Exploring symmetries through the action on the Torsors of the eight connected components Janus Symplectic Group

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Abstract

In this article, we explore the Poincaré group, identified as the isometry group of Minkowski space and considered here as a dynamic group. We rely on the work of J.M. Souriau to trace how, through the action of the group on its momentum space, quantities such as energy, momentum, and spin emerge as purely geometric entities. Continuing this line of thought, we integrate the antichronous movements resulting from the complete group. The group is then extended to a five-dimensional configuration, interpreted as a geometric manifestation of the existence of electric charges and the symmetry between matter and antimatter. This leads us to the formulation of the Janus group, incorporating $\boldsymbol{CPT}$ symmetry. Finally, we demonstrate that these developments represent the beginnings of a geometric interpretation of Andrei Sakharov's model, which proposes to locate primordial antimatter in an antichronous sheet of the universe, thus offering an interpretation of the baryonic asymmetry of the universe in Cosmology.

Contents

1 The physico-mathematical foundations of this approach

The French mathematician Jean-Marie Souriau, who passed away in 2012, used to say, *"A little mathematics takes you away from physics, but a lot of it brings you back"*. In his work, he provided an example of such a statement by revealing the physical quantities like energy, momentum, and spin as objects of pure geometry, representing a brilliant application of symplectic geometry. He is one of the few who excelled both as a high-level mathematician and an excellent physicist. In his work *Structure of Dynamical Systems* [23] (today, we prefer to use the term *symplectic groups*), he constructs the action of the Poincaré group on the dual of its Lie algebra, known as the momentum space. It is a vector space of the same dimension as the group, which is 10. He then organizes its components according to:

- A scalar, energy
- A 3-vector momentum
- A 3-vector spin
- A 3-vector to which he gives the name *"passage"*

These components of momentum then define motions in Minkowski space, where the Poincaré group is the isometry group. These motions are divided into classes, and Souriau establishes a connection between particles and classes of motions. He shows that the components of the 3-vector passage can be canceled by choosing a coordinate system that accompanies the particle in its motion. The remaining physical quantities are the first three. Their emergence can also be interpreted as an application of Noether's theorem:

- The scalar energy is then associated with the subgroup of temporal translations.
- The 3-vector momentum with the subgroup of spatial translations.
- The 3-vector spin (unquantized) with the Lorentz subgroup, around which the Poincaré group is constructed.

But at the end of this approach, a surprise awaited the physicist. The *Lorentz group* is defined by:

$$\mathcal{L}or := \{L \in \mathrm{GL}(4,\mathbb{R}),\ \tau(L)L = I_4\}\,.$$

with:

$$\tau(L) := I_{1,3}L^T I_{1,3} \quad , \quad I_{1,k} := \begin{pmatrix} -1 & 0 \\ 0 & I_k \end{pmatrix} \ (k \in \mathbb{N}).$$

We extend the map τ to vectors of $\mathbb{R}^4$, by setting for all $X \in \mathbb{R}^4$:

$$\tau(X) := X^T I_{1,3}$$

The Lorentz group has four connected components (see [5], [17] and [18]):

- $\mathcal{L}or_n$ is the neutral component (its *restricted subgroup*), does not invert either space or time *i.e.* defined by:

$$\mathcal{L}or_n := \{L \in \mathcal{L}or,\ \det(L) = 1 \ \wedge \ [L]_{00} \geq 1\}$$

- $\mathcal{L}or_s$ inverts space *i.e.* defined by:

$$\mathcal{L}or_s := \{L \in \mathcal{L}or,\ \det(L) = -1\ \wedge\ [L]_{00} \geq 1\}$$

- $\mathcal{L}or_t$ inverts time but not space *i.e.* defined by:

$$\mathcal{L}or_t := \{L \in \mathcal{L}or,\ \det(L) = 1\ \wedge\ [L]_{00} \leq -1\}$$

- $\mathcal{L}or_{st}$ inverts both space and time *i.e.* defined by:

$$\mathcal{L}or_{st} := \{L \in \mathcal{L}or,\ \det(L) = -1\ \wedge\ [L]_{00} \leq -1\}$$

We have:

$$\mathcal{L}or = \mathcal{L}or_n\ \sqcup\ \mathcal{L}or_s\ \sqcup\ \mathcal{L}or_t\ \sqcup\ \mathcal{L}or_{st}. \tag{1}$$

The first two components are grouped together to form the subgroup called *"orthochronous"*:

$$\mathcal{L}or_o = \mathcal{L}or_n\ \sqcup\ \mathcal{L}or_s \tag{2}$$

It includes **P**-symmetry, which poses no problem for physicists who know that there are photons of *"right"* and *"left"* helicity whose motions are derived from this symmetry. This corresponds to the phenomenon of the polarization of light.

The last two components form the subset *"retrochronous"* or *"antichronous"*, whose components invert time:

$$\mathcal{L}or_a = \mathcal{L}or_t\ \sqcup\ \mathcal{L}or_{st} \tag{3}$$

We have:

$$\mathcal{L}or = \mathcal{L}or_o\ \sqcup\ \mathcal{L}or_a \tag{4}$$

The *Poincaré group* is defined by:

$$\mathcal{P}oin := \left\{ \begin{pmatrix} L & D \\ 0 & 1 \end{pmatrix},\ L \in \mathcal{L}or\ \wedge\ D \in \mathbb{R}^4 \right\}, \tag{5}$$

it inherits the properties of the Lorentz group and thus has four connected components. We then distinguish the subgroup of the complete Poincaré group, constructed from the orthochronous components of the Lorentz group. And we define all components (like Lorentz group):

$$\forall \alpha \in \{n, s, t, st, o, a\},\ \mathcal{P}oin_\alpha := \left\{ \begin{pmatrix} L_\alpha & D \\ 0 & 1 \end{pmatrix},\ L_\alpha \in \mathcal{L}or_\alpha\ \wedge\ D \in \mathbb{R}^4 \right\}. \tag{6}$$

We have the same decomposition like (1), (2), (3) and (4).

The classification of motions yields two classes corresponding to the movements of photons and particles with a positive mass m. Souriau summarizes his study by providing a summary of the group's action on its momentum (see [23] chapter 13).

We can define the moment matrix M and the stress–energy vector P as follows:

$$M := \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \quad , \quad P := \begin{pmatrix} E \\ p \end{pmatrix}$$

with ℓ the angular momentum of M, g the relativist barycenter of M, p the linear momentum of P, and E the energy of P.

The action is written (see [23] equation 13.107) for all $L \in \mathcal{L}or$:

$$M' = LM\tau(L) + C\tau(P)\tau(L) - CLP\tau(C) \tag{7}$$

$$P' = LP \tag{8}$$

We have:

$$\mathcal{L}or_t = -\mathcal{L}or_s \quad \mathcal{L}or_{st} = -\mathcal{L}or_n \tag{9}$$

Then, it is possible to write the complete Poincaré group as:

$$\mathcal{P}oin := \left\{ \begin{pmatrix} \lambda L_o & D \\ 0 & 1 \end{pmatrix},\ L_o \in \mathcal{L}or_o \ \wedge \ D \in \mathbb{R}^4 \ \wedge \ \lambda \in \{\pm 1\} \right\}. \tag{10}$$

The action of the complete group is then written as follows for all $L := \lambda L_o \in \mathcal{L}or$:

$$M' = L_o M\tau(L_o) + \lambda C\tau(P)\tau(L_o) - CLP\tau(C)$$

$$P' = \lambda L_o P$$

It's then observed that the *retrochronous* components reverse the energy and, consequently, the mass, as noted by J.M. Souriau ((14.67) of page 198 [23]).

In the past, we have seen an example where P. Dirac suggested the use of an electric charge symmetry. The existence of particles with opposite electric charges was thus directly implied by an extension of the theory. This involved postulating the existence of positrons. Fortunately, the existence of such particles was quickly confirmed by C.A. Anderson's observations[1], which earned him the Nobel Prize in 1936.

We are in 1970. J.M. Souriau's theoretical framework raised the possibility of particles with negative energy, which were categorized into two classes:

- Particles endowed with a negative mass m
- Photons endowed with negative energy.

In conclusion, the author indicated potential measures to circumvent the emergence of particles with negative mass, one of which was to decide that only the orthochronous components of the Poincaré group should pertain to the realm of physics.

[1] To be precise, this observation did not follow P. Dirac's deduction in the sense that, in 1923, the Russian D. Skobeltzyn was the first to make this observation.

2 When the theory of dynamic groups illuminates the traveled path

The application of the coadjoint action of a symplectic group on the dual of its Lie algebra, initiated by the mathematician Jean-Marie Souriau, has shed light on specific aspects of the approach followed by physics. The orbit method is due to Kirillov ([4], [8], [6], [7], [13], [14], [16], [24], [27] and [28]).

Thus, the restricted Lorentz symplectic group, limited to its two orthochrone components, translates, through the invariance properties that result from it, the aspects of special relativity. In 1970, J-M Souriau established that the analysis of the components of its moment makes it possible to shed light on the geometric nature of a spin (not quantized): see [23] and [22]. He uses for this purpose symplectic methods ([11], [9], [25] and [26]). In the theory of symplectic groups, we show a classification in terms of movements. At this stage, the action of these elements reversing space finds its illustration in the phenomenon of polarization of light, where any *"right"* photon can be converted into a *"left"* photon.

By operating the product of this group by that of the spatio-temporal translations, we obtain the restricted Poincaré symplectic group, always limited to its two orthochrone components. In its moment, we first find the energy related to the subgroup of temporal translations. Then the momentum, linked to the spatial translations, the two being linked by the invariance of the modulus of the energy-momentum four-vector under the action of the Lorentz group.

By adding a translation along a fifth dimension to the restricted Poincaré group, we form a Lie group to which we will give the name *Restricted Kaluza Group* ([1], [2], [3], [12], [15]). This group is not the 15-dimensional Kaluza group associated with a 5-dimensional Lorentzian manifold but a new 11-dimensional group, including 5-dimensional space-time translation. This new dimension endows the momentum with an additional scalar that can be identified with the electric charge q, which may be positive, negative, or zero, and is still not quantized. We then bring out the geometric translation according to a scalar ϕ due to endowing the masses with an invariant electric charge. Then, by bringing in a new symmetry reflecting the inversion of the fifth dimension, synonymous with an inversion of the scalar from q to $-q$, we double the number of its connected components from 2 to 4. The action on the moment then links this new symmetry to the inversion of the electric charge q. We thus deduce the geometric modeling of charge conjugation or **C**-*Symmetry*, which translates the matter-antimatter symmetry introduced by Dirac. It's then logical to name this new extension, the *Restricted Janus Group*.

By introducing a new symmetry to the previous group, which we describe as **T**-*Symmetry* and which converts matter into antimatter with negative mass – a concept we could name *antimatter in the Feynman sense* – we build the *Janus Symplectic Group*. Thus, we double the number of connected components from four to eight, grouped into two subsets: *"Orthochronous"*, conserving time and energy properties, and *"Antichronous"*, reversing time and energy. Therefore, we bring forth the geometric translation of endowing masses with an invariant electric charge. As the Jean-Marie Souriau demonstrated as early as 1970, a pioneer in the theory of symplectic groups ([23], [10], [22]), this approach has allowed key elements, which have marked the progress of relativistic physics, to be given a purely geometric nature.

In relation to the world of physics, wouldn't the role of mathematics be to illuminate the path traveled? Conversely, could it be possible that the exploration of new symmetries, accompanying this decoding using symplectic groups, contains more than what we thought we put into it? That it could designate new paths to follow?

This is what we will consider with the Janus Symplectic Group with charge symmetry, by integrating the antichronous components of the Lorentz group, resulting from its simple axiomatic definition, with the obvious repercussions on the Poincaré group and its extensions.

3 Janus Symplectic Group

Let $\tilde{\mathbf{T}} := I_{1,3}$, $\tilde{\mathbf{P}} := -\tilde{\mathbf{T}}$ and:

$$\forall \lambda, \nu \in \{0,1\}, \quad \mathcal{L}or\left(\tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda}\right) := \left\{L_n \tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda},\ L_n \in \mathcal{L}or_n\right\}.$$

Then, there are 4 connected components of $\mathcal{L}or$, given by[2]

$$\mathcal{L}or_n = \mathcal{L}or\left(\tilde{\mathbf{P}}^{0}\tilde{\mathbf{T}}^{0}\right) \qquad \mathcal{L}or_s = \mathcal{L}or\left(\tilde{\mathbf{P}}^{1}\tilde{\mathbf{T}}^{0}\right)$$
$$\mathcal{L}or_t = \mathcal{L}or\left(\tilde{\mathbf{P}}^{0}\tilde{\mathbf{T}}^{1}\right) \qquad \mathcal{L}or_{st} = \mathcal{L}or\left(\tilde{\mathbf{P}}^{1}\tilde{\mathbf{T}}^{1}\right)$$

and we have the decomposition:

$$\mathcal{L}or = \bigsqcup_{\nu,\lambda \in \{0,1\}} \mathcal{L}or\left(\tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda}\right) \tag{11}$$

Then, we define the Janus symplectic group.

Definition 3.1. The *Janus symplectic group* is defined as the subgroup of $\mathrm{GL}(6,\mathbb{R})$:

$$\mathcal{J}an := \left\{ \begin{pmatrix} L & 0 & D \\ 0 & (-1)^{\eta} & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \eta \in \{0,1\} \ \wedge\ \phi \in \mathbb{R} \ \wedge\ L \in \mathcal{L}or \ \wedge\ D \in \mathbb{R}^4 \right\}$$

The Janus symplectic group is therefore a subgroup of the group of isometries in dimen-

[2] Equalities are shown by double inclusion. For example, let's demonstrate that $\mathcal{L}or_s = \mathcal{L}or\left(\tilde{\mathbf{P}}^{1}\tilde{\mathbf{T}}^{0}\right)$. Take $L \in \mathcal{L}or_s$ ($\det(L) = -1$ et $[L]_{00} \geq 1$). Then we have $\det(L\tilde{\mathbf{P}}) = -1$ and $[L\tilde{\mathbf{P}}]_{00} \geq 1$ *i.e.*, we have $L_n := L\tilde{\mathbf{P}} \in \mathcal{L}or_n$. Since $\tilde{\mathbf{P}}^{-1} = \tilde{\mathbf{P}}$, we can conclude that $L = L_n\tilde{\mathbf{P}} \in \mathcal{L}or\left(\tilde{\mathbf{P}}^{1}\tilde{\mathbf{T}}^{0}\right)$. The inclusion in the other direction is trivial.

sion 5 given by[3]:

$$\mathrm{Aff}(\mathcal{O}(1,4)) := \left\{ \begin{pmatrix} L & D' \\ 0 & 1 \end{pmatrix},\ L \in \mathcal{O}(1,4)\ \wedge\ D' \in \mathbb{R}^5 \right\}$$

with $\tau_{1,4}(L) := I_{1,4} L^T I_{1,4}$ and $\mathcal{O}(1,4) := \{L \in \mathrm{GL}(5,\mathbb{R}),\ \tau_{1,4}(L)L = I_5\}$. The elements of $\mathrm{Aff}(\mathcal{O}(1,4))$ are the elements which preserve the distance between two events (*pentavectors*) $X := (t,x,y,z,\xi)$ and $X' := (t',x',y',z',\xi')$ given by:

$$d(X,X') := c^2(t-t')^2 - (x-x')^2 - (y-y')^2 - (z-z')^2 - (\xi-\xi')^2$$

This fifth dimension is of space type (we note the variable ξ). Each dimension is therefore associated with a symmetry, there are three types of symetries:

- the $\mathbf{T}$-*symmetry*;
- the $\mathbf{P}_x$-*symmetry*, $\mathbf{P}_y$-*symmetry*, $\mathbf{P}_z$-*symmetry* grouped together what we call the $\mathbf{P}$-*symmetry*;
- the ξ-*symmetry* corresponding to the $\mathbf{C}$-*symmetry* (the charge conjugation).

This space of dimension 5 is a Minkowski metric space to which we have added one dimension, it has the metric $I_{1,4}$.

We also define the *restricted Janus group* is the subgroup of $\mathcal{J}an$ given by:

$$\mathcal{J}an_n := \left\{ \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \phi \in \mathbb{R}\ \wedge\ L_n \in \mathcal{L}or_n\ \wedge\ D \in \mathbb{R}^4 \right\}$$

Let:

$$\mathbf{C} := \begin{pmatrix} I_4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \mathbf{P} := \begin{pmatrix} \tilde{\mathbf{P}} & 0 \\ 0 & I_2 \end{pmatrix} \quad , \quad \mathbf{T} := \begin{pmatrix} \tilde{\mathbf{T}} & 0 \\ 0 & I_2 \end{pmatrix}.$$

We have:

$$\forall \lambda,\eta,\nu \in \{0,1\},\quad \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix} \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda = \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore by equation (11):

$$\mathcal{J}an = \left\{ \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \lambda,\eta,\nu \in \{0,1\}\ \wedge\ \phi \in \mathbb{R}\ \wedge\ L_n \in \mathcal{L}or_n\ \wedge\ D \in \mathbb{R}^4 \right\}.$$

[3] $\mathrm{Aff}(\mathcal{O}(1,4))$ is the affine group associated with $\mathcal{O}(1,4)$, it is also defined by the semi-direct product $\mathrm{Aff}(\mathcal{O}(1,4)) := \mathcal{O}(1,4) \ltimes \mathbb{R}^5$. We can also define the symplectic Janus group as being the affine group associated with the subgroup of $\mathcal{O}(1,4)$ given by:

$$\mathcal{E}lec := \left\{ \begin{pmatrix} L & 0 \\ 0 & (-1)^\eta \end{pmatrix},\ \eta \in \{0,1\}\ \wedge\ L \in \mathcal{L}or \right\}$$

called the *symplectic electric group* and we have $\mathcal{J}an := \mathrm{Aff}(\mathcal{E}lec)$.

Definition 3.2. (i) The ***CPT**-group* is the subgroup $\mathcal{K}$ of $\mathcal{J}an$ of order 8 generated by $\mathbf{C}$, $\mathbf{P}$ and $\mathbf{T}$ *ie*:

$$\mathcal{K} := \left\{\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda,\ \eta,\nu,\lambda \in \{0,1\}\right\} = \{I_6, \mathbf{T}, \mathbf{P}, \mathbf{PT}, \mathbf{C}, \mathbf{CT}, \mathbf{CP}, \mathbf{CPT}\}.$$

(ii) For all $\mathbf{X} \in \mathcal{K}$, the ***X**-component of $\mathcal{J}an$* is:

$$\mathcal{J}an\,(\mathbf{X}) := \{JX,\ J \in \mathcal{J}an_n\}.$$

Thus, we have:

$$\mathcal{J}an\left(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda\right) = \left\{\begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \phi \in \mathbb{R}\ \wedge\ L_n \in \mathcal{L}or_n\ \wedge\ D \in \mathbb{R}^4\right\}.$$

These 8 components are the 8 connected components of $\mathcal{J}an$, we have the decomposition:

$$\mathcal{J}an = \bigsqcup_{\mathbf{X}\in\mathcal{K}} \mathcal{J}an\,(\mathbf{X}) = \bigsqcup_{\eta,\nu,\lambda\in\{0,1\}} \mathcal{J}an\left(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda\right).$$

The group $\mathcal{L}or$ is the Lie group of dimension 6 and its Lie algebra is:

$$\mathfrak{lor} := \mathcal{A}(1,3) := \{\Lambda \in \mathcal{M}(4,\mathbb{R}),\ \tau_{1,3}(\Lambda) = -\Lambda\}$$

Then, the group $\mathcal{J}an$ is a Lie group of dimension 11 and its Lie algebra is:

$$\mathfrak{jan} = \left\{\begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix},\ \Lambda \in \mathcal{A}(1,3)\ \wedge\ \Gamma \in \mathbb{R}^4\ \wedge\ \varepsilon \in \mathbb{R}\right\}.$$

We have two caracterisations[4]:

$$\left(\mathbb{R}^5\right)^* = \left\{\begin{pmatrix}\Gamma\\ \varepsilon\end{pmatrix} \longmapsto -\begin{pmatrix} P^T & q\end{pmatrix} I_{1,4} \begin{pmatrix}\Gamma\\ \varepsilon\end{pmatrix} = -\tau(P)\Gamma - q\varepsilon,\ \begin{pmatrix}P\\ q\end{pmatrix} \in \mathbb{R}^5\right\}$$
$$\mathcal{A}(1,3)^* = \left\{\Lambda \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda),\ M \in \mathcal{A}(1,3)\right\}$$

Then, we have:

$$\mathfrak{jan}^* = \left\{\left\{\ M \mid P \mid q\ \right\} : \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) - \tau(P)\Gamma - q\varepsilon,\ M \in \mathcal{A}(1,3)\ \wedge\ P \in \mathbb{R}^4\ \wedge\ q \in \mathbb{R}\right\}^{5}.$$

The *action of the group $\mathcal{J}an$ on* $\mathfrak{jan}^*$ is defined by the coadjoint representation *i.e.*, for

[4] For all $\beta \in \mathbb{R}^*$, the application Φ_β which to $M \in \mathcal{A}(1,3)$ associates the linear form $\Lambda \longmapsto \beta\mathrm{Tr}(M\Lambda)$ is an isomorphism of $\mathcal{A}(1,3)$ to $\mathcal{A}(1,3)^*$. Taking $\{A_{kl} := -E_{kl} + [I_{1,3}]_{ll}[I_{1,3}]_{kk}E_{lk},\ k,l \in \{1,\dots,4\},\ k < l\}$ the canonical basis of $\mathcal{A}(1,3)$, we have $\Phi_{-1/2}(A_{kl})(A_{kl}) = 1$, hence the choice of $\beta := -1/2$.

[5] The elements of $\mathfrak{jan}^*$ are called **torsors**.

any $a \in \mathcal{J}an$ and any $\mu \in \mathfrak{jan}^*$, we denote this action by:

$$a \bullet \mu := \mathrm{Ad}^*_a(\mu).$$

with

$$\begin{array}{rccc} \mathrm{Ad}^*: & \mathcal{J}an & \longrightarrow & \mathrm{Aut}(\mathfrak{jan}^*) \\ & a & \longmapsto & \mathrm{Ad}^*_a : \mu \longmapsto \left(Z \longmapsto \mu\left(a^{-1} Z a\right)\right) \end{array}$$

Proposition 3.1. Let:

$$a := \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{J}an \quad , \quad \left\{ \ M \ \middle| \ P \ \middle| \ q \ \right\} \in \mathfrak{jan}^*.$$

We have:

$$\begin{aligned} & a \bullet \left\{ \ M \ \middle| \ P \ \middle| \ q \ \right\} \\ = & \left\{ \ LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \ \middle| \ LP \ \middle| \ (-1)^\eta q \ \right\}. \end{aligned}$$

Proof. We have:

$$\begin{aligned} & \left(a \bullet \left\{ \ M \ \middle| \ P \ \middle| \ q \ \right\}\right) \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\ = & \left\{ \ M \ \middle| \ P \ \middle| \ q \ \right\} \left(a^{-1} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} a \right) \\ = & \left\{ \ M \ \middle| \ P \ \middle| \ q \ \right\} \left(\begin{pmatrix} \tau(L) & 0 & -\tau(L)D \\ 0 & (-1)^\eta & (-1)^{\eta+1}\phi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \right) \\ = & \left\{ \ M \ \middle| \ P \ \middle| \ q \ \right\} \begin{pmatrix} \tau(L)\Lambda L & 0 & \tau(L)(\Lambda D + \Gamma) \\ 0 & 0 & (-1)^\eta \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\ = & -\frac{1}{2}\mathrm{Tr}\left(M\tau(L)\Lambda L\right) - \tau(P)\tau(L)(\Lambda D + \Gamma) - (-1)^\eta q \varepsilon \\ = & -\frac{1}{2}\mathrm{Tr}\left[\left(LM\tau(L) + 2D\tau(P)\tau(L)\right)\Lambda\right] - \tau(LP)\Gamma - (-1)^\eta q\varepsilon \\ = & -\frac{1}{2}\mathrm{Tr}\left[\left(LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D)\right)\Lambda\right] - \tau(LP)\Gamma - (-1)^\eta q\varepsilon \\ = & \left\{ \ LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \ \middle| \ LP \ \middle| \ (-1)^\eta q \ \right\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$\square$

To describe the Lie algebra of $\mathcal{J}an$, we can also use the isomorphism of Lie algebras[6]:

$$\begin{array}{cccc} j: & (\mathbb{R}^3, \wedge) & \longrightarrow & (\mathcal{A}(3), [\ ,\ ]) \\ & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \end{array} .$$

with $\wedge$ the cross product on $\mathbb{R}^3$ and $\mathcal{A}(3)$ the vector space of antisymmetric matrices of size 3. Then, we have:

$$\mathfrak{jan} = \left\{ \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix},\ \Lambda \in \mathcal{A}(1,3)\ \wedge\ \ \Gamma \in \mathbb{R}^4\ \wedge\ \varepsilon \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \beta, w, \gamma \in \mathbb{R}^3\ \wedge\ v, \varepsilon \in \mathbb{R} \right\}.$$

Therefore, for all $\left\{\ M \,\middle|\, P \,\middle|\, q\ \right\} \in \mathfrak{jan}^*$, there are unique $\ell, g, p \in \mathbb{R}^3$ and $E, q \in \mathbb{R}$ such as:

$$\begin{aligned} \left\{\ M \,\middle|\, P \,\middle|\, q\ \right\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} &= \left\{\ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \,\middle|\, \begin{pmatrix} E \\ p \end{pmatrix} \,\middle|\, q\ \right\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} \right) - \begin{pmatrix} E & p^T \end{pmatrix} I_{1,3} \begin{pmatrix} v \\ \gamma \end{pmatrix} - q\varepsilon \\ &= \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon \end{aligned}$$

We denote this last equality as:

$$\left\{\ \ell \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The dual $\mathfrak{jan}^*$ has the following descriptions:

$$\left\{ \left\{\ \ell \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} : \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \longmapsto \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon,\ \ \ell, g, p \in \mathbb{R}^3\ \wedge\ E, q \in \mathbb{R} \right\}.$$

Definition 3.3. Let

$$\mu := \left\{\ M \,\middle|\, P \,\middle|\, q\ \right\} := \left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} \in \mathfrak{jan}^*$$

with relations:

$$M = \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \quad , \quad P = \begin{pmatrix} E \\ p \end{pmatrix}.$$

[6] We have for all $u, v \in \mathbb{R}^3$: $u \wedge v = j(u)(v)$ and $j(u \wedge v) = [j(u), j(v)] = j(u)j(v) - j(v)j(u)$.

(i) The matrix $M := M(\mu) \in \mathcal{A}(1,3)$ is called the *moment matrix associated with* μ. The vector $\ell := \ell(\mu) \in \mathbb{R}^3$ is called the *angular momentum of* M, and the vector $g := g(\mu) \in \mathbb{R}^3$ is the *relativist barycenter of* M.

(ii) (a) The vector $P := P(\mu) \in \mathbb{R}^4$ is called the *stress–energy vector associated with* μ. The vector $p := p(\mu) \in \mathbb{R}^3$ is called the *linear momentum of* P, and the scalar $E := E(\mu) \in \mathbb{R}$ is called the *energy of* P.

(b) The *first Casimir number associated with* μ is defined by:

$$C_1 := C_1(\mu) := P^T I_{1,3} P = E^2 - p^2.$$

(c) The *mass associated to* μ is defined by :

$$m := m(\mu) := \operatorname{sign}(E)\sqrt{C_1} = \operatorname{sign}(E)\sqrt{E^2 - p^2}.$$

(iii) The scalar $q := q(\mu) \in \mathbb{R}$ is called the *electric charge associated with* μ.

We deduce a simple expression of the action of the **CPT**-group $\mathcal{K}$ on the torsors of $\mathfrak{jan}^*$.

Corollary 3.2. Let $\left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} \in \mathfrak{jan}^*$. For all $\lambda, \eta, \nu \in \{0,1\}$, we have:

$$(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} = \left\{\ l \,\middle|\, (-1)^{\lambda+\nu} g \,\middle|\, (-1)^\nu p \,\middle|\, (-1)^\lambda E \,\middle|\, (-1)^\eta q\ \right\}.$$

Proof. We apply the Proposition 3.1 with $a := \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda$:

$$\begin{aligned}
(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} &= (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{\ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \,\middle|\, \begin{pmatrix} E \\ p \end{pmatrix} \,\middle|\, q\ \right\} \\
&= \left\{\ \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu \,\middle|\, I_{1,3} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \begin{pmatrix} E \\ p \end{pmatrix} \,\middle|\, (-1)^\eta q\ \right\} \\
&= \left\{\ \begin{pmatrix} 0 & (-1)^{\lambda+\nu} g^T \\ (-1)^{\lambda+\nu} g & j(\ell) \end{pmatrix} \,\middle|\, \begin{pmatrix} (-1)^\lambda E \\ (-1)^\nu p \end{pmatrix} \,\middle|\, (-1)^\eta q\ \right\} \\
&= \left\{\ l \,\middle|\, (-1)^{\lambda+\nu} g \,\middle|\, (-1)^\nu p \,\middle|\, (-1)^\lambda E \,\middle|\, (-1)^\eta q\ \right\}
\end{aligned}$$

□

So we have:

$$\begin{aligned}
\mathbf{C} \bullet \left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} &= \left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, -q\ \right\} \\
\mathbf{P} \bullet \left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} &= \left\{\ l \,\middle|\, -g \,\middle|\, -p \,\middle|\, E \,\middle|\, q\ \right\} \\
\mathbf{T} \bullet \left\{\ l \,\middle|\, g \,\middle|\, p \,\middle|\, E \,\middle|\, q\ \right\} &= \left\{\ l \,\middle|\, -g \,\middle|\, p \,\middle|\, -E \,\middle|\, q\ \right\}
\end{aligned}$$

Corollary 3.3. Let $\mu \in \mathfrak{jan}^*$. For all $\lambda, \eta, \nu \in \{0,1\}$, we have:

$$P\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) = \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda P(\mu)$$
$$C_1\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) = C_1(\mu)$$
$$m\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) = (-1)^\lambda m(\mu)$$

Proof. Let $\mu := \left\{ \ l \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \middle| \ q \ \right\} \in \mathfrak{jan}^*$. We have for the stress-energy tensor:

$$P(\mathbf{P} \bullet \mu) = P\left(\left\{ \ l \ \middle| \ -g \ \middle| \ -p \ \middle| \ E \ \middle| \ q \ \right\}\right) = \begin{pmatrix} E \\ -p \end{pmatrix} = \tilde{\mathbf{P}} P(\mu)$$
$$P(\mathbf{T} \bullet \mu) = P\left(\left\{ \ l \ \middle| \ -g \ \middle| \ p \ \middle| \ -E \ \middle| \ q \ \right\}\right) = \begin{pmatrix} -E \\ p \end{pmatrix} = \tilde{\mathbf{T}} P(\mu)$$
$$P(\mathbf{P} \bullet \mu) = P\left(\left\{ \ l \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \middle| \ -q \ \right\}\right) = \begin{pmatrix} E \\ p \end{pmatrix} = P(\mu)$$

for the first Casimir number:

$$C_1\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) = P(\mu)^T \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \tilde{\mathbf{P}}^\nu \tilde{\mathbf{P}}^\lambda P(\mu) = P(\mu)^T I_{1,3} P(\mu) = C_1(\mu)$$

for the mass:

$$m\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right) = \operatorname{sign}\left(E\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right)\right) \sqrt{C_1\left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu\right)} = \operatorname{sign}((-1)^\lambda E) \sqrt{C_1(\mu)} = (-1)^\lambda m(\mu)$$

□

Therefore the elements variable by these actions are:

$$P\left(\mathbf{P} \bullet \mu\right) = \tilde{\mathbf{P}} P(\mu) \qquad P\left(\mathbf{T} \bullet \mu\right) = \tilde{\mathbf{T}} P(\mu) \qquad m\left(\mathbf{T} \bullet \mu\right) = -m(\mu) \tag{12}$$

and we have the following table:

		$\lambda = 0$	$\lambda = 1$
$\eta = 0$	$\nu = 0$	• neutral symetry • $a = I_6$ • $\mu' = \{\ell, g, p, E, q\}$	• $\mathbf{T}$ – symetry • $a = \mathbf{T}$ • $\mu' = \{\ell, -g, p, -E, q\}$
	$\nu = 1$	• $\mathbf{P}$ – symetry • $a = \mathbf{P}$ • $\mu' = \{\ell, -g, -p, E, q\}$	• $\mathbf{PT}$ – symetry • $a = \mathbf{PT}$ • $\mu' = \{\ell, g, -p, -E, q\}$
$\eta = 1$	$\nu = 0$	• $\mathbf{C}$ – symetry • $a = \mathbf{C}$ • $\mu' = \{\ell, g, p, E, -q\}$	• $\mathbf{CT}$ – symetry • $a = \mathbf{CT}$ • $\mu' = \{\ell, -g, p, -E, -q\}$
	$\nu = 1$	• $\mathbf{CP}$ – symetry • $a = \mathbf{CP}$ • $\mu' = \{\ell, -g, -p, E, -q\}$	• $\mathbf{CPT}$ – symetry • $a = \mathbf{CPT}$ • $\mu' = \{\ell, g, -p, -E, -q\}$

Figure 1: This table lists the 8 values of $\mu' := (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{ \; l \; \middle| \; g \; \middle| \; p \; \middle| \; E \; \middle| \; q \; \right\}$ for all $\lambda, \eta, \nu \in \{0, 1\}$.

4 Discussion & Conclusion

In this paper, we have performed a double extension of the restricted Poincaré group limited to its orthochronous components, which are classically used in physics. This extension also includes the transition from the four-dimensional Minkowski spacetime to a new space of the same dimension, to which we have added a translation along an additional fifth dimension to form a new Lie group. The existence of this additional subgroup results in the invariance of a scalar, identified as the electric charge. A symmetry is introduced along this fifth dimension, and we have shown that this leads to the inversion of the electric charge. This provides a geometric representation of the symmetry between matter and antimatter.

In 1905, physics made a spectacular leap forward when Albert Einstein introduced the theory of special relativity. This theory was based on the idea that time, multiplied by a constant c with the dimensions of speed, specifically the speed of light, became a coordinate similar to the other three spatial dimensions, integrated into the geometry of Minkowski space. In 1915, with the publication of his field equation, Einstein was able to explain phenomena such as the precession of Mercury's perihelion and the deflection of light by the sun.

At that time, the concept of an expanding cosmos was unthinkable, yet it was confirmed by the observations of Edwin Hubble and the theories of Friedmann, thus establishing the foundations of the Big Bang theory. This theory revealed that the universe had experienced, in its distant past, conditions of extreme density and temperature.

Simultaneously, quantum mechanics provided a new approach to deciphering microphysical phenomena, with Paul Dirac soon introducing the concept of antimatter. According to this theory, when the universe was one thousandth of a second old, it consisted of equal parts matter and antimatter, coexisting with short-wavelength photons. A balance was formed where the annihilations of matter-antimatter pairs were compensated by the creation of new pairs from gamma photons. However, as the universe expanded, these annihilations should have prevailed, logically leading to the total disappearance of matter. In 1967, the discovery of the cosmic microwave background at 2.7 K both homogeneous and isotropic, reinforced the Big Bang theory, identifying these photons as those produced by matter-antimatter annihilations, whose wavelengths had stretched along with the expansion of the universe to centimeter dimensions.

However, this discovery did not explain why one in a million matter particles had survived annihilation, nor why its antimatter counterpart, described as primordial, remained unobserved. Initially, it was assumed that half of the observed galaxies could be composed of antimatter. But this environment quickly proved to be globally collisional on the scale of the age of the universe. In such a context, a single collision between a matter galaxy and an antimatter galaxy would have produced a detectable gamma-ray flux by our observation instruments. The absence of such detection led to the conclusion that, for an unknown reason, half of the universe had been lost.

Beyond this half-century, no model had been proposed to explain such a paradox until 1967, when Sakharov suggested that in what should be considered one of the two sheets of a twin universe, linked by the initial singularity of the Big Bang (Figure 2), the rate of production of baryons from quarks was lower than that of the production of antibaryons from antiquarks. Our observable universe would thus be composed, in addition to many photons from annihilations, of baryons and free antiquarks.

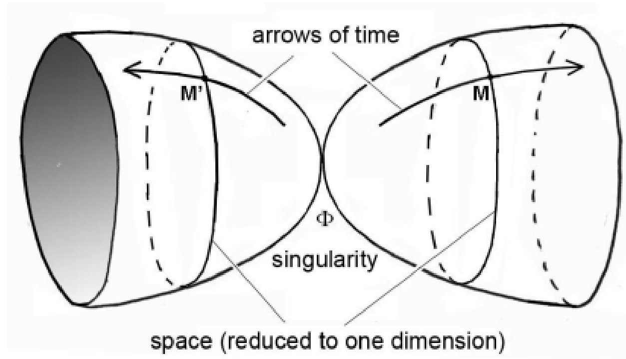

Figure 2: Sakharov Cosmological Model

For symmetry reasons, Sakharov postulated the existence of a twin universe where the situation was symmetrical: this universe contained photons from annihilations, as well as antibaryons and free quarks. For more than half a century, this model has remained the only contender. Sakharov also envisaged that this second universe would be symmetric to ours with $\boldsymbol{PT}$ symmetry, that is, enantiomorphic and with an opposing arrow of time. This twin universe would also be composed of antimatter, which suggests a $\boldsymbol{C}$ symmetry. ([19],[20],[21]).

In summary, these two universes are symmetric according to the $\boldsymbol{CPT}$ transformation. The Janus model, based on this group-theoretical approach, represents this $\boldsymbol{CPT}$ symmetry. Indeed, according to the construction around the complete Poincaré group, which includes its antichronous components, the time symmetry $\boldsymbol{T}$ automatically induces charge symmetry $\boldsymbol{C}$. In 1970, J.M. Souriau demonstrated through his theory of dynamical groups ([23]) that $\boldsymbol{T}$ symmetry led to the inversion of energy (see equation (14.67) of [23]) and mass (see equation (14.24) of [23]). Thus, according to the present work, which represents an extension of this theory, Sakharov's universe would include photons with negative energy, antimatter particles with negative mass, and a corresponding residue of quarks with negative energy. In the Sakharov model, the particles of the two universes do not interact. Whereas in the Janus model, these two sheets of the universe are folded onto each other, forming a structure akin to a covering (Figure 3).

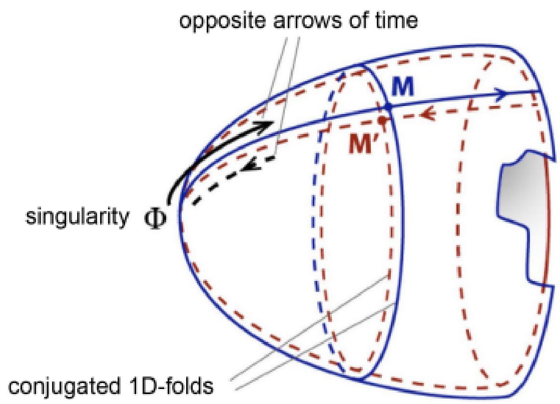

Figure 3: Janus Cosmological Model

Particles with opposite masses could then interact gravitationally. A subsequent article will present the system of coupled field equations that models this interaction. Currently, the standard model of cosmology fails to explain the recent data from the Hubble and James Webb space telescopes, creating a significant crisis among experts. The Janus model provides a solution to this crisis, requiring a change in geometric paradigm.

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