reasonable restriction $\rho \geq 0$ for all $p \geq 0$.] The point at which the pressure reaches zero is the star's surface; the value of r there is the star's radius, R; and the value of m there is the star's total mass-energy, M. Having reached the surface, renormalize Φ by adding a constant to it everywhere, so that it obeys the boundary condition (23.28e). The result is a relativistic stellar model whose structure functions Φ, m, ρ, p, n satisfy the equations of structure.

Notice that for any fixed choice of the equations of state $p = p(n)$, $\rho = \rho(n)$, the stellar models form a one-parameter sequence (parameter p_c). Once the central pressure has been specified, the model is determined uniquely.

The next chapter describes a variety of realistic stellar models constructed numerically by the above prescription. For an idealized stellar model constructed analytically, see Box 23.2.

EXERCISE

Exercise 23.8. NEWTONIAN STARS OF UNIFORM DENSITY

Calculate the structures of uniform-density configurations in Newtonian theory. Show that the relativistic configurations of Box 23.2 become identical to the Newtonian configurations in the weak-gravity limit. Also show that there are no mass or radius limits in Newtonian theory.

(continued on page 612)

Box 23.2 RELATIVISTIC MODEL STAR OF UNIFORM DENSITY

For realistic equations of state (see next chapter), the equations of stellar structure (23.28) cannot be integrated analytically; numerical integration is necessary. However, analytic solutions exist for various idealized and *ad hoc* equations of state. One of the most useful analytic solutions [Karl Schwarzschild (1916b)] describes a star of uniform density,

$$\rho = \rho_0 = \text{constant for all } p. \tag{1}$$

It is not necessary to indulge in the fiction of "an incompressible fluid" to accept this model as interesting. Incompressibility would imply a speed of sound, $v = (dp/d\rho)^{1/2}$, of unlimited magnitude, therefore in excess of the speed of light, and therefore in contradiction with a central principle of special relativity ("principle of causality") that no physical effect can be propagated at a speed $v > 1$. (If a source could cause an effect so quickly in one local Lorentz frame, then there would exist another local Lorentz frame in which the effect would occur before the source had acted!) However, that the part of the fluid in the region of high pressure has the same density as the part of the fluid in the region of low pressure is an idea easy to admit, if only one thinks of the fluid having a composition that varies from one

r value to another ("hand-tailored"). Whether one thinks along this line, or simply has in mind a globe of water limited in size to a small fraction of the dimensions of the earth, one has in Schwarzschild's model an instructive example of hydrostatics done in the framework of Einstein's theory.

The mass equation (23.28a) gives immediately

$$m = \left\{ \begin{matrix} (4\pi/3)\rho_0 r^3 & \text{for } r < R \\ M = (4\pi/3)\rho_0 R^3 & \text{for } r > R \end{matrix} \right\}. \tag{2}$$

from which follows the length-correction factor in the metric

$$\frac{d(\text{proper distance})}{dr} = e^{\Lambda} = [1 - 2m(r)/r]^{-1/2}. \tag{3}$$

When for ease of visualization the space geometry (r, ϕ) of an equatorial slice through the star is viewed as embedded in a Euclidean 3-geometry (z, r, ϕ) [see §23.8], the "lift" out of the plane $z = 0$ is

$$z(r) = \begin{cases} (R^3/2M)^{1/2}[1 - (1 - 2Mr^2/R^3)^{1/2}] & \text{for } r \leq R, \\ (R^3/2M)^{1/2}[1 - (1 - 2M/R)^{1/2}] + [8M(r - 2M)]^{1/2} - [8M(R - 2M)]^{1/2} & \\ & \text{for } r \geq R. \end{cases} \tag{4}$$

The knowledge of $m(r)$ from (2) allows the equation of hydrostatic equilibrium (23.28b) to be integrated to give the pressure:

$$p = \rho_0 \left\{ \frac{(1 - 2Mr^2/R^3)^{1/2} - (1 - 2M/R)^{1/2}}{3(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}} \right\} \text{ for } r < R. \tag{5}$$

The pressure in turn leads via (23.28e) to the time-correction factor in the metric.

$$\frac{d(\text{proper time})}{dt} = e^{\Phi} = \left\{ \begin{matrix} \frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2} & \text{for } r < R \\ (1 - 2M/r)^{1/2} & \text{for } r > R \end{matrix} \right\}. \tag{6}$$

Several features of these uniform-density configurations are noteworthy. (1) For fixed energy density, ρ_0, the central pressure

$$p_c = \rho_0 \left\{ \frac{1 - (1 - 2M/R)^{1/2}}{3(1 - 2M/R)^{1/2} - 1} \right\}, \tag{7}$$

increases monotonically as the radius, R, increases—and, hence, also as the mass, $M = (4\pi/3)\rho_0 R^3$, and the ratio ("strength of gravity")

$$2M/R = (8\pi/3)\rho_0 R^2 \qquad \hat{R} \tag{8}$$

increase. This is natural, since, as more and more matter is added to the star, a greater and greater pressure is required to support it. (2) The central pressure becomes infinite when M, R, and $2M/R$ reach the limiting values

$$R_{\text{lim}} = (9/4)M_{\text{lim}} = (3\pi\rho_0)^{-1/2}, \tag{9}$$

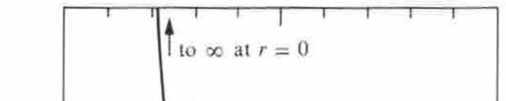
$$(2M/R)_{\text{lim}} = 8/9. \tag{10}$$

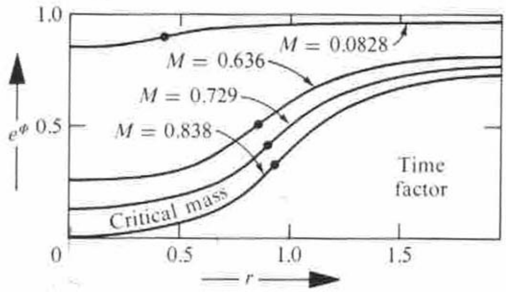
No star of uniform density can have a mass and radius exceeding these limits. These limits are purely relativistic phenomena; no such limits occur in Newtonian theory. (3) Inside the star the space geometry (geometry of a hypersurface $t = constant$) is that of a three-dimensional spherical surface with radius of curvature

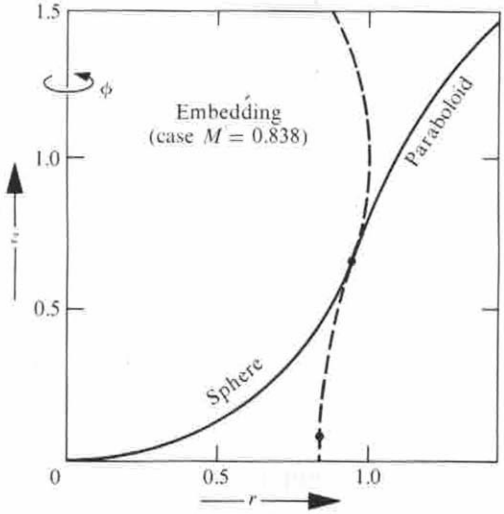
$$a = (3/8\pi\rho_0)^{1/2}. \tag{11}$$

[See equation (4), above.] Outside the star the (Schwarzschild) space geometry is that of a three-dimensional paraboloid of revolution. The interior and exterior geometries join together smoothly. All these details are shown in the following three diagrams. There all quantities are given in the following geometric units (to convert mass in g or density in g/cm³ into mass in cm or density in cm^{-2}, multiply by 0.742×10^{-28} cm/g): lengths, in units $(3/8\pi\rho_0)^{1/2}$; pressure, in units ρ_0; mass, in units $(3/32\pi\rho_0)^{1/2}$.

$ds^2 = -e^{2\Phi} dt^2 \rightarrow$ p. 594







$0{,}838 = \left(\sqrt{\frac{8}{9}}\right)^3$

masse critique