Contribution of the kinetic theory of gases to the dynamics of galaxies

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Classical attempts to construct a galaxy model, in a stationary and axisymmetric situation, consist of giving a gravitational field and injecting it into the collisionless Boltzmann equation to deduce the solution distribution function f. We will do exactly the opposite, by assimilating the galaxy to a self-gravitating point-mass system. The velocity distribution function is then the solution of an integrodifferential equation. Taking into account the Newtonian character of the potential, we can replace it with the system consisting of the Vlasov equation, written in terms of residual velocity, and the Poisson equation. We then give Log f the form of a polynomial of degree 2, such that one of the axes of the velocity ellipsoid points towards the center of the system. This single constraint gives the evolution of the axes in space, these being equal to the center of the galaxy (Maxwell-Boltzmann distribution). Moving away from the center, the axis pointing in this direction remains constant while the transverse axes tend to zero at infinity. We then construct the macroscopic velocity field by excluding any vortex structure. This field then tends towards a solid body rotation at the center. The velocity tends towards a remote plateau, which is then consistent with the observational data.

Keywords: Galactic dynamics, ellipsoid of velocities, Vlasov equation, elliptical solution, Evolution of the velocity at the periphery, confinement, dark halo, Janus cosmological Model

I. INTRODUCTIN

Today, there is an approach to dynamics that can be considered classic. It can be found in chapter 4 of J.Binney and S.Tremaine's basic work "galactic dynamics" [1]. Galaxies, considered as self-gravitating mass-point systems, are modeled using a velocity distribution function:

$$f(t, x, y, z, u, v, w) \equiv f(t, \mathbf{r}, \mathbf{v}) \tag{1}$$

where

$$\mathbf{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad and \qquad \mathbf{v} \equiv \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

In this classical approach, this function is then normalized according to :

$$\int f(t, \mathbf{r}, \mathbf{v}) d^3\mathbf{r} d^3\mathbf{v} = 1 \tag{2}$$

A function f is then assumed to obey the collisionless Boltzmann equation, which is written :

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \Phi}{\partial \mathbf{r}}.\frac{\partial f}{\partial \mathbf{v}} = 0 \tag{3}$$

where $\Phi(\mathbf{r})$ in an arbitrarily given potential field corresponding to the problem.

In what follows, we'll use skinny letters to denote scalars and bold letters to denote vectors and matrices. The approach then focuses on Jeans' theorem:

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Steady-state solutions of the Boltzmann collisionless equation depends on the phase space only through integrals of motion in a given potential, and any function of the integrals yields in a steady state solution of the collisionless Boltzmann equation.

Thus, the distribution function associated with the simplest model, that of the isothermal sphere, is presented as a special case of a polytropic model. In the treatment of this problem by S.Chandrasekhar [2], the approach is radically different, in the sense that this distribution function, spherically symmetric in velocity space, is presented as a special case of an elliptic function. In fact, the collisionless Boltzmann equation can be written in terms other than absolute velocity components:

$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{4}$$

We'll refer to ourselves as $\mathbf{C}$, the residual velocity:

$$\mathbf{C} = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \tag{5}$$

It's a velocity that kinetic gas theorists refer to as the "thermal agitation velocity". It's important to be clear about the notations we're going to use. Thus, the letter n designates the number of mass points per unit volume, deduced from the distribution function f by :

$$n(\mathbf{r}) = \int f(\mathbf{r}, \mathbf{v}) \, d^3\mathbf{v} \tag{6}$$

And not by $\int f d^3\mathbf{v} = 1$. Multiplying by the mass m of the components gives the mass density $\rho = nm$. The macroscopic velocity is :

$$\mathbf{c}_0 = \frac{1}{n(\mathbf{r})} \int \mathbf{v} \, f \, d^3\mathbf{v} \ \equiv <\mathbf{v}> \tag{7}$$

And residual velocity is defined by:

$$\mathbf{C} = \mathbf{v} - \mathbf{c}_0 \tag{8}$$

We will follow the approach chosen by S.Chandrasekhar in his book. Unfortunately, he never uses vector notation. So his collisionless Boltzmann equation becomes, by adopting our notations :

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial \Psi}{\partial \mathbf{x}}\frac{\partial f}{\partial u} - \frac{\partial \Psi}{\partial \mathbf{y}}\frac{\partial f}{\partial v} - \frac{\partial \Psi}{\partial \mathbf{z}}\frac{\partial f}{\partial w} = 0 \tag{9}$$

We can write it in a more compact way:

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \Psi}{\partial \mathbf{r}}.\frac{\partial f}{\partial \mathbf{v}} = 0 \tag{10}$$

Chandrasekhar then decided to concentrate on elliptic functions where $Logf$ is in the form of a polynomial of degree 3 according to the components U, V, W of the residual velocity. And he writes (with his notations):

$$Logf = aU^2 + bV^2 + cW^2 + 2fVW + 2gWU + 2hUV - 2\Delta_1 U - 2\Delta_2 V - 2\Delta_3 W - \chi \tag{11}$$

He therefore has ten unknown functions to determine. He then writes the Boltzmann collisionless equation in terms of the velocity of agitation (U, V, W). He then obtains a third-degree polynomial containing twenty terms. Since this equation must be independent of these residual velocity components, this leads him to a system of twenty second-order differential equations, the simple writing of which takes several pages. To this he adds the Poisson equation of the Newtonian gravitational potential:

$$\Delta\Psi = 4\pi G\rho \tag{12}$$

The construction of these elliptical solutions, where the logarithm of the distribution function is a polynomial of degree two as a function of the residual velocity components, and where the masses gravitate in their own gravitational field, therefore implies the construction of the solution emanating from a system of twenty-one non-linear differential equations. The potential being Newtonian, we derive the distribution function according to :

$$\Psi = Gm \int_x \frac{n(\mathbf{r})}{r} d^3r = Gm \int_x \frac{d^3r}{r} \int_v f(\mathbf{r}, \mathbf{v})\, d^3v \tag{13}$$

The gravitational field g, this time considered as created by the distribution function f , is then defined by:

$$\mathbf{g} = -\frac{\partial \Psi}{\partial r} = -Gm \frac{\partial}{\partial \mathbf{r}} \left[\int_x \frac{d^3\mathbf{r}}{r} \int_v f(\mathbf{r}, \mathbf{v})\, d^3v \right] \tag{14}$$

Mathematically speaking, we can summarize Chandrasekhar's approach by saying that he looked for functions f such that their logarithm is expressed as a polynomial of degree two as a function of the residual velocity, which is a solution of the integrodifferential equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{r}} - Gm \left(\frac{\partial}{\partial \mathbf{r}} \left[\int_x \frac{d^3\mathbf{r}}{r} \int_v f(\mathbf{r}, \mathbf{v}) d^3\mathbf{v} \right] \right) . \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{15}$$

The introduction of the solution, according to Chandrasekhar, in the form of a polynomial of degree 2, transforms the collisionless Boltzmann equation into a polynomial of degree three. If we restrict ourselves to a stationary solution :

- By cancelling out the degree-three terms, we deduce the evolution of the components of the velocity ellisoid.
- By cancelling out terms of degree two, we deduce the macroscopic velocity field.
- By cancelling out the unit-order terms, we deduce the potential in space and hence the density.
- When stationary, there are no zero-order terms.

At the end of this calculation, we don't have to ask whether the gravitational potential is compatible with such a solution, since it is an integral part of it. As we can see, Chandrasekhar's approach is the reverse of the conventional one. As early as the 1970s, we took up the question of how to construct these elliptical solutions, taking advantage of the computational techniques introduced in 1939 by S.Chapman and T.G.Cowling in their book "The Mathematical Theory of Non-Uniform Gases" [3]. At the time, their notations were remarkably elegant and compact. Scalars are indicated by thin letters, and vectors by bold ones. But they also introduced dyadics, which are simple matrices composed using vectors:

$$\mathbf{ab} = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix} \tag{16}$$

Gradient vectors are then themselves considered as dyadics, formed from the vector $\nabla = \dfrac{\partial}{\partial \mathbf{r}}$, being the position vector (x, y, z) and, for example, the gravitational potential Ψ. The scalar product of two vectors is always denoted $\mathbf{a}.\mathbf{b}$. But the authors introduce the scalar product of two dyadics noted $\mathbf{w} : \mathbf{w}'$ and defined by :

$$\mathbf{w} : \mathbf{w}' = \sum_\alpha \sum_\beta w_{\alpha\beta}\, w_{\beta\alpha} = \mathbf{w}' : \mathbf{w} \tag{17}$$

The product of a dyadic matrix and a vector is also noted:

$$(\mathbf{ab}).\mathbf{d} \qquad \textit{and} \qquad \mathbf{d}.(\mathbf{ab}) \tag{18}$$

It is then possible to use the set of theorems related to this algebra of dyadics, such as:

$$(\mathbf{ab}).\mathbf{d} = \mathbf{a}(\mathbf{b}.\mathbf{d}) \qquad\qquad \mathbf{d}(\mathbf{ab}) = (\mathbf{d}.\mathbf{a})\mathbf{b} \tag{19}$$

$$\mathbf{ab} : \mathbf{cd} = \mathbf{a}.(\mathbf{b}.\mathbf{cd}) = \mathbf{a}.\{(\mathbf{b}.\mathbf{c})\mathbf{d}\} = (\mathbf{a}.\mathbf{d})(\mathbf{b}.\mathbf{c}) \tag{20}$$

$$\mathbf{ab} : \mathbf{cd} = \mathbf{ac} : \mathbf{bd} \tag{21}$$

$\mathbf{w}$ being a dyadic :

$$\mathbf{b}.(\mathbf{a}.\mathbf{w}) = \mathbf{ba} : \mathbf{w} \tag{22}$$

The authors also introduce a “mobile operator”:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{c}_0.\frac{\partial}{\partial \mathbf{r}} \tag{23}$$

The tedious collisionless Boltzmann equation then becomes:

$$\frac{\partial \ln(f)}{\partial t} + \mathbf{c}_0.\frac{\partial \ln(f)}{\partial \mathbf{r}} + \mathbf{C}.\frac{\partial \ln(f)}{\partial \mathbf{r}} - \left(\frac{\partial \Psi}{\partial \mathbf{r}} + \mathbf{c}_0.\frac{\partial \mathbf{c}_0}{\partial \mathbf{r}}\right).\frac{\partial \ln(f)}{\partial \mathbf{C}} - \frac{\partial \ln(f)}{\partial \mathbf{C}}\mathbf{C} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} = 0 \tag{24}$$

In steady state:

$$\mathbf{c}_0.\frac{\partial \ln(f)}{\partial \mathbf{r}} + \mathbf{C}.\frac{\partial \ln(f)}{\partial \mathbf{r}} - \left(\frac{\partial \Psi}{\partial \mathbf{r}} + \frac{\partial \mathbf{c}_0}{\partial t} + \mathbf{c}_0.\frac{\partial \mathbf{c}_0}{\partial \mathbf{r}}\right).\frac{\partial \ln(f)}{\partial \mathbf{C}} - \frac{\partial \ln(f)}{\partial \mathbf{C}}\mathbf{C} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} = 0 \tag{25}$$

The terms $\frac{\partial \ln(f)}{\partial \mathbf{C}}\mathbf{C}$ and $\frac{\partial \mathbf{c}_0}{\partial \mathbf{r}}$ are dyadic matrices composed from two vectors. Note the remarkable compactness of this formulation. If we don’t use it, the last term of the equation is itself transformed into nine terms, etc. $\frac{\partial \mathbf{c}_0}{\partial \mathbf{r}}$ is therefore the velocity gradient matrix, not to be confused with its divergence, which is written as :

$$\frac{\partial}{\partial \mathbf{r}}.\mathbf{c}_0 \tag{26}$$

Macroscopic parameters are defined as stochastic quantities. We have already seen the density n, above, and the macroscopic velocity $\mathbf{c}_0$. We define the pressure matrix $\mathbf{p}$, which is a dyadic, as:

$$\mathbf{p} = m \int \mathbf{CC}\, f\, d^3\mathbf{v} = nm < \mathbf{CC} > \tag{27}$$

With the dyadic:

$$\mathbf{CC} = \begin{pmatrix} U^2 & UV & UW \\ UV & V^2 & VW \\ UW & VW & W^2 \end{pmatrix} \tag{28}$$

The matrix is symmetrical. There is a third macroscopic quantity which, in fluid mechanics, is the absolute temperature T , which is then defined as the mean value of the kinetic energy associated with the thermal agitation velocity:

$$\frac{3}{2}k_B T = \frac{1}{2}\rho < C^2 >= \frac{1}{2}m \int C^2 f d^3\mathbf{v} \tag{29}$$

k_B being Boltzmann’s constant. From the pressure matrix, we define a scalar pressure:

$$p = \rho\, tr < \mathbf{CC} >= \rho < C^2 >= nk_B T \tag{30}$$

And Boyle’s law comes into play again. We add the heat flux vector, defined as the transport of thermal agitation energy:

$$\mathbf{q} = \frac{1}{2}m \int C^2\, \mathbf{C}\, f\, d^3\mathbf{v} = \frac{1}{2}\,\rho\, < C^2\mathbf{C} > \tag{31}$$

Entropy is also defined as the mean value of the quantity $Logf$:

$$s = \frac{1}{n}\int f\ \ln(f)\ d^3\mathbf{v}\ = < \ln(f) > \tag{32}$$

The collisionless Boltzmann equation also provides the conservation equations, which are greatly simplified by the use of dyadics:

$$\frac{\partial \rho}{\partial t} + \mathbf{c}_0 . \frac{\partial \rho}{\partial \mathbf{r}} + \rho \frac{\partial}{\partial \mathbf{r}} . \mathbf{c}_0 = 0 \tag{33}$$

$$\frac{\partial}{\partial \mathbf{r}} . \mathbf{p} + \rho \left(\frac{\partial \Psi}{\partial \mathbf{r}} + \frac{\partial \mathbf{c}_0}{\partial t} + \mathbf{c}_0 . \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} \right) \tag{34}$$

$$\frac{\partial T}{\partial t} + \mathbf{c}_0 . \frac{\partial T}{\partial \mathbf{r}} + \frac{2}{3 k_B n} \left(\mathbf{p} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{r}} . \mathbf{q} \right) \tag{35}$$

The term $\mathbf{p} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}}$ represents the work of the pressure forces. Of course these equations simplify when the medium is in a state of local thermodynamic equilibrium, i.e. when the distribution function has Maxwellian form:

$$f^0 = n \left(\frac{m}{2 \pi k_B T} \right)^{3/2} e^{-\frac{m C^2}{2 k_B T}} \tag{36}$$

In other words, $\ln(f)$ is a spherical polynomial. The equations then become Euler's equations. If the medium is homogeneous and uniform, the unsteady solution gives the Friedman equation, and we find [4] the Newtonian cosmology discovered in 1934 by Milne and Mc Crea [5]. We will use this particular case of the Maxwellian solution, already treated by Chandrasekhar [2] to show the path followed when we center the solution on the simple form of the distribution function. We have :

$$\ln(f^0) = Cst - \frac{m C^2}{2 k_B T} + \ln\left(\frac{n}{T^{3/2}}\right) \tag{37}$$

We introduce this form of the distribution function into the collisionless Boltzmann equation in a stationary situation. The equation becomes:

$$(\mathbf{C} + \mathbf{c}_0) . \left(\frac{m C^2}{2 k_B T} \frac{\partial T}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{r}} \ln\left(\frac{n}{T^{3/2}}\right) \right) + \frac{m \mathbf{C}}{k_B T} . \left(\frac{\partial \Psi}{\partial \mathbf{r}} + \mathbf{c}_0 . \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} \right) + \frac{m}{k_B T} \mathbf{C}\mathbf{C} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} = 0 \tag{38}$$

We have a single third-order term:

$$\frac{m C^2}{2 k_B T} \left(\mathbf{C} . \frac{\partial T}{\partial \mathbf{r}} \right) = 0 \tag{39}$$

The medium is therefore isothermal. Transposed to astrophysics, this means that the dispersion of mass-point velocities is constant throughout space. We now turn to second-order terms. In dyadic algebra, this is written :

$$\mathbf{C}\mathbf{C} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} = 0 \tag{40}$$

By expanding, we have nine terms. The solution to this system of differential equations is:

$$\mathbf{c}_0 = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} \tag{41}$$

Assuming $\mathbf{v}_0 = 0$, we obtain a solid-body rotation with constant angular velocity ω . The unit order terms give :

$$\frac{\partial \ln(n)}{\partial \mathbf{r}} + \frac{m}{k_B T} \left(\frac{\partial \Psi}{\partial \mathbf{r}} + \mathbf{c}_0 . \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} \right) = 0 \tag{42}$$

Equation which states that the gravitational field is balanced by the combination of pressure force plus centrifugal force. We have the following equation:

$$\mathbf{c}_0 = \omega \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{43}$$

Let's introduce the vector:

$$\mathbf{u} = \omega \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \tag{44}$$

We get :

$$\frac{\partial}{\partial \mathbf{r}} \cdot \left(\ln(n) + \frac{m}{k_B T} \left(\Psi - \frac{1}{2} u^2 \omega^2 \right) \right) \tag{45}$$

Whence:

$$n = n_0 \; e^{\frac{m}{k_B T} \Psi + \frac{m}{2 k_B T} u^2 \omega^2} \tag{46}$$

Combining with the Poisson equation:

$$\Delta \Psi = 4 \, \pi \, G \, n_0 \; e^{\frac{m}{k_B T} \Psi + \frac{m}{2 k_B T} u^2 \omega^2} \tag{47}$$

From which we derive the gravitational potential and mass distribution.

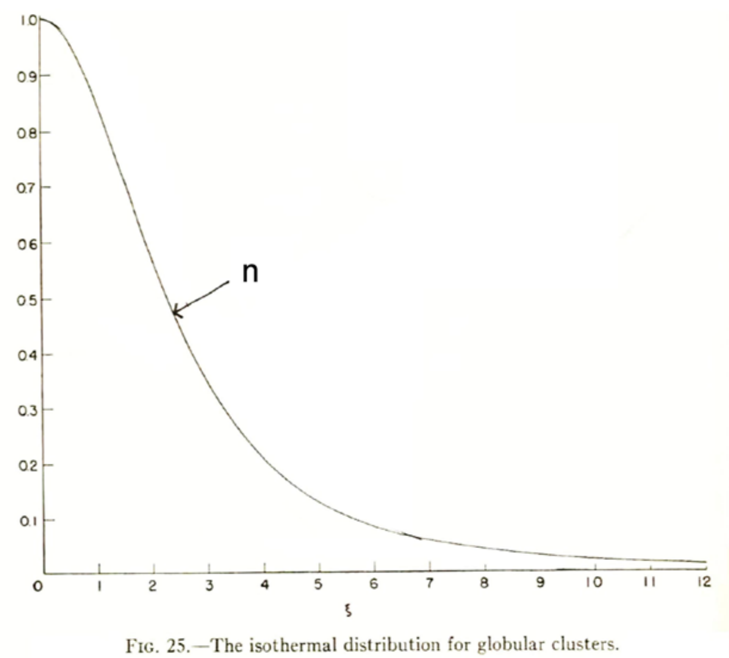

FIG. 1. Mass distribution of an isothermal, spheroidal formation, according to Chandrasekhar

Which is nothing other than the result obtained by Chandrasekhar, with a heavier handwriting. After this long preamble, we are now in a position to understand the strategy of the present article, whose project is to construct a model of a galaxy as an exact stationary solution of the collisionless Boltzmann equation, having as its sole starting point the form of the distribution function, and exploiting the computational technique of dyadic algebra. Under these conditions, $\ln(f)$ becomes a particular polynomial of order two, depending on the components (U, V, W) of the residual velocity. By introducing this expression into the equation written in terms of the components of the residual velocity, and setting its various polynomials to zero, we obtain a system of twenty non-linear differential equations, to which Poisson's equation is added. As with the Maxwellian solution, the third-order terms will give the evolution of the velocity ellipsoid in space. The second-order terms will give the velocity field and the first-order terms, combined with the Poisson equation, the gravitational potential and mass distribution. In stationary mode, there are no zero-order terms. So there's no need to ask whether this potential is compatible with this distribution function, since it follows from it as an integral part of a solution, based solely on the form of this function.

II. EXTENSION TO A STEADY AXISYMMETRIC ELLIPTICAL SOLUTION

The hypothesis that the logarithm of the distribution function is a polynomial of degrees 2 depending on the components of the residual velocity, such that one of the axes of the ellipsoid points towards the origin was proposed in 1972 by J.P.Petit [6], then taken up again in 1974 in reference [7]. In this model one of the principal axes of the velocity ellipsoid points towards the geometric center of the system, which corresponds to figure 2:

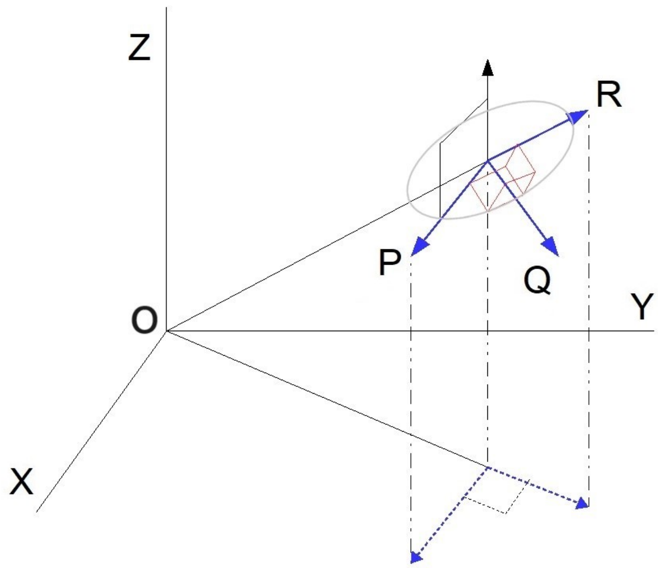

FIG. 2. One of the axes of the velocity ellipsoid pointing towards the geometric center

In this figure the vector $\mathbf{P}$ is parallel to the plane $(z = 0)$. We can define speed dispersions according to:

$$\begin{aligned} \sigma_P &= \langle\, C_P^2 \,\rangle \\ \sigma_Q &= \langle\, C_Q^2 \,\rangle \\ \sigma_R &= \langle\, C_R^2 \,\rangle \end{aligned} \tag{48}$$

Still following reference [6]:

$$\ln(f) = \ln(B) - \frac{m}{2k_B H}\mathbf{C}^2 + a\,(\mathbf{C}.\mathbf{r})^2 + \alpha\,[\mathbf{C}.(\mathbf{k}\times\mathbf{r})]^2 \tag{49}$$

The functions B, H, α, a are functions of the position and will then have to be determined. We only have data referring to the velocity ellipsoid in the vicinity of the Sun. Its major axis then does not point towards the center but in its vicinity. We will attribute this small deviation, called the vertex deviation, to the influence of the local fluctuation of the gravitational field linked to the presence of the spiral arm. When we introduce the form of the distribution function (49) into equation (25), the latter becomes a polynomial of degree 3 as a function of the components (U, V, W) of the residual velocity. Since this equation must be satisfied regardless of these components, we then obtain a system of twenty partial differential equations to which we must add Poisson's equation (12).

III. THE TERMS OF DEGREE THREE GIVE THE EVOLUTION OF THE VELOCITY ELLIPSOID [4]

Let's group these terms together:

$$-\frac{m}{2k_B}C^2C.\frac{\partial}{\partial r}(\frac{1}{H}) + 2a(C.r).C.C + C.\frac{\partial a}{\partial r}(C.r)^2 + C.\frac{\partial \alpha}{\partial r}[C.(k\times r)]^2 = 0 \tag{50}$$

$$
\begin{aligned}
&-\frac{m}{2k_B}\left(C_x^2+C_y^2+C_z^2\right)\left(C_x\frac{\partial}{\partial x}\left(\frac{1}{H}\right)+C_y\frac{\partial}{\partial y}\left(\frac{1}{H}\right)+C_z\frac{\partial}{\partial z}\left(\frac{1}{H}\right)\right)\\
&+2a(C_x^2+C_y^2+C_z^2)(xC_x+yC_y+zC_z)\\
&+\left(C_x\frac{\partial a}{\partial x}+C_y\frac{\partial a}{\partial y}+C_z\frac{\partial a}{\partial z}\right)(xC_x+yC_y+zC_z)^2\\
&+\left(C_x\frac{\partial \alpha}{\partial x}+C_y\frac{\partial \alpha}{\partial y}+C_z\frac{\partial \alpha}{\partial z}\right)(-yC_x+xC_y)^2=0
\end{aligned}
\tag{51}
$$

Whence :

$$C_x^3:-\frac{m}{2k_B}\frac{\partial}{\partial x}\left(\frac{1}{H}\right)+2ax+x^2\frac{\partial a}{\partial x}+y^2\frac{\partial \alpha}{\partial x}=0 \tag{52}$$

$$C_y^3:-\frac{m}{2k_B}\frac{\partial}{\partial y}\left(\frac{1}{H}\right)+2ay+y^2\frac{\partial a}{\partial y}+x^2\frac{\partial \alpha}{\partial y}=0 \tag{53}$$

$$C_z^3:-\frac{m}{2k_B}\frac{\partial}{\partial z}\left(\frac{1}{H}\right)+2az+z^2\frac{\partial a}{\partial z}=0 \tag{54}$$

$$C_x^2C_y:-\frac{m}{2k_B}\frac{\partial}{\partial y}(\frac{1}{H})+2ay+x^2\frac{\partial a}{\partial y}+2xy\frac{\partial a}{\partial x}+y^2\frac{\partial \alpha}{\partial y}-2xy\frac{\partial \alpha}{\partial x}=0 \tag{55}$$

$$C_y^2C_x:-\frac{m}{2k_B}\frac{\partial}{\partial x}\left(\frac{1}{H}\right)+2ax+y^2\frac{\partial a}{\partial x}+2xy\frac{\partial a}{\partial y}+x^2\frac{\partial \alpha}{\partial x}-2xy\frac{\partial \alpha}{\partial y}=0 \tag{56}$$

$$C_x^2C_z:-\frac{m}{2k_B}\frac{\partial}{\partial z}\left(\frac{1}{H}\right)+2az+x^2\frac{\partial a}{\partial z}+2xz\frac{\partial a}{\partial x}+y^2\frac{\partial \alpha}{\partial z}=0 \tag{57}$$

$$C_y^2C_z:-\frac{m}{2k_B}\frac{\partial}{\partial z}\left(\frac{1}{H}\right)+2az+y^2\frac{\partial a}{\partial z}+2yz\frac{\partial a}{\partial x}+x^2\frac{\partial \alpha}{\partial z}=0 \tag{58}$$

$$C_z^2C_x:-\frac{m}{2k_B}\frac{\partial}{\partial x}\left(\frac{1}{H}\right)+2ax+z^2\frac{\partial a}{\partial x}+2xz\frac{\partial a}{\partial z}=0 \tag{59}$$

$$C_z^2C_y:-\frac{m}{2k_B}\frac{\partial}{\partial y}\left(\frac{1}{H}\right)+2ay+z^2\frac{\partial a}{\partial y}+2yz\frac{\partial a}{\partial z}=0 \tag{60}$$

$$C_xC_yC_z:2yZ\frac{\partial a}{\partial x}+2xz\frac{\partial a}{\partial y}+2xy\frac{\partial a}{\partial z}-2xy\frac{\partial \alpha}{\partial z} \tag{61}$$

Assuming that a and α do not depend on r or z, we get [8]:

$$C_x^3\equiv C_z^2C_x\equiv C_y^2C_x:-\frac{m}{2k_B}\frac{\partial}{\partial x}\left(\frac{1}{H}\right)+2ax=0 \tag{62}$$

$$C_y^3\equiv C_x^2C_y\equiv C_z^2C_y:-\frac{m}{2k_B}\frac{\partial}{\partial y}\left(\frac{1}{H}\right)+2ay=0 \tag{63}$$

$$C_z^3\equiv C_x^2C_z\equiv C_y^2C_z:-\frac{m}{2k_B}\frac{\partial}{\partial z}\left(\frac{1}{H}\right)+2az=0 \tag{64}$$

In addition we have:

$$\rho^2 = x^2 + y^2 \Rightarrow \begin{cases} \dfrac{\partial \rho^2}{\partial x} = 2x \\ \dfrac{\partial \rho^2}{\partial y} = 2y \end{cases} \tag{65}$$

We get :

$$C_x^3 \equiv C_z^2 C_x \equiv C_y^2 C_x : -\frac{m}{2k_B}\frac{\partial}{\partial \rho^2}\left(\frac{1}{H}\right) + a = 0 \tag{66}$$

$$C_y^3 \equiv C_x^2 C_y \equiv C_z^2 C_y : -\frac{m}{2k_B}\frac{\partial}{\partial \rho^2}\left(\frac{1}{H}\right) + a = 0 \tag{67}$$

$$C_z^3 \equiv C_x^2 C_z \equiv C_y^2 C_z : -\frac{m}{2k_B}\frac{\partial}{\partial z^2}\left(\frac{1}{H}\right) + a = 0 \tag{68}$$

That is, after integration:

$$\frac{\partial}{\partial \rho^2}\left(\frac{1}{H}\right) = \frac{m}{2k_B}a \Rightarrow \frac{1}{H} = \frac{m}{2k_B}a\rho^2 + f_1(z^2) \tag{69}$$

$$\frac{\partial}{\partial z^2}\left(\frac{1}{H}\right) = \frac{m}{2k_B}a \Rightarrow \frac{1}{H} = \frac{m}{2k_B}az^2 + f_2(\rho^2) \tag{70}$$

The function f_1 depends only on z^2. So, if we differentiate (69) with respect to z^2 we can write:

$$\frac{\partial}{\partial z^2}\left(\frac{m}{2k_B}a\rho^2 + f_1(z^2)\right) = \frac{\partial}{\partial z^2}f_1(z^2) = \frac{\partial}{\partial z^2}\left(\frac{1}{H}\right) = \frac{m}{2k_B}a \tag{71}$$

Thus :

$$f_1(z^2) = \frac{m}{2k_B}a\rho^2 + k_z \tag{72}$$

Whence :

$$\frac{1}{H} = \frac{m}{2k_B}a\rho^2 + \frac{m}{2k_B}az^2 + k_z \tag{73}$$

Originally we have $\dfrac{1}{H} = k_z$ that we set equal to the constant $\dfrac{1}{H} = \dfrac{1}{T_0}$ which gives:

$$\frac{1}{H} = \frac{m}{2k_B}a\rho^2 + \frac{m}{2k_B}az^2 + \frac{1}{T_0} \tag{74}$$

By introducing the characteristic length r_o we get:

$$r_0^2 = \frac{m}{2ak_BT_0} \Rightarrow H = \frac{T_0}{1 + \dfrac{r^2}{r_0^2}} \tag{75}$$

We will now decompose the residual velocity vector C according to its projections on the P, Q, R axes.

$$C_r = C.R = C.\frac{r}{\|r\|} = \frac{C.r}{\|r\|} \tag{76}$$

$$C_p = C.P = C.\frac{k \times r}{\|k \times r\|} = \frac{C.(k \times r)}{\|k \times r\|} \tag{77}$$

$$C_q = C - C_r - C_p \tag{78}$$

$$\|k \times r\|^2 = (|k \times r).(|k \times r) = (k.k)(r.r) - (k.r)(k.r) = r^2 - z^2 = \rho^2 \tag{79}$$

$$\ln(f) = \ln(B) + a_r C_r^2 + a_p C_p^2 + a_q C_q^2 \tag{80}$$

$$\ln(f) = \ln(B) + \frac{(a_r - a_q)}{r^2}(C.r)^2 + \frac{(a_p - a_q)}{\rho^2}[C.(k \times r)]^2 + a_q C^2 \tag{81}$$

By identifying:

$$a_q = -\frac{m}{2k_B H} \tag{82}$$

$$a = \frac{(a_r - a_q)}{r^2} \rightarrow a_q = -\frac{m}{2k_B H} + ar^2 \tag{83}$$

$$\alpha = \frac{(a_p - a_q)}{\rho^2} \rightarrow a_p = -\frac{m}{2k_B H} + a\rho^2 \tag{84}$$

Let's ask :

$$\rho_0^2 = \frac{m}{2\alpha k_B T_0} \tag{85}$$

We get :

$$a_r = -\frac{m}{2k_B T_0} \tag{86}$$

$$a_p = -\frac{m}{2k_B T_0}\left(1 + \frac{r^2}{r_0^2} - frac\rho^2\rho_0^2\right) \tag{87}$$

$$a_q = -\frac{m}{2k_B T_0}\left(1 + \frac{r^2}{r_0^2}\right) \tag{88}$$

We can then express the velocity distribution function as a function of the coordinates (C_r, C_p, C_q) :

$$f = f_0 \; e^{\left(-\frac{m}{2k_B T_0}\left[C_r^2 + C_p^2\left(1 + \frac{r^2}{r_0^2} - \frac{\rho^2}{\rho_0^2}\right) + C_q^2\left(1 + \frac{r^2}{r_0^2}\right)\right]\right)} \tag{89}$$

with :

$$f_0 = n\left(\frac{m}{2\pi k_B T_0}\right)^{\frac{3}{2}}\left(1 + \frac{r^2}{r_0^2}\right)^{\frac{1}{2}}\left(1 + \frac{r^2}{r_0^2} - \frac{\rho^2}{\rho_0^2}\right)^{\frac{1}{2}} \tag{90}$$

Which gives us the axes of the velocity ellipsoid:

$$\sigma_r = \sqrt{\frac{2k_B T_0}{m}} \tag{91}$$

$$\sigma_p = \sqrt{\frac{2k_B T_0}{m}\frac{1}{1 + \frac{r^2}{r_0^2} - \frac{\rho^2}{\rho_0^2}}} \tag{92}$$

$$\sigma_p = \sqrt{\frac{2k_BT_0}{m}\frac{1}{1+\frac{r^2}{r_0^2}}} \tag{93}$$

We see that:

$$\sigma_p \leq \sigma_q \leq \sigma_r \tag{94}$$

At the center of the system, the distribution of Maxwellian speeds:

$$\sigma_p = \sigma_q = \sigma_r \tag{95}$$

The component σ_r is independent of r. If we want to take into account the observational data, these give $\sigma_p \approx \sigma_q$. This will mean, in our solution, that $\frac{1}{\rho_0^2} \ll \frac{1}{r_0^2}$ i.e. $\alpha \ll a$. At large distance the transverse axes tend towards zero, while the radial axis retains its value.

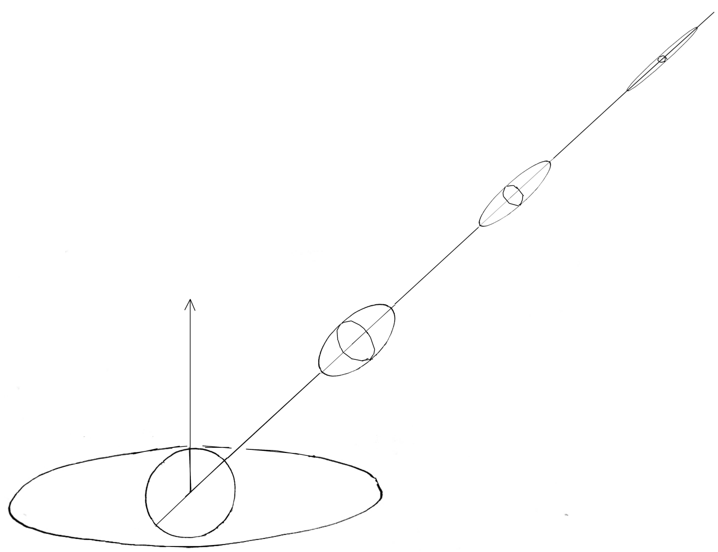

FIG. 3. Evolution of the axes of the velocity ellipsoid

In doing so, we are only repeating identically the calculations of references [6] and [7]. In 2016 reference [8] introduced a symmetry that had escaped the authors, and which leads to an interesting result.

IV. THE TERMS OF ORDER 2 PROVIDE THE MACROSCOPIC VELOCITY FIELD [8]

These are the terms from the expression:

$$\mathbf{c}_0.\frac{\partial \ln(f)}{\partial \mathbf{r}} - \frac{\partial \ln(f)}{\partial \mathbf{C}}\mathbf{C} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} = 0 \tag{96}$$

Mathematically, the velocity field, resulting from hypothesis (48) must emanate from equation (96). In all cases, the gradient $\frac{\partial \ln(f)}{\partial \mathbf{r}}$, located in a plane containing oz, due to the axisymmetry, contains a priori components parallel and perpendicular to the oz axis. Similarly, there should be components $\mathbf{c}_{0r}$ and $\mathbf{c}_{0z}$ also located in a plane passing through this axis. But the non-zero nature of these components does not seem compatible with a physical situation. Indeed, these two components, depending on their sign, translate movements of distance and approach with respect to, respectively, the oz and or axes. Movements that cannot correspond, exclusively, either to the distance or to the approach, because this would translate either to a loss or to a gain of matter. There would therefore be no conservation of mass. These components must therefore, in this velocity field, have values of both signs. The only velocity field ensuring the conservation of mass corresponds to 3D helical trajectories falling on a family of surfaces having the topology of nested tori.

If we opt for the assumption that this term is non-zero, we're out of the realm of galactic dynamics. The solution, with its macroscopic velocity field in the form of helical trajectories inscribed on a family of surfaces with the oz axis as the axis of symmetry and the horizontal plane as the plane of symmetry, then refers to the trajectories of electrically charged particles in a tokamak (we would then have to consider two Vlasov equations, one for electrons and one for hydrogen ions). The law of velocity variation with distance from the axis would then follow a different law. But these are two completely different physics problems.

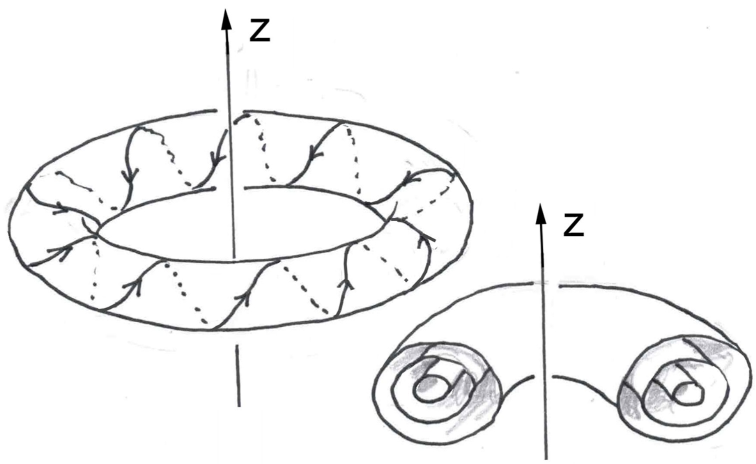

FIG. 4. 3D velocity fields on a family of surfaces with nested torus topology

The projection of these trajectories along the plane passing through the oz axis corresponds to the figure 5.

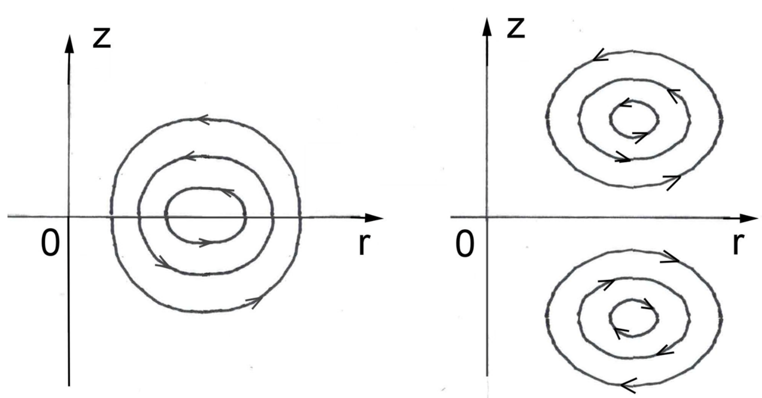

FIG. 5. Projection of velocity fields with non-zero $\mathbf{c}_{0z}.\frac{\partial \ln(f)}{\partial \mathbf{r}}$

But if we add the natural hypothesis of a symmetry with respect to the plane $z = 0$ this excludes the left field. Continuing to consider the existence of non-zero $\mathbf{c}_{0r}$ and $\mathbf{c}_{0z}$ components we should then consider the structure of the figure on the right. But, a simple physicist's intuition suggests that the most plausible solution would be the one where these components $\mathbf{c}_{0r}$ and $\mathbf{c}_{0z}$ are simply zero. In these conditions the term $\mathbf{c}_0.\nabla_r \ln(f)$ becomes zero and the equation with the terms of order two is reduced to:

$$\frac{\partial \ln(f)}{\partial \mathbf{C}}.\mathbf{C} : \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} = Tr(\mathbf{AB}) = 0 \tag{97}$$

Let's calculate the dyadic $\mathbf{B}$:

$$\mathbf{B} = \frac{\partial \mathbf{c}_0}{\partial \mathbf{r}} = \begin{pmatrix} \frac{\partial c_{0x}}{\partial x} & \frac{\partial c_{0y}}{\partial x} & 0 \\ \frac{\partial c_{0x}}{\partial y} & \frac{\partial c_{0y}}{\partial y} & 0 \\ \frac{\partial c_{0x}}{\partial z} & \frac{\partial c_{0y}}{\partial z} & 0 \end{pmatrix} = \begin{pmatrix} -y\frac{\partial \omega}{\partial x} & x\frac{\partial \omega}{\partial x} + \omega & 0 \\ -y\frac{\partial \omega}{\partial y} - \omega & x\frac{\partial \omega}{\partial y} & 0 \\ -y\frac{\partial \omega}{\partial z} & x\frac{\partial \omega}{\partial z} & 0 \end{pmatrix} \tag{98}$$

Let's calculate the dyadic $\mathbf{A}$:

$$A = \frac{\partial \ln(f)}{\partial \mathbf{C}}.C \tag{99}$$

$$\begin{aligned} A &= -\frac{m}{2k_B H}C.C + 2a(C.r).r.C + 2\alpha[C.(k\times r)].(k\times r) \\ &= A_1 + A_2 + A_3 \end{aligned} \tag{100}$$

$$A_1 = -\frac{m}{2k_B H}\begin{pmatrix} C_x^2 & C_xC_y & C_xC_z \\ C_yC_x & C_y^2 & C_yC_z \\ C_zC_x & C_zCy & C_z^2 \end{pmatrix} \tag{101}$$

$$A_2 = 2a(xC_x + yC_y + zC_z)\begin{pmatrix} xC_x & xC_y & xC_z \\ yC_x & yC_y & yC_z \\ zC_x & zC_y & zC_z \end{pmatrix} \tag{102}$$

$$\begin{aligned} A_3 &= 2\alpha\begin{pmatrix} y^2C_x - xyC_y \\ -xyC_x + x^2C_y \\ 0 \end{pmatrix}(C_xC_yC_z) \\ &= 2\alpha\begin{pmatrix} y^2C_xC_x - xyC_yC_x & y^2C_xC_y - xyC_yC_y & y^2C_xC_z - xyC_yC_z \\ -xyC_xC_x + x^2C_yC_x & -xyC_xC_y + x^2C_yC_y & -xyC_xC_z + x^2C_yC_z \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{103}$$

Consider the following two matrices

$$\mathbf{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} \quad and \quad \mathbf{B} = \begin{pmatrix} B_{xx} & B_{xy} & B_{xz} \\ B_{yx} & B_{yy} & B_{yz} \\ B_{zx} & B_{zy} & B_{zz} \end{pmatrix} \tag{104}$$

The trace of their matrix product is given by

$$Tr(\mathbf{AB}) = (A_{xx}B_{xx} + A_{xy}B_{yx} + A_{xz}B_{zx}) + (A_{yx}B_{xy} + A_{yy}B_{yy} + A_{yz}B_{zy}) + (0) \tag{105}$$

Let's calculate each of the terms:

$$\begin{aligned} A_{xx} &= -\frac{m}{2k_B H}C_x^2 + 2a(x^2C_x^2 + yxC_xC_y + zxC_xC_z) + 2\alpha(y^2C_x^2 - xyC_yC_x) \\ &= C_x^2\left(-\frac{m}{2k_B H} + 2ax^2 + 2\alpha y^2\right) + C_xC_y2(a-\alpha)xy + C_xC_z2axz \end{aligned} \tag{106}$$

$$\begin{aligned} A_{yy} &= -\frac{m}{2k_B H}C_y^2 + 2a(xC_xyC_y + yC_yyC_y + zC_zyC_y) + 2\alpha(-xyC_xCy + x\text{š}C_yC_y) \\ &= C_y^2\left(-\frac{m}{2k_B H} + 2ay^2 + 2\alpha x^2\right) + C_xC_y2(a-\alpha)xy + C_yC_z2ayz \end{aligned} \tag{107}$$

$$\begin{aligned} A_{xy} &= -\frac{m}{2k_B H}C_xC_y + 2a(xC_xxC_y + yC_yxC_y + zC_zxC_y) + 2\alpha(y^2C_xCy - xyC_yC_y) \\ &= C_xC_y\left(-\frac{m}{2k_B H} + 2\alpha x^2 + 2\alpha y^2\right) + C_y^22(a-\alpha)xy + C_yC_z2axz \end{aligned} \tag{108}$$

$$\begin{aligned} A_{yx} &= -\frac{m}{2k_BH}C_yC_x + 2a(xC_xyC_x + yC_yyC_x + zC_zyC_x) + 2\alpha(-xyC_xC_x + x^2C_yCx) \\ &= C_x^2 2(a-\alpha)xy + C_xC_y\left(-\frac{m}{2k_BH} + 2\alpha y^2 + 2\alpha x^2\right) + C_xC_z 2ayz \end{aligned} \tag{109}$$

$$\begin{aligned} A_{xz} &= -\frac{m}{2k_BH}C_xC_z + 2a(xC_xxC_z + yC_yxC_z + zC_zxC_z) + 2\alpha(y^2C_xCz - xyC_yC_z) \\ &= C_z^2 2axz + C_xC_z\left(-\frac{m}{2k_BH} + 2ax^2 + 2\alpha y^2\right) + C_yC_z 2(a-\alpha)xy \end{aligned} \tag{110}$$

$$\begin{aligned} A_{yz} &= -\frac{m}{2k_BH}C_yC_z + 2a(xC_xyC_z + yC_yyC_z + zC_zyC_z) + 2\alpha(-xyC_xCz + x^2C_yC_z) \\ &= C_z^2 2axz + C_xC_z 2(a-\alpha)xy + CyCz\left(-\frac{m}{2k_BH} + 2ay^2 + 2\alpha x^2\right) \end{aligned} \tag{111}$$

The terms C_x^2 come from $A_{xx}B_{xx}$ and from $A_{yx}B_{xy}$

$$\left(-\frac{m}{2k_BH} + 2ax^2 + 2\alpha y^2\right)\left(-y\frac{\partial\omega}{\partial x}\right) + 2(a-\alpha)xy\left(x\frac{\partial\omega}{\partial x} + \omega\right) \tag{112}$$

The terms C_y^2 come from $A_{xy}B_{yx}$ and from $A_{yy}B_{yy}$

$$\left(-\frac{m}{2k_BH} + 2ay^2 + 2\alpha x^2\right)\left(x\frac{\partial\omega}{\partial y}\right) + 2(a-\alpha)xy\left(-y\frac{\partial\omega}{\partial y} - \omega\right) \tag{113}$$

The terms C_z^2 come from $A_{xz}B_{zx}$ and from $A_{yz}B_{zy}$

$$2axz\left(-y\frac{\partial\omega}{\partial z}\right) + 2ayz\left(x\frac{\partial\omega}{\partial z}\right) \tag{114}$$

The terms C_xC_y come from $A_{xy}B_{yx}, A_{xx}B_{xx}, A_{yx}Bxy, A_{yy}B_{yy}$

$$\begin{aligned} &\left(-\frac{m}{2k_BH} + 2ax^2 + 2\alpha y^2\right)\left(-y\frac{\partial\omega}{\partial y} - \omega\right) + 2(a-\alpha)xy\left(-y\frac{\partial\omega}{\partial x}\right) \\ &+ \left(-\frac{m}{2k_BH} + 2ay^2 + 2\alpha x^2\right)\left(+x\frac{\partial\omega}{\partial x} + \omega\right) + 2(a-\alpha)xy\left(+x\frac{\partial\omega}{\partial y}\right) = 0 \end{aligned} \tag{115}$$

The terms C_xC_z come from $A_{xx}B_{zx}, A_{xx}B_{xx}, A_{yx}Bxy, A_{yz}B_{yz}$

$$\begin{aligned} &(2axz)\left(-y\frac{\partial\omega}{\partial x}\right) + \left(-\frac{m}{2k_BH} + 2ax^2 + 2\alpha y^2\right)\left(-y\frac{\partial\omega}{\partial z}\right) \\ &+(2axy)\left(x\frac{\partial\omega}{\partial x} + \omega\right) + 2(a-\alpha)xy\left(x\frac{\partial\omega}{\partial z}\right) = 0 \end{aligned} \tag{116}$$

The terms C_yC_z come from $A_{xy}B_{yx}, A_{xz}B_{zx}, A_{yy}Byy, A_{yz}B_{zy}$

$$\begin{aligned} &(2axz)\left(-y\frac{\partial\omega}{\partial x} - \omega\right) + 2(a-\alpha)xy\left(-y\frac{\partial\omega}{\partial z}\right) \\ &+(2ayz)\left(x\frac{\partial\omega}{\partial y}\right) + \left(-\frac{m}{2k_BH} + 2ay^2 + 2\alpha x^2\right)\left(x\frac{\partial\omega}{\partial z}\right) = 0 \end{aligned} \tag{117}$$

Let us now exploit the fact that ω only depends on ρ^2 and z^2. Thus, from (65) we obtain:

$$\frac{\partial\omega}{\partial x} = \frac{\partial\omega}{\partial\rho^2}\frac{\partial\rho^2}{\partial x} = 2x\frac{\partial\omega}{\partial\rho^2} \tag{118}$$

$$\frac{\partial\omega}{\partial y} = \frac{\partial\omega}{\partial\rho^2}\frac{\partial\rho^2}{\partial y} = 2y\frac{\partial\omega}{\partial\rho^2} \tag{119}$$

$$\frac{\partial\omega}{\partial z} = \frac{\partial\omega}{\partial z^2}\frac{\partial z^2}{\partial z} = 2z\frac{\partial\omega}{\partial z^2} \tag{120}$$

The equation C_x^2 becomes:

$$\frac{\partial\ln(\omega)}{\partial\rho^2} = -\frac{(a-\alpha)}{\left(\frac{m}{k_BH} - 2\alpha\rho^2\right)}) \tag{121}$$

Placing ourselves in the particular context where a and α are constant throughout space, we obtain:

$$\begin{aligned}\frac{\partial\ln(\omega)}{\partial\rho^2} &= -\frac{1}{2}\frac{2(a-\alpha)}{\left(\frac{m}{k_BT_0} - 2(a-\alpha)\rho^2 + 2az^2\right)} \\ &= -\frac{1}{2}\frac{\partial}{\partial\rho^2}\left[ln\left(\frac{m}{k_BT_0} + 2(a-\alpha)\rho^2 + 2az^2\right)\right]\end{aligned} \tag{122}$$

Whence:

$$\omega = \frac{\omega_{\rho_0}(z^2)}{\sqrt{\frac{m}{k_BT_0} + 2(a-\alpha)\rho^2 + 2az^2)}} \tag{123}$$

On the same way as before, the equations C_xC_z give

$$\frac{\partial ln\omega}{\partial z^2} = -\frac{a}{\left(\frac{m}{k_BH} - 2\alpha\rho^2\right)} \tag{124}$$

By taking up the hypothesis that a and α are constant , we get :

$$\omega = \frac{\omega_{z_0}(\rho^2)}{\sqrt{\frac{m}{k_BT_0} + 2(a-\alpha)\rho^2 + 2az^2)}} \tag{125}$$

Which gives the rotation speed at a point:

$$v = \rho.\omega = \rho.\frac{\omega_0}{\sqrt{\frac{m}{k_BT_0} + 2(a-\alpha)\rho^2 + 2az^2)}} \tag{126}$$

In the diametrical plane of the galaxy:

$$v = \frac{\omega_0\ \rho}{\sqrt{\frac{m}{k_BT_0} + 2(a-\alpha)\rho^2)}} \tag{127}$$

Taking into account that $\alpha \ll a$

$$v \approx \frac{\omega_0\ \rho}{\sqrt{\frac{m}{k_BT_0} + 2a\rho^2)}} \tag{128}$$

Which gives a linear growth near the center of symmetry (solid body rotation) and a constant speed plateau at the periphery:

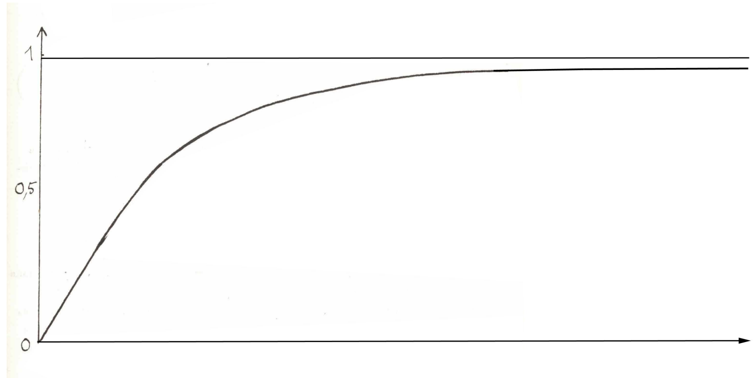

FIG. 6. Rotation curve [8]

V. COMPARISON WITH OBSERVATIONAL DATA

While the curves derived from observational data do show a velocity plateau at the periphery, their behavior near the center often includes a velocity peak. This is only absent in relatively rare cases, such as the NGC 128 galaxy.

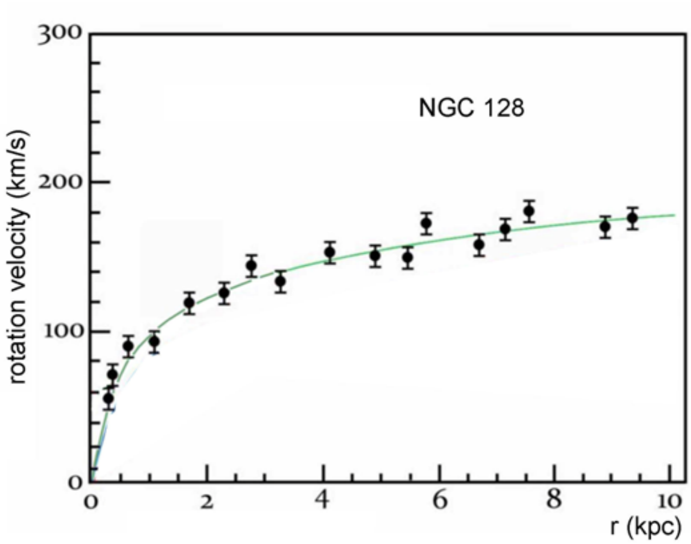

FIG. 7. Velocity curve of galaxy NGC 128

In contrast, rotation curves near the center of galaxies are generally very irregular.

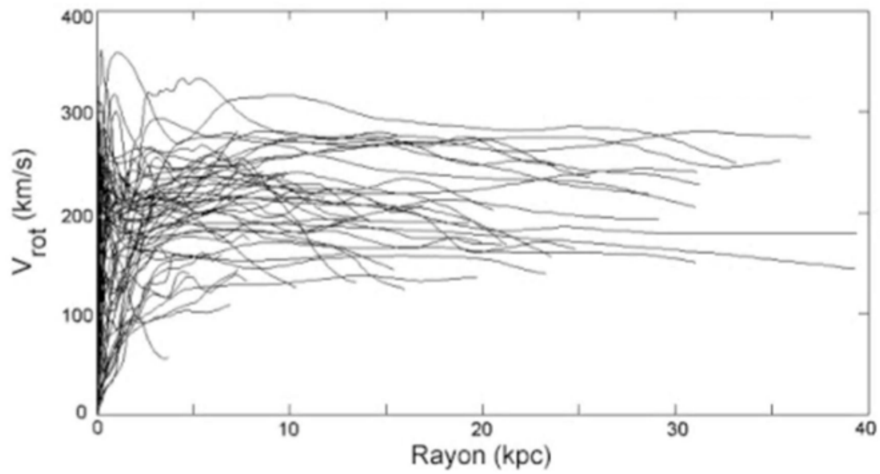

FIG. 8. Rotation curves of galaxies

Those who originally interpret these overspeed as being due to the action of a dark matter halo are then obliged to endow this distribution with a peak, at the center of the galaxy, so that the excursion of the gravitational field can counterbalance the centrifugal force. But it's also possible that these velocity excursions are the result of cannibalism by galaxies with smaller masses. Like the cannibal galaxy that absorbs them, these mini-galaxies are sets of non-collisionless mass-dots. Subjected to the potential well of the larger galaxy, they then fall towards their central part. In so doing, they retain their angular momentum, as these point-mass ensembles cannot transfer it to the cannibal galaxy by viscous exchange, or encounters. In contrast, the NGC 128 galaxy would not have experienced any absorption of mini-galaxies. At most, it could have merged with similar galaxies, in terms of mass distribution and velocity field.

VI. ABOUT THE POTENTIAL AND MASS DISTRIBUTION

The cancellation of the unit order terms corresponds to the equation:

$$\mathbf{C}.\frac{\partial \ln(f)}{\partial \mathbf{r}} - \left(\frac{\partial \Psi}{\partial \mathbf{r}} + \mathbf{c}_0.\frac{\partial \mathbf{c}_0}{\partial \mathbf{r}}\right).\frac{\partial \ln(f)}{\partial \mathbf{C}} = 0 \tag{129}$$

The density field is then constructed by combining it with Poisson's equation. We'll present the complete numerical solution in a later article. This potential is necessarily compatible with the form chosen for the distribution function, since it is an integral part of the solution. This construction is not straightforward, as it requires iterations to evaluate the term $\dfrac{\partial^2 \Psi}{\partial z^2}$. Nevertheless, we can draw conclusions about the shape of the potential and the values of the density. We know that the distribution function tends towards Maxwellian form at the center. The equation then reduces to :

$$\frac{\partial \ln(n)}{\partial \mathbf{r}} + \left(\frac{\partial \Psi}{\partial \mathbf{r}}\right)\frac{m}{k_B T_0} \tag{130}$$

The equation of the potential at the center is then:

$$\Delta \Psi = Cst\ e^{-\dfrac{m\Psi}{k_B T_0}} \tag{131}$$

Let's be satisfied with reduced variables and an expression of the potential in arbitrary units. The form of the potential near the center is :

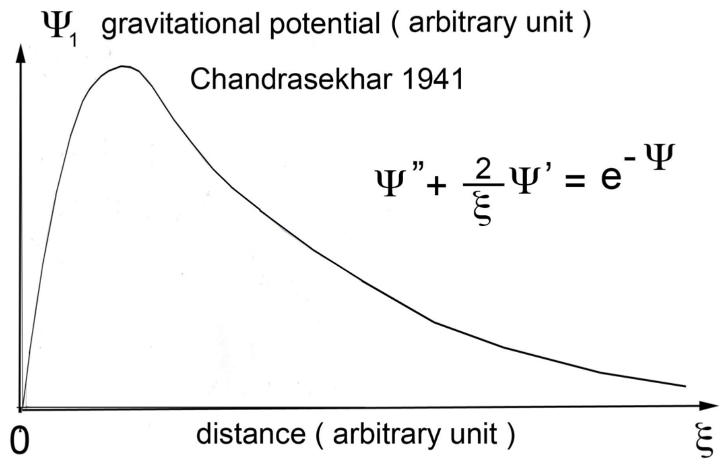

FIG. 9. Gravitational potential in spherical symmetry (Chandrasekhar [2]

VII. CONCLUSION

We have opted for a completely different approach to the one considered classical, taking up that of Chandrasekhar, based entirely on the choice of an elliptical distribution function. We have extended this method by introducing rotation (axisymmetric stationary system). The distribution function is associated with a velocity ellipsoid. The only assumption, of a geometrical nature, is that one of the axes of this ellipsoid in all space points to the system's center of symmetry. The mathematical tool of dyadic algebra is then used to carry out the calculations. Introducing the elliptic form of the distribution function into the collisionless Boltzmann equation, written in terms of the components of the residual velocity, transforms it into a third-order polynomial. Cancellation of the third-order terms then gives the evolution of the axes of the ellipsoid. We then find that the three axes become equal at the center, i.e. the velocity distribution becomes Maxwelian. The transverse axes tend towards zero at infinity, while the radial velocity dispersion remains constant. The fact that the transverse axes are smaller than the radial velocity dispersion axis is consistent with observational data from the vicinity of the Sun. Cancellation of the second-order terms gives the velocity field. This tends towards solid-body rotation at the center and a plateau at infinity. It then remains to calculate the gravitational field by combining the cancellation of the unit-order terms and Poisson's equation. We refer this to a future article.

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[1] J. Binney and S. Tremaine. *Galactic Dynamics.* Princeton Series in Astrophysics, 2008.
[2] S. Chandrasekhar. *Principles of stellar dynamics.* Dover publications, 1941.
[3] S. Chapman and T.G. Cowling. *The mathematical theory of non uniform gazes.* Chambridge University Press, 1939.
[4] J.P. Petit and G. Monnet. Entropie maximale et cosmologie de friedman. *Comptes Rendus de l'Académie des Sciences de Paris*, série A:1245–1248, 1975.
[5] W.H. McCrea and E.A. Milne. Newtonian Universes and the Curvature of Space. *The Quarterly Journal of Mathematics*, os-5:73–80, 1934.
[6] J.P. Petit. Modèle tridimensionnel et instationnaire de système stellaire auto-gravitant. application aux amas globulaires et aux galaxies (three-dimensional and unsteady model of self-gravitating stellar system. application to globular clusters and galaxies). *Note aux Comptes Rendus de l'Académie des Sciences de Paris*, série B:373–376, 1972.
[7] J.P. Petit and G. Monnet. Axisymmetrical elliptical solution of the couple vlasov plus poisson equations. *International 1974 Meeting on Galactic Dynamics at the High Studies Institute, Bures sur Yvette, France*, 1974.
[8] G. D'Agostini. Solution elliptique de l'équation de vlasov. *Private Communication*, 2016.