := \delta(r) := \begin{cases} \kappa P(r) + \Lambda + \Phi(r) & \text{if } r \leq R \\ \Lambda + \frac{\Phi_0}{r^4} & \text{if } r > R. \end{cases}$$

Let's define:

$$R_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4}.$$

For $r > R$, we have:

$$h = \left(1 - \frac{R_s}{r} - \frac{\Lambda r^2}{3} + \frac{R_Q^2}{r^2}\right) \mathrm{d}t \otimes \mathrm{d}t - \left(1 - \frac{R_s}{r} - \frac{\Lambda r^2}{3} + \frac{R_Q^2}{r^2}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r - r^2 h_\Omega$$

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