About JANUS

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April 25, 2023

Abstract

In this work, we discuss the JANUS model in the context of the Galilean mechanics, consider a covering of the space-time with two disjoint sheets corresponding respectively to the negative and positive mass sectors. The space-time is equipped with a single metric and connection representing the gravitation of which the sources are obtained by push-forwarding the flux of both positive and negative masses onto the space-time. The asymmetry between both sectors regarding Coriolis' force is a means to know in which sector we are living.

Keywords JANUS model · negative mass · time reversal operator · Newton-Cartan theory

Mathematics subject classification 83C25 · 83F05 · 70G45

1 Introduction

In this work, we discuss the JANUS model [6] in the context of the Galilean mechanics. To avoid the runaway effect, we build a model for which the masses of the same sign attract each other according to Newton's law, while masses of opposite signs repel each other according to anti-Newton's law.

Negative masses are positive masses that go back in time. The keystone idea is to consider a covering of the space-time with two disjoint sheets corresponding respectively to the negative and positive mass sectors. However, unlike JANUS model [6], we need on the space-time a single metric and connection representing the gravitation of which the sources are obtained by push-forwarding the flux of both positive and negative masses onto the space-time.

According to Élie Cartan's idea, the motion of a free falling particle (test-particle) is such that its linear 4-momentum is parallel-transported. At the classical limit of weak fields, we recover Newton's law and anti Newton's law.

The negative and positive masses are living within two disjoint sectors. The photons travel at the same velocity (to respect the causality principle) in distinct sectors, so negative masses are not visible from the positive mass sector and conversely. The unique interaction between positive and negative masses is in the space-time through the gravitation.

This approach is a milestone to build a relativistic cosmological model able to explain, amongst other puzzling experimental observations, the existence of dark matter.

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2 Galilean Mechanics

Before we go any further, we can verify our ideas at the approximation of weak fields. For this aim, we use the theoretical framework of asymptotic expansion of general relativity with Galilean covariance proposed in [4]. To begin with, we recall some basic concepts of this approach.

The space-time will be consider as a differential manifold $\mathcal{M}$ of dimension 4. A point $\boldsymbol{X} \in \mathcal{M}$ represents an event. The 4-column vector of its coordinates $(X^\alpha)_{0\leq\alpha\leq 3}$ in a chosen local chart will be denoted X, $X^0 = t$ being the time and $X^i = x^i$ for $1 \leq i \leq 3$ being the spatial coordinates, so we can write

$$X = \begin{pmatrix} t \\ x \end{pmatrix}$$

In the sequel, the intrinsic, coordinate-free objects are denoted by a capital letter while their representations in local charts are denoted by a normal letter. Let V (resp. V') the components of a tangent vector $\overrightarrow{\boldsymbol{V}}$ in a local chart X (resp. X'). The affine transformations $V' = PV + V_0$, where $P \in \mathbb{GL}(4)$ and $V_0 \in \mathbb{R}^4$, preserving the distances, the time durations, the uniform straight motions and the oriented volumes are called Galilean transformations. In what follows, we are only interested by the linear ones

$$P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix} \tag{1}$$

where $u \in \mathbb{R}^3$ is the velocity of transport, or Galilean boost, and $R \in \mathbb{SO}(3)$ is a spatial rotation. The set of all these transformations is a Lie subgroup $\mathbb{GAL}$ of the affine group $\mathbb{GA}\,(4)$ called Galilei group. The Lie subgroup of the linear Galilean transformations (1) is denoted $\mathbb{GAL}_0$.

Theorem. A necessary and sufficient condition for the Jacobian matrix $P = \frac{\partial X'}{\partial X}$ of a coordinate change $X \mapsto X'$ being a linear Galilean transformation is that this change is compound of a rigid body motion and a clock change:

$$x' = (R\,(t))^T\,(x - x_0\,(t)), \quad t' = t + t_0 \tag{2}$$

where $t \mapsto R\,(t) \in \mathbb{SO}(3)$ and $t \mapsto x_0(t) \in \mathbb{R}^3$ are smooth mappings, and $t_0 \in \mathbb{R}$ is a constant. Then the velocity of transport is given by $u = \omega\,(t) \times\,(x - x_0\,(t)) + \dot{x}_0\,(t)$ where ω is Poisson's vector such that: $\dot{R} = j\,(\omega)\,R$ (where for two vectors u and v, $j\,(u)\,v = u \times v$).

For the proof, the reader is referred to ([3], p. 339-341, Theorem 16.4).

The relation between local charts:

$X \;\sim\; X' \quad \Leftrightarrow \quad X$ and X' are deduced one from each other by a transition map (2)

is an equivalence relation. An equivalence class under $\sim$ is called a Galilean atlas and its elements are called Galilean charts or Galilean coordinate systems or Galilean reference frames. In a physical point of view, the importance of these charts lies in the fact that they are the coordinate systems in which the observers measure the durations and distances.

3 Bi-matter space-time as a covering

To modelize the existence of two species of mater, we introduce a covering $\boldsymbol{\pi} : \mathcal{C} \mapsto \mathcal{M}$ of the space-time with two disjoint sheets $\mathcal{M}_+$ and $\mathcal{M}_-$ called sectors, each of which populated with particles with mass (baryons, leptons) or without mass (photons). These two populations do not interact directly. In particular, photons emitted by particles of one sector cannot be

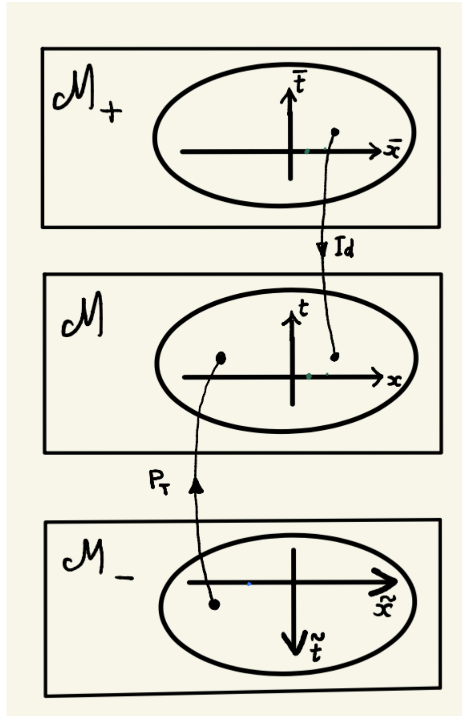

Figure 1: Bi-matter covering

received by particles of the other sector. The only interaction between massive particles of different sectors is through the gravitation, *i.e.* a connection that equips the space-time $\mathcal{M}$.

We claim that (Figure 1):

- $\mathcal{M}$, $\mathcal{M}_+$ and $\mathcal{M}_-$ are equipped respectively with Galilean atlases $\mathcal{A}$, $\mathcal{A}_+$ and $\mathcal{A}_-$.
- For every event $\boldsymbol{X}$ of the space-time, let $\boldsymbol{X}_+$ (resp. $\boldsymbol{X}_-$) the event of $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) such that $\pi(\boldsymbol{X}_+) = \boldsymbol{X}$ (resp. $\pi(\boldsymbol{X}_-) = \boldsymbol{X}$). For every local chart X of $\mathcal{A}$, there exists a local chart $\bar{X}$ of $\mathcal{A}_+$ such that the projection $\boldsymbol{\pi}$ is represented in the local charts X and $\bar{X}$ by

  $$t = \bar{t}, \qquad x = \bar{x}$$

  X and $\bar{X}$ will be called **twin local charts** of $\mathcal{M}$ and $\mathcal{M}_+$. If there is no confusion, the coordinates in the positive sector will be denoted latter on without bar and the projection $\boldsymbol{\pi}$ will be locally represented by the **identity map**. Likewise, for every local chart X of $\mathcal{A}$, there exists a local chart $\tilde{X}$ of $\mathcal{A}_-$ such that the projection $\boldsymbol{\pi}$ is represented in the local charts X and $\tilde{X}$ by the **T-reversal** P_T

  $$t = -\tilde{t}, \qquad x = \tilde{x}$$

  X and $\tilde{X}$ will be called **twin local charts** of $\mathcal{M}$ and $\mathcal{M}_-$.

4 Gravitation field equations

The particles of both sectors do not interact directly, but the space-time is the theater of operations where they interact through the gravitation.

In absence of massive particles, the space-time is flat and endowed with Minkowski metric represented in a Galilean chart by

$$\eta = \begin{pmatrix} c^2 & 0 \\ 0 & -I \end{pmatrix}$$

In the approximation of a weak gravitational field, the space-time is curved and the metric becomes

$$G = \eta + h = \begin{pmatrix} c^2 + 2\,\phi & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} c^2 & 0 \\ 0 & -I \end{pmatrix} + \begin{pmatrix} 2\,\phi & 0 \\ 0 & 0 \end{pmatrix}$$

where ϕ is the gravitation potential from which the gravity is generated by

$$g = -grad\,\phi$$

The field equation can be deduced from the Hilbert-Einstein principle of which the functional is at the weak field approximation

$$I_{HE} = \int_{\mathcal{M}} \left[\frac{1}{4\,\pi\,k_N} \left(c^2 + \phi\right) div\,g + \rho\,c^2 - \rho \left(\frac{1}{2} \parallel v^2 \parallel - \phi \right) \right] \mathrm{d}X^4$$

where k_N is Newton's gravitational constant, ρ is the density of mass and v is the velocity [4]. The variation with respect to the potential gives

$$div\,g = -4\,\pi k_N\,\rho$$

then

$$\Delta\,\phi = 4\,\pi k_N\,\rho \tag{3}$$

The 4-velocity is the tangent vector $\overrightarrow{\boldsymbol{U}}$ represented in a Galilean chart by

$$U = \dot{X} = \begin{pmatrix} 1 \\ v \end{pmatrix},$$

The density of mass is the time component of the 4-vector of mass flux $\overrightarrow{\boldsymbol{N}} = \rho\,\overrightarrow{\boldsymbol{U}}$. It is obtained as the sum of the push-forward of the corresponding mass fluxes $\overrightarrow{\boldsymbol{N}}_+$ and $\overrightarrow{\boldsymbol{N}}_-$ of both sectors

$$\overrightarrow{\boldsymbol{N}} = \boldsymbol{\pi}_*(\overrightarrow{\boldsymbol{N}}_+) + \boldsymbol{\pi}_*(\overrightarrow{\boldsymbol{N}}_-)$$

In the twin local charts X and $\bar{X}$, $\overrightarrow{\boldsymbol{N}}_+$ and its push-forward are represented by the same column vector of components

$$N_+ = \pi_*(N_+) = \begin{pmatrix} \rho_+ \\ \rho_+ v_+ \end{pmatrix}$$

where $\rho_+ \geq 0$. For the negative sector, we consider the twin local charts X and $\tilde{X}$. The mass flux $\overrightarrow{\boldsymbol{N}}_-$ is represented in the local chart $\tilde{X}$ by

$$\tilde{N}_- = \begin{pmatrix} \rho_- \\ \rho_- v_- \end{pmatrix}$$

where $\rho_- \geq 0$. Its push-forward is represented in the local chart X by

$$N_- = \pi_*(\tilde{N}_-) = P_T \tilde{N}_- = \begin{pmatrix} -\rho_- \\ \rho_- v_- \end{pmatrix}$$

Then the total flux of mass is represented in the local chart X by

$$N = \tilde{N}_+ + \tilde{N}_- = \begin{pmatrix} \rho_+ - \rho_- \\ \rho_+ v_+ + \rho_- v_- \end{pmatrix}$$

The solution of Poisson's equation (3) is

$$\phi = \phi_+ + \phi_-, \qquad \phi_\pm = \mp \int \frac{k_N\, \rho_\pm(x', t)}{\parallel x - x' \parallel}\, \mathrm{d}^3 x'$$

that leads to the decomposition of the gravity into

$$g = g_+ + g_-, \qquad g_\pm = -grad\, \phi_\pm = \mp \int \frac{k_N\, \rho_\pm(x', t)}{\parallel x - x' \parallel^2}\, \frac{x - x'}{\parallel x - x' \parallel}\, \mathrm{d}^3 x' \tag{4}$$

where the effect of the acceleration g_+ is attractive and g_- is repulsive on a positive mass.

5 Equations of motion

In General Relativity, the gravitation is a connection. Let us consider its classical version. A covariant differential or connection of a vector field is $\nabla V^\alpha = dV^\alpha + \Gamma^\alpha_\beta(dX)\, V^\beta$ where, using Christoffel symbols, the elements of the connection matrix Γ are $\Gamma^\alpha_\beta(dX) = \Gamma^\alpha_{\mu\beta} dX^\mu$.

At each group of transformation is associated a family of connections and the corresponding geometry. We call Galilean connections the symmetric connections on the tangent bundle $T\mathcal{M}$ associated to Galilei group [8, 9, 5]. In a Galilean chart, the connection matrix is valued in the Lie algebra $\mathfrak{gal}_0$ of $\mathbb{GAL}_0$ [2, 3]. In absence of Coriolis' effect, we have

$$\Gamma = \begin{pmatrix} 0 & 0 \\ -g\, d\, t & 0 \end{pmatrix} \tag{5}$$

where $g^j = -\Gamma^j_{00}$ are the components of the gravity ([1], [7]).

For a spinless particle of mass m, the linear 4-momentum $\overrightarrow{\boldsymbol{T}} = m \overrightarrow{\boldsymbol{U}}$ is represented in a Galilean chart by

$$T = m\, U = \begin{pmatrix} m \\ p \end{pmatrix} \tag{6}$$

According to Élie Cartan's prescription [1], the motion of a free falling particle (test-particle) is such that its linear 4-momentum is parallel-transported

$$\nabla_U T = \dot{T} + \Gamma(U)\, T = 0 \tag{7}$$

In a Galilean chart, this equation itemizes

$$\dot{m} = 0, \qquad \dot{p} = m\, g \tag{8}$$

For a particle of mass $m_+ > 0$ and velocity v in the positive sector, its 4-momentum is $\overrightarrow{\boldsymbol{T}} = \boldsymbol{\pi}_*(\overrightarrow{\boldsymbol{T}}_+)$ In the twin local charts X and $\bar{X}$, $\overrightarrow{\boldsymbol{T}}_+$ and its push-forward are represented by the same column vector of components

$$T = \pi_*(\bar{T}_+) = \begin{pmatrix} m_+ \\ m_+ v \end{pmatrix} \tag{9}$$

The push-forward is represented in the Figure 2. The equation of motion (8) of a particle of the positive sector reads

$$m_+ \dot{v} = m_+ g$$

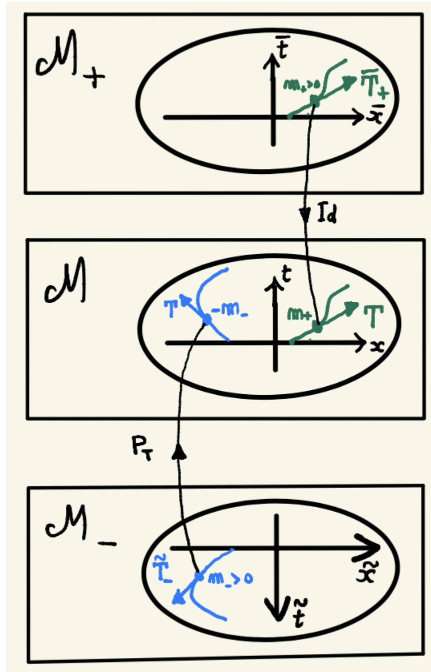

Figure 2: Push-forward of T for positive and negative sectors

then, according to the decomposition of the gravity (4),

$$\dot{v} = g_+ + g_-$$

hence the particle of the positive sector is attracted by particles of the same sector (Newton's law) and repelled by particles of the other one (anti Newton's law).

For a particle of mass $m_- > 0$ and velocity v in the negative sector, we consider the twin local charts X and $\tilde{X}$. The 4-momentum $\overrightarrow{\boldsymbol{T}}_-$ is represented in the local chart $\tilde{X}$ by

$$\tilde{T}_- = \begin{pmatrix} m_- \\ m_- v \end{pmatrix}$$

Its push-forward is represented in the local chart X by

$$T = \pi_*(\tilde{T}_-) = P_T \tilde{T}_- = \begin{pmatrix} -m_- \\ m_- v \end{pmatrix} \tag{10}$$

The equation of motion (8) of a particle of the negative sector reads

$$m_- \dot{v} = -m_- g \tag{11}$$

then, according to the decomposition of the gravity (4),

$$\dot{v} = -g_+ - g_-$$

hence the particle of the negative sector is attracted by particles of the same sector and repelled by particles of the other one.

At the classical limit of weak fields, whatever the sector, the law of conservation of the 4-momentum allows to recover Newton's and anti-Newton's law.

6 The asymmetry between both sectors, a means to know in which sector we are living

Up to now, we have considered the simplest situations in which only occurs the universal gravitation interaction. Now it is time for a new actor to burst in on the scene, Coriolis' force. The most general form of a Galilean connection is [3]

$$\Gamma = \begin{pmatrix} 0 & 0 \\ j\,(\Omega)\,d\,x - g\,d\,t & j\,(\Omega)\,d\,t \end{pmatrix}$$

where Ω is a 3-column vector associated by the mapping j^{-1} to the skew-symmetric matrix, the elements of which being $\Omega^i_j = \Gamma^i_{j0}$. The spinning vector Ω can be interpreted as representing Coriolis' effects [7, 3]. In a Galilean chart, this equation itemizes

$$\dot{m} = 0, \qquad \dot{p} = m\,(g - \Omega \times v) - \Omega \times p = 0 \tag{12}$$

In the positive sector, (9) leads to the equation of motion

$$m_+ \dot{v} = m_+ (g - 2\,\Omega \times v)$$

then a particle of the positive sector is sensitive to Coriolis' force represented by the last term.

In the negative sector, owing to (10) the equation of motion

$$m_- \dot{v} = -m_- g$$

is identical to (11), then a particle of the negative sector is insensitive to Coriolis' effect. This reveals the asymmetry between both sectors concerning the spinning. As, according to what the observations show, our mater is sensitive to Coriolis' effect, showing that we are living in the positive sector.

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