The black hole model goes with an analytic extension of the space-time.

Jean-Pierre Petit[1] , Gilles d'Agostini

Key words : spacetime geodesics, real geodesics, virtual geodesics, timeline, wormhole, Flamm surface

Abstract : We show that the existence of geodesics inside the Schwarzschild sphere goes with the extension of real spacetime to a complex portion, the sphere becoming a cobord, and is an analytical extension of this geometry.

1 – **Introduction**.

From 1905 onwards, a new discipline developed, which can be described as geometric cosmology, in the sense that mathematicians and geometers introduced a way of approaching the description of the universe using the tools of geometry. Its first success was, in 1905, the interpretation of the invariance of the speed of light, with its corollary the abandonment of the idea that this light, which Maxwell had shown to be a wave, propagated in a rigid and fixed medium, the support of all things, which had been given the name of ether.

In his 1905 article, A. Einstein [1] adopted a more kinetic than geometric approach, although it achieved its intended goal and accounted for the results of Michelson and Morley [2]. H. Minkowski established the geometric context in 1909 in his famous lecture entitled "Space and Time" (Raum und Zeit) [3]. The three excerpts that follow summarize its essential elements. The first contains the definition of length in this space, invariant under a change of coordinates and essentially positive, real.

[1] Jean-pierre.petit@manaty.net

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Einfüngen is das folgende Grundprinzip:

Die Substanz die in eimen Punkt gefunden wurde, an dem sich die Universtät gerade befindet, könnte von einer Wahl die Raumzeit übernommen werden, die wir alle als Repos in Betracht ziehen

Das Axiome bedeutet, das jeder Punkt vorliegt:

$$c^2dt^2 - dx^2 - dy^2 - dz^2$$

Es ist Tag jeden Tag positiv, oder Sie möchten wissen dass die Vitesse v jeden Tag etwas ist, was c.

Let us now introduce the following fundamental axiom:

Substance at any point of the Universe can, by suitable choice of spacetime, always be considered at rest

This axiom means that, at each point :

$$c^2dt^2 - dx^2 - dy^2 - dz^2$$

is always positive, or, equivalently ,that the velocity v is always less than c .

Fig.1 : The elementary length, in Minkowski space [3].

In the following figure, proper time is defined, also essentially real and positive.

Stellen Sie sich nun vor, dass durch einen Punkt P (x,y,z,t) im Universum die Weltlinie eines Punktes verlaäuft. Das Zeitvektorelement dx,dy,dz entlang dieser Kurve entspricht dem Ausdruck

$$d\tau = \frac{1}{c}\sqrt{c^2\,dt^2 - dx^2 - dy^2 - dz^2}.$$

Das Integral $\int d\tau = \tau$ dieser Größe entlang der Weltlinie ausgehend von einem beliebigen Punkt P, wird als Eigenzeit im Punkt P bezeichnet.

Now imagine, at a point in the univers *P(x,x,z,t)* the wordline of a material point passing through it. The time vector element *dx, dy,dz, dt* along this curve corresponds to the expression:

$$d\tau = \frac{1}{c}\sqrt{c^2\,dt^2 - dx^2 - dy^2 - dz^2}.$$

The integral $\int d\tau = \tau$ of this quantity taken along the the wordline from any point P, will be called the proper time.

Fig.2 : Proper time, as defined by Minkowski [3].

The behavior of material points, endowed with mass, is inscribed along world lines:

§1. Bewegungungsgleichungen eines materiellen Punktes im statischen Gravitationsfeld.

Germäβ der gängigen Relativitätstheorie bewegt sich ein Punkt in Abwesenheit von Kräften gemäβ der Gleichung

(1) $$\delta\left\{\int ds\right\} = \delta\left\{\int\sqrt{-dx^2 - dy^2 - dz^2 + c^2dt^2}\right\} = 0.$$

Diese Gleichung besagt, dass sich der materielle Punkt geradlinig und gleichförmig bewegt. Yjos ist die Frage der Bewegung in Form des Hamiltonschen Prinzips.

§1. Equations of Motion of the Material Point in the Static Gravitational Field

According to the customary theory of relativity,[5] in the absence of forces a point moves according to the equation

(1) $$\delta\left\{\int ds\right\} = \delta\left\{\int\sqrt{-dx^2 - dy^2 - dz^2 + c^2dt^2}\right\} = 0.$$

For this equation states that the material point moves rectilinearly and uniformly. This is the equation of motion in the form of Hamilton's principle; for we can also

Fig.3 : Minkowski space world lines are geodesics of spacetime [3].

This represents the evolution initiated in 1744 by Pierre Louis de Maupertuis, through his principle of least action: "Nature follows the paths of least expenditure" (in particular the shortest paths) [4]. In 1854, Bernhard Riemmann [5] introduced the concept of the metric and saw geodesics as lines of shortest path. In 1900, Gregorio Ricci-Cubastro and Tullio Levi-Civita [6] laid the foundations of differential geometry. In 1917, Tullio Levi-Civita [7] precisely redefined what world lines in physics are by giving a mathematical definition of covariant geodesics, which are no longer lines representing the shortest path, but curves along which the tangent vector travels parallel to itself.

$$\frac{d^2x^i}{dp^2} - \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0 \tag{1}$$

In parallel, the calculus of variation, with Leonhard Euler in 1744 [8] and then Pierre Louis Lagrange in 1788 [9], established itself as one of the major tools of physics. In 1909, as shown in Figure 3, Hermann Minkowski [3] identified the calculus of variation as the tool for constructing world lines, as geodesics of spacetime, in the sense of Tullio Levi-Civita. However, he did not have time to pursue this further, succumbing to peritonitis four months later. In 1913, M. Grossman [10], very interested in these tools of modern geometry, helped Einstein become familiar with them. Both generalized the variational approach proposed by Minkowski, according to the equation of variations:

$$\delta\left\{\int ds\right\} = \delta\left\{\int\sqrt{g_{\mu\nu}\,dx_\mu dx_\nu}\right\} = 0 \tag{2}$$

Thus, things are clear and the problem is perfectly posed. The variables belong to $\mathbb{R}^4$ and the length s, in other words up to the coefficient c, the proper time \tau are fundamentally real quantities. The 1913 article contains elements foreshadowing the imminent emergence of the field equation the following year, as well as the expression of the source quantities of the gravitational field in tensor form. Under the influence of Gustav Mie [11] (all these people communicated closely with one another, either verbally or by correspondence), Einstein was impressed by the idea that such a fundamental theory must arise from action. On November 25, 1915, he published the article [12] presenting the field equation, which immediately became the basis of general relativity. In an article he submitted five days earlier, David Hilbert [13] arrived at the same result using a variational approach. His Lagrangian was then constructed using two scalars: the Ricci scalar, which he called K, and a Lagrangian describing matter. Everything was then in place for the first solutions to this equation to appear. First came the initial approximate solution constructed by Einstein in an article published alongside his field equation [14]. Then, in January and February 1916, Karl Schwarzschild published, one after the other, two exact solutions to Einstein's equation: a stationary, spherically symmetric solution, first without a forcing term [15], and then with a forcing term [16]. This work was quickly taken up and commented on by Ludwig Flamm [17], Johannes Droste [18], and Hermann Weyl [19]. All their approaches conformed to the principles established by Minkowski, Einstein, and Grossman (Figures 1, 2, 3). Even though the concept of the metric signature was not yet in use at the time, they all explicitly opted for (+ - - -). Below is the Gram matrix presented by Einstein in [14].

Wir gehen nun in solcher Weise vor. Die $g_{\mu\nu}$ seien in »nullter Näherung« durch folgendes, der ursprünglichen Relativitätstheorie entsprechende Schema gegeben

$$\left.\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{matrix}\right\}, \qquad (4)$$

Traduction :

We will now proceed in this manner. The $g_{\mu\nu}$ are given, to a zeroth approximation, the following schema, corresponding to the original theory of relativity.

Fig.4 : The Gram matrix, after Einstein [14]

2 – **Schwarzschild's external metric** [15].

As shown in reference [20], the initial formulation of the exterior metric solution, in its initial coordinates $\left(t\,, r = \sqrt{\,x^2 + y^2 + z^2\,}\,, \theta\,, \varphi \right) \in \mathbb{R}^4$ is:

(3) $$ds^2 = \frac{(r^3+\alpha^3)^{1/3} - \alpha}{(r^3+\alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4\, dr^2}{(r^3+\alpha^3)[(r^3+\alpha^3)^{1/3} - \alpha]}$$

$$-\,(r^3 + \alpha^3)^{2/3}(\,d\theta^2 + sin^2\theta d\varphi^2)$$

This metric is regular, and we can then plot the curves showing the evolution of the metric potentials as a function of r:

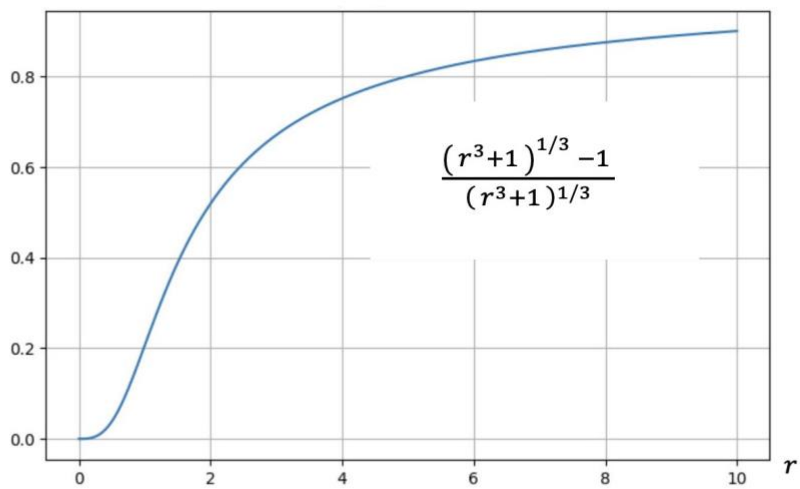

Fig.5 : Function g_{tt} with coordinates $(\,t\,, r\,, \theta\,, \varphi\,)$

When $r \to +0$ the term g_{tt} tends to zero

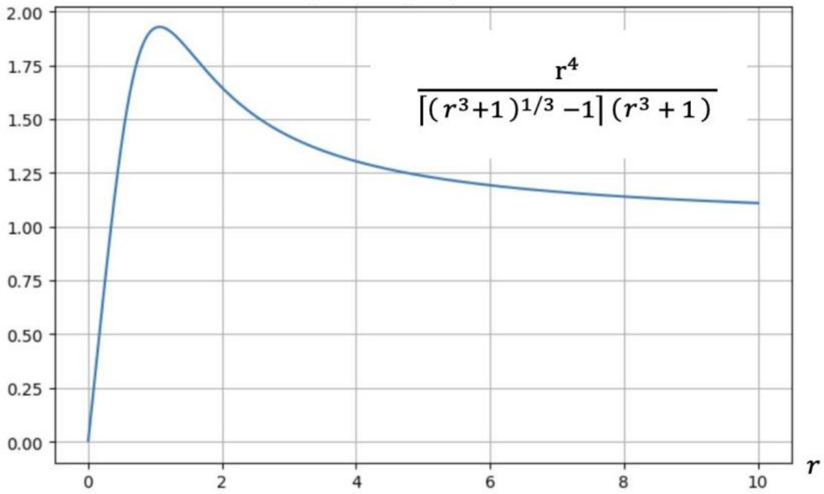

Fig.6 : Function - g_{rr} in coordinates (t, r, θ, φ)

When $r \to 0$ the term g_{rr} tends to zero.

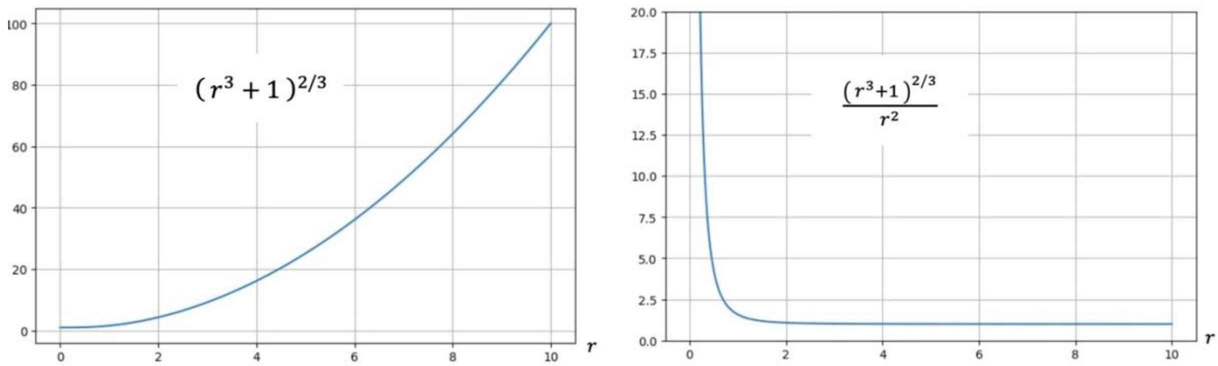

Fig.7 : Function - $g_{\theta\theta}$ in coordinates (t, r, θ, φ)

Up to a factor on the radial coordinate, the metric tends towards the Lorentz metric at infinity. We see that regardless of the values of the variables, these functions remain positive. To use today's terminology, the signature $(+ - - -)$ is defined and invariant. We obtain complete consistency between:

(4) $$(t, r, \theta, \varphi) \in \mathbb{R}^4 \ \varphi \ \leftrightarrow \ ds^2 \geq 0$$

Et, vice-versa :

(5) $$ds^2 < 0 \longrightarrow (t, r, \theta, \varphi) \notin \mathbb{R}^4$$

To obtain an imaginary length *ds*, the set of coordinates must cease to be real numbers. The hypersurface is no longer located in real spacetime, but in a form of complex extension.

H. Minkowski, in defining his space, had emphasized the fundamental invariance (see Figure 1), meaning that the speed could not exceed c. With general relativity, this same physical constraint extends this to a curved space, with the causal constraint:

(6) $$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \;\geq\; 0$$

The metric solutions constructed by K. Schwarzschild, including the exterior metric [15], possess this fundamental property, which implies that the proper time of τ is everywhere real and positive. Examination shows that the geometric object described by Schwarzschild's exterior metric solution is non-contractile. The following integral has a minimal value:

(7) $$A \;=\; \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sqrt{-\,g_{\theta\theta}} \;\geq\; 4\pi\alpha^2$$

We have shown that the construction of the isometric embedding of sections at constant t and θ, in a frame of reference $\left(R \;=\; (r^3 \;+\; \alpha^2)^{1/3} \;,\; \varphi \right)$ the object reveals the Flamm surface, carrying topological information. In the initial Schwarzschild coordinate system $(\, t \, , r \, , \theta \, , \varphi \,)$ the topology is that of a manifold with boundary, this boundary being the sphere of minimum area $4\pi\alpha^2$. We will perform a change of variable that will create the second sheet of the surface, making the boundary disappear, which transforms into a throat sphere.

(8) $$r \;=\; \alpha\,[\,(\,1 \;+\; log\; ch\rho\,)^3 \;-\; 1\,]$$

The metric becomes:

(9) $$ds^2 = \frac{ln\; ch\rho}{1+ln\; ch\rho} c^2 dt^2 - \alpha^2 \left[\frac{1+ln\; ch\rho}{ln\; ch\rho} th^2\rho\; d\rho^2 + (1 + ln\; ch\rho)^2 (d\theta^2 + sin^2\theta\; d\varphi^2) \right]$$

Here again, the term terme $\;g_{\rho\rho} \;\rightarrow\; -2\;$. We then have the two sheets, one corresponding to $\rho \;>\; 0\;$ and the other to $\rho \;<\; 0\;$. For the throat sphere: $\rho \;=\; 0$. Here again we find this minimum area $4\pi\alpha^2$ at this point. The metric potentials are:

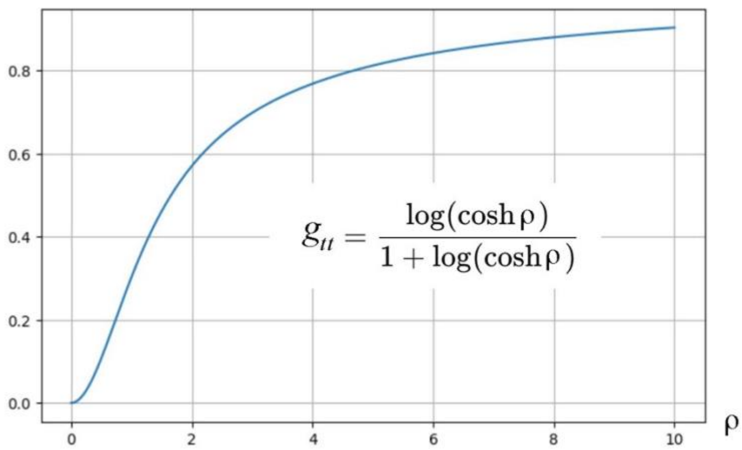

Fig.8 : Function g_{tt} the representation $(\, t \, , \rho \, , \theta \, , \varphi \,)$

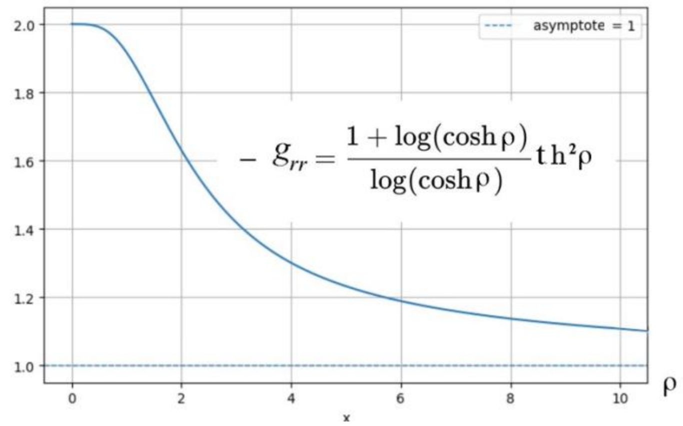

Fig.9 : Function $g_{\rho\rho}$ the representation $(t , \rho , \theta , \varphi)$

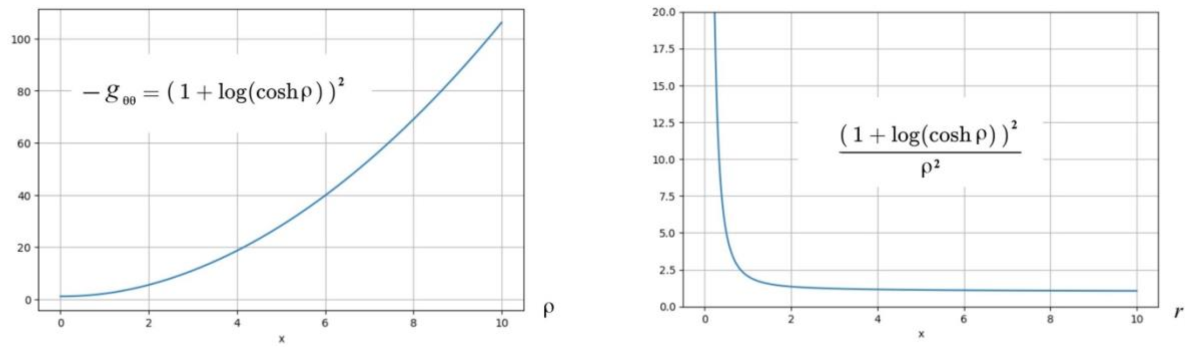

Fig.10 : Function $g_{\theta\theta}$ the representation $(t , \rho , \theta , \varphi)$

Here again the signature is $(+ \ - \ - -)$, defined and invariable.

3 – **Construction of the spatio-temporal geodesics of the exterior metric**.

Schwarzschild utilise la méthode variationnelle :

ON THE GRAVITATIONAL FIELD OF A MASS POINT ACCORDING TO EINSTEIN'S THEORY †

BY K. SCHWARZSCHILD

(Communicated January 13th, 1916

TRANSLATION‡ AND FOREWORD BY

S. Antoci* and A. Loinger'

§1. In his work on the motion of the perihelion of Mercury (see Sitzungsberichte of November 18th, 1915) Mr. Einstein has posed the following problem:

Let a point move according to the prescription:

$$\begin{cases} \delta \int ds = 0, \\ \quad where \\ ds = \sqrt{\Sigma g_{\mu\nu} dx_\mu dx_\nu} \quad \mu, \nu = 1, 2, 3, 4, \end{cases} \qquad (1)$$

Fig.11 : Schwarzschild's variational Technique [15].

Using the two representations introduced previously, this leads Schwarzschild to the variational equation:

(10)

$$\delta \int \sqrt{\frac{(r^3+\alpha^3)^{1/3}-\alpha}{(r^3+\alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4\, dr^2}{(r^3+\alpha^3)[(r^3+\alpha^3)^{1/3}-\alpha]} - (r^3+\alpha^3)^{2/3}(d\theta^2 + sin^2\theta d\varphi^2)}$$

(11)

$$\delta \int \sqrt{\frac{(r^3+\alpha^3)^{1/3}-\alpha}{(r^3+\alpha^3)^{1/3}} c^2 \left(\frac{dt}{\mathrm{dp}}\right)^2 - \frac{r^4}{(r^3+\alpha^3)[(r^3+\alpha^3)^{1/3}-\alpha]} \left(\frac{dr}{dp}\right)^2 - (r^3+\alpha^3)^{2/3}\left(\left(\frac{d\theta^2}{dp}\right) + sin^2\theta\left(\frac{d\theta^2}{dp}\right)\right)}\ dp$$

Posing :

(12) $$\frac{dt}{dp} = \dot{p} \qquad \frac{dr}{dp} = \dot{r} \qquad \frac{d\theta}{dp} = \dot{\theta} \qquad \frac{d\varphi}{dp} = \dot{\varphi}$$

Where p is a parameter identifying a point on a curve drawn in this space. This gives the equation of variations:

(13)

$$\delta \int \sqrt{\frac{(r^3+\alpha^3)^{1/3}-\alpha}{(r^3+\alpha^3)^{1/3}} c^2 \dot{t}^2 - \frac{r^4}{(r^3+\alpha^3)[(r^3+\alpha^3)^{1/3}-\alpha]} \dot{r}^2 - (r^3+\alpha^3)^{2/3}(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2)} = 0$$

This clearly falls within the Lagrangian problem:

(14) $$\delta \int L(x^i, \dot{x}^i)\ dp$$

Where :
(15)

$$L = \sqrt{\frac{(r^3+\alpha^3)^{1/3}-\alpha}{(r^3+\alpha^3)^{1/3}}c^2\dot{t}^2 - \frac{r^4}{(r^3+\alpha^3)[(r^3+\alpha^3)^{1/3}-\alpha]}\dot{r}^2 - (r^3+\alpha^3)^{2/3}(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2)}$$

Which leads to Lagrange equations :

(16) $$\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0$$

Which provide solution curves $t(p)$, $r(p)$, $\theta(p)$, $\varphi(p)$,, which will be inscribed in the real field for all values of these functions such that the quantity under radical remains real.

Introduce the new Lagrangian :
(17) $$\Lambda = L^2$$

The Lagrange equations, constructed from this new Lagrangian, will give identical solution curves if:

(18) $$\frac{d}{dp}\left(\frac{\partial \Lambda}{\partial \dot{x}^i}\right) - \frac{\partial \Lambda}{\partial x^i} = 2\frac{dL}{dp}\frac{\partial L}{\partial \dot{x}^i} + 2L\left(\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}\right) = 0$$

That is to say, if $\frac{dL}{dp} = 0$, therefore if the parameter p is the length s, dividing equation (3) by ds gives:

(19) $$1 = \frac{(r^3+\alpha^3)^{1/3}-\alpha}{(r^3+\alpha^3)^{1/3}}c^2\dot{t}^2 - \frac{r^4}{(r^3+\alpha^3)[(r^3+\alpha^3)^{1/3}-\alpha]}\dot{r}^2 - (r^3+\alpha^3)^{2/3}(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2)$$

Therefore, the function L, when expressed with the parameter s, is constant. Schwarzschild thus concludes that he can replace his equation of variations with:
(20)

$$\delta\int\left(\frac{(r^3+\alpha^3)^{1/3}-\alpha}{(r^3+\alpha^3)^{1/3}}c^2\dot{t}^2 - \frac{r^4}{(r^3+\alpha^3)[(r^3+\alpha^3)^{1/3}-\alpha]}\dot{r}^2 - (r^3+\alpha^3)^{2/3}(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2)\right)ds = 0$$

Schwarzschild knows he will have no problem with his parametric solutions *t(s), r(s), q(s),j(s)* since, because $g_{\mu\nu}\, dx^\mu dx^\nu \geq 0$ (velocities v less than c) . The quantity under the radical remains non-negative.

But the calculation with a Lagrangian in this form is cumbersome, and seems unnecessarily complicated to him, insofar as his goal is to find the approximate solution published by Einstein [14] a month earlier, which accounts for the very small advance of Mercury's perihelion. He then introduces what he calls, not as a new variable but as a simple intermediate quantity (*Hilfsgrobe*).

(21) $$R = (r^3 + \alpha^3)^{1/3} \quad or \quad r = (R^3 - \alpha^3)^{1/3}$$

In the letter he sent to Einstein on December 22, 1915 [21] he hastened to clarify:

(22) $$R = r\left(1 + \frac{\alpha^3}{3\,r^3} + \dots\right)$$

Adding : Sind Kein *"erlaubten" Koordinaten mit denen man die Feldgleichungen bilden dürfte* (R, θ, φ are not "allowed" coordinates thanks to which we must construct the field equations). Indeed, by opting for the variational equation

(23)

$$\delta \int \left[\left(1 - \frac{\alpha}{R}\right) c^2 \dot{t}^2 - \frac{\dot{R}^2}{1 - \frac{\alpha}{R}} - R^2\left(\dot{\theta}^2 + sin^2\theta\, \dot{\varphi}^2\right) \right] ds$$

That is to say, for the Lagrangian:

(24)

$$L = \left(1 - \frac{\alpha}{R}\right) c^2 \dot{t}^2 - \frac{\dot{R}^2}{1 - \frac{\alpha}{R}} - R^2\left(\dot{\theta}^2 + sin^2\theta\, \dot{\varphi}^2\right)$$

By admitting solutions from Lagrange's equations such as $R < \alpha$, we break the Minkowski-Einstein axiomatic imperative of non-negativity of the square of the length element ds (of the proper time element $d\tau$) and of invariance of the signature $(+ - - -)$. We then consider integrating parametric solutions in the form of real functions R and φ (in this projection-based presentation of the solution) equipped with purely imaginary length and time. This extension of spacetime thus reflects its extension, beyond the real field, into a complex domain.

Physicists negotiate this by concluding:

> - *Within the Schwarzschild sphere, the variables of time and space exchange their respective roles.*

This does not occur if we consider the change of variable (8) where the non-negativity of the square of the element of length {ds}^2 is preserved and where the signature $(+ - - -)$ remains invariant regardless of the values of the new variables. This then allows us to construct the geodesic curves from the variational equation:

(25)

$$\delta \int \left[\frac{ln\, ch\rho}{1 + ln\, c\rho} c^2 \dot{t}^2 - \alpha^2 \left[\frac{1 + ln\, ch\rho}{ln\, ch\rho} th^2\rho\, d\rho^2 + (1 + ln\, ch\rho)^2 (d\theta^2 + sin^2\theta\, d\varphi^2) \right] \right] ds = 0$$

The geodesics, corresponding to all real values of ρ , going from $-\infty$ to $+\infty$, have the following appearance and become free of artifacts [22].

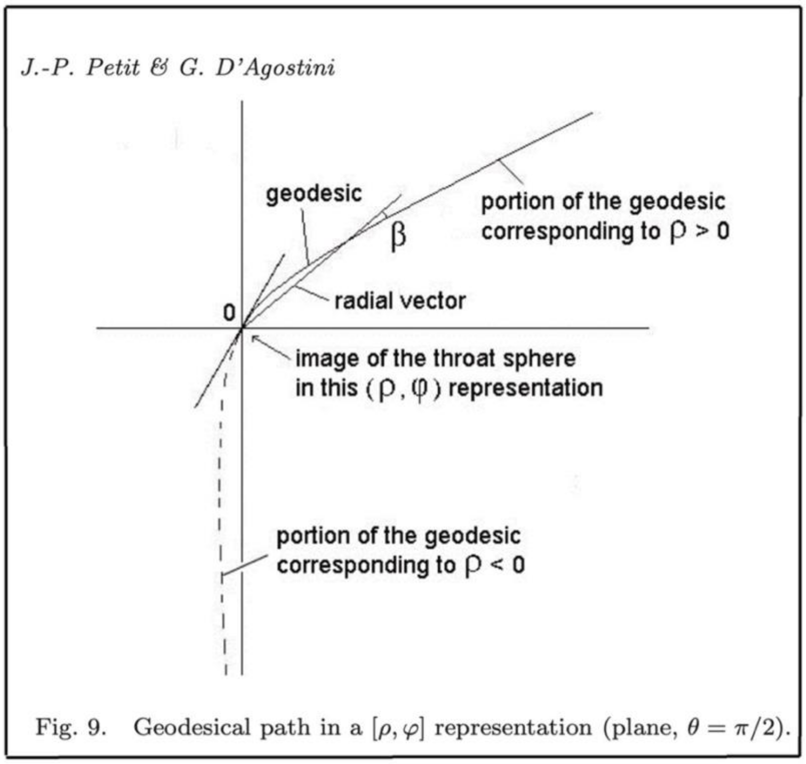

Fig. 12: Projection onto the plane(ρ , φ) of a geodesic of the Schwarzschild geometry in the representation (t , ρ , θ, φ) [20]

This situation may seem disconcerting if we cling to the idea of lengths expressed in meters. In this geometry of spacetime, the only quantity upon which all measurement is based is proper time τ. We have the illusion that at the center of this figure lies a sphere of area $4\pi\alpha^2$, representing square meters, reduced to a point. But everything changes and becomes conceivable when we consider that this integral expression does not represent square meters but squared seconds.

In the original Schwarzschild representation (t , r , θ, φ), the curve has a similar shape. It is a projection of a spacetime geodesic. Below, in these two modes of representation:

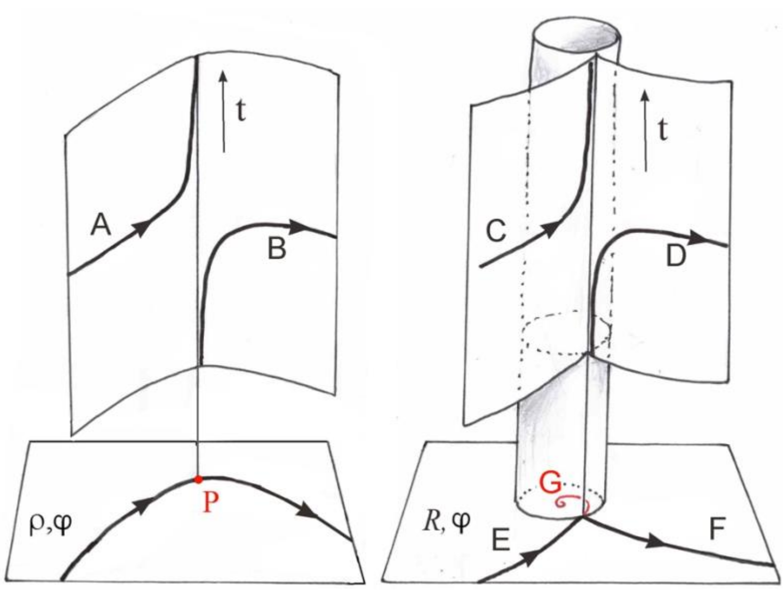

Fig.13 : Spacetime geodesics and 2D projections

In the representation $(t, \rho, \theta, \varphi)$ (the representation (t, r, θ, φ) is analogous), the projections onto the planes (ρ, φ) and (r, φ) give real curves. In (t, ρ, φ), along branches A and B, the reference mass reaches or leaves the Schwarzschild sphere, reduced to point P (but whose area is $4\pi\alpha^2$), in infinite time. In the representation (t, R, φ), the reference mass traveling along branch C reaches the Schwarzschild sphere in infinite time. Conversely, following branch C, it leaves this same sphere in infinite time. In this diagram, resulting from an isometric embedding, the image of the Schwarzschild sphere in the plane (R, φ) becomes a circle with a perimeter of $2\pi\alpha$. This pseudo-length is, in fact, a proper time quantity and must therefore be expressed in seconds.

In the plane (R, φ), branch E is extended by the real red curve G, which spirals indefinitely towards the center of the circle. This is the projection of the geodesic field extension into a complex portion of spacetime, where along these portions, proper length and time become purely imaginary. This extension into a complex field, which goes hand in hand with its analytic continuation [23], is the alternative to treating the solution where real proper length and time are retained, and where the non-contractibility of the geometric object must then be taken into account, with a different topology, and where a wormhole is obtained in which two Minkowski spacetimes are then connected by a throat sphere.

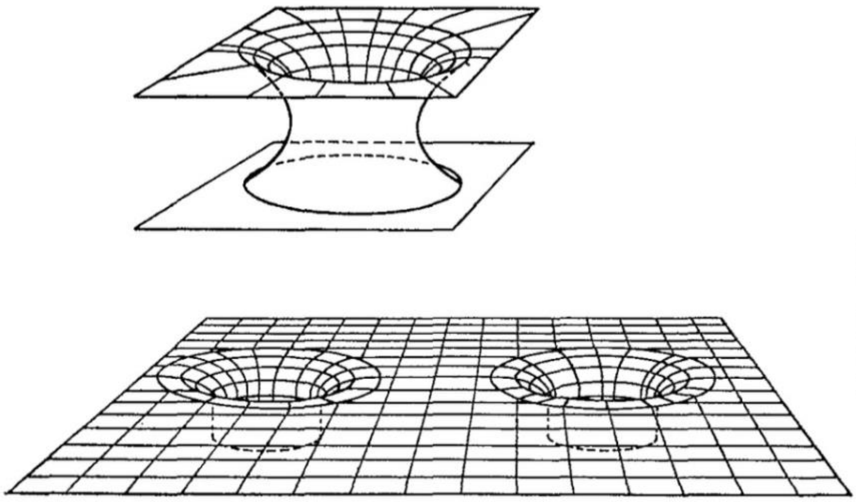

FIG. 1. Two interpretations of the 3-dimensional "maximally extended Schwarzschild metric" at the time $T=0$. Above: A connection or bridge in the sense of Einstein and Rosen between *two* otherwise Euclidean spaces. Below: A wormhole in the sense of Wheeler connecting two regions in *one* Euclidean space, in the limiting case where these regions are extremely far apart compared to the dimensions of the throat of the wormhole.

Fig.14 : The wormhole geometry [23].

4 – **Examples of extensions in a** 2D- **complex field.**

It seems useful to give examples of the extension of 2D objects, defined by a metric, which is expressed in a certain coordinate system. The closest example is the Flamm surface, defined by the metric:

(26) $$ds^2 = \frac{dR^2}{1-\frac{\alpha}{R}} - R^2 d\varphi^2 \qquad R = (r^3 + \alpha^3)^{1/3}$$

Still with the same constraint $R \geq \alpha$. Flamm [17] observes that this object can be embedded in $\mathbb{R}^3$ and that it is then generated by the rotation of a horizontal parabola. :

(27) $$z = \pm 2\sqrt{\alpha(r - \alpha)}$$

This corresponds to an isometric embedding in a space (z, ρ, φ). The length of a curve segment corresponds to the integral:

(28) $$\int \sqrt{\frac{\dot{r}^2}{1-\frac{\alpha}{r}} + r^2\dot{\varphi}^2} \quad ds$$

As we saw above, if the affine parameter allowing us to locate the position of the points on the curve and the length s, we obtain the Lagrange equations by basing the calculus of variations on :

(29) $$\delta \int \left(\frac{\dot{r}^2}{1-\frac{\alpha}{r}} + r^2\dot{\varphi}^2 \right) ds = 0$$

But this eliminates the constraint of non-negativity of the quantity under the radical. The equations giving the solution curves are the same:

(30) $$\frac{d\varphi}{dr} = \pm \frac{h}{r^2} \frac{1}{\sqrt{\left(1-\frac{\alpha}{r}\right)\left(1-\frac{h^2}{r^2}\right)}}$$

hese curves were first plotted in 2015 [22], 99 years after the publication of Flamm's article.

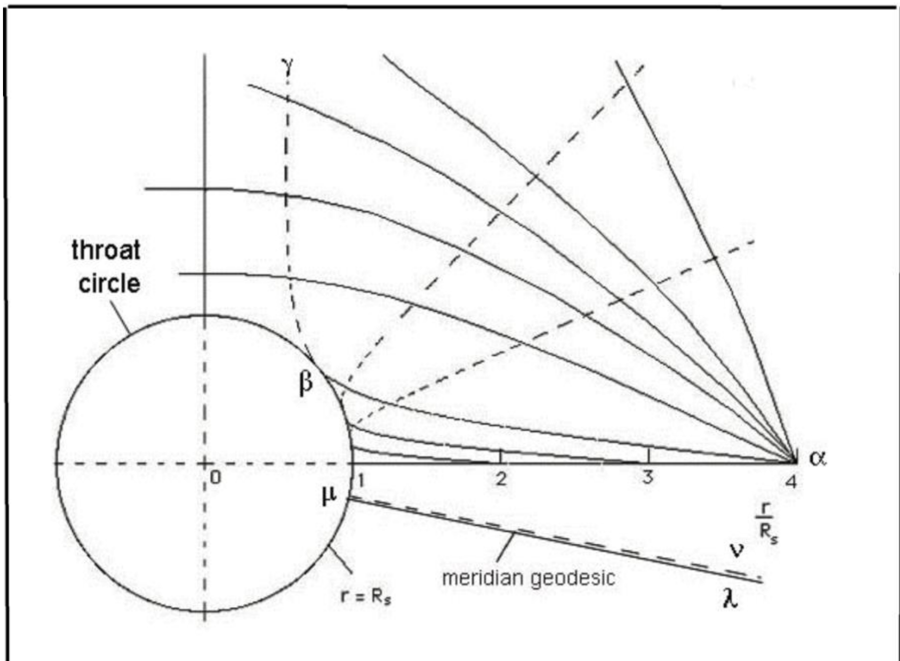

Fig. 4. Projection of geodesics.

Fig.15 : Projection (r, φ) of the geodesics of the Flamm surface [22].

We see that for $r > \alpha$ we obtain real curves for $|h| < \alpha$. But for $r < \alpha$ we also obtain real curves with \$|h| > \alpha$, which are spirals, which start tangent to the circle and wind in a spiral, the length is purely imaginary along these curves.

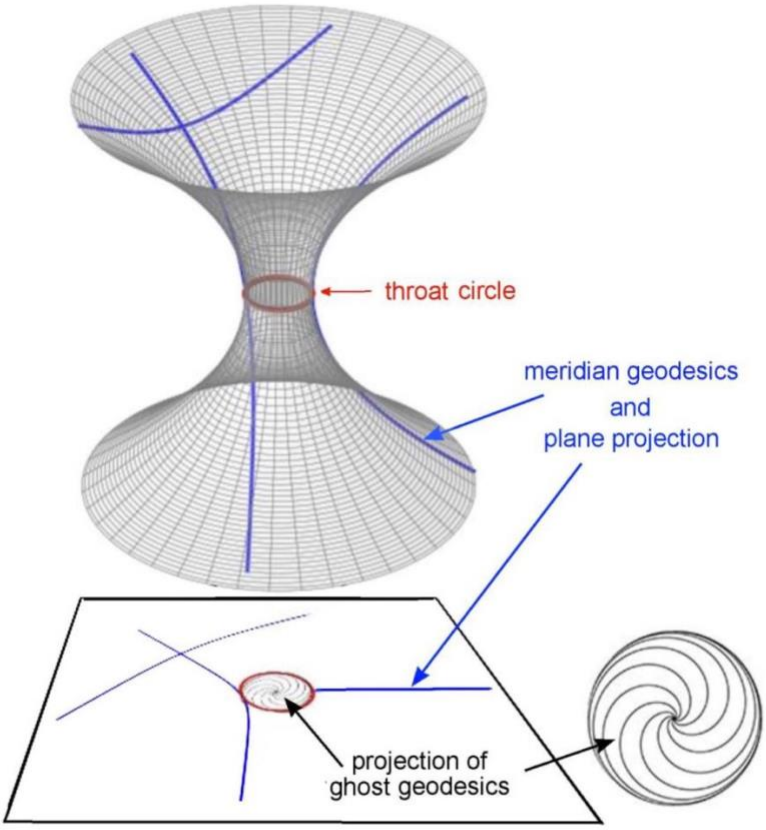

Fig. 16: Flamm surface, real geodesics and projection
of the set of solutions to the Lagrange equations

We usually consider the object described by the metric (26) only in a real form, such that we only retain from the Lagrange equations the curves corresponding to $R > \alpha$, which we can construct in an isometric representation by embedding it in Euclidean space, whose points are located by the coordinates (R, φ, z). But if we imagine that this object belongs to a complex manifold, we are no longer constrained by the imperative $ds^2 > 0$, which would then only concern a part of this complex 2-surface, in this case its real part. It would then extend into a portion of a complex surface, along which $ds^2 < 0$, *where the real throat circle then plays the role of coboundary*. The solution curves of the Lagrange equations, corresponding $0 < R < \alpha$ (indicated at the bottom right of Figure 16) are only projections into the real subplane of curves that are a priori inscribed on the complex part of the surface.

We cannot go any further, as it is impossible to consider an isometric embedding of this portion of the surface in three-dimensional space. These curves can therefore be described as "virtual geodesics".

The appearance of these virtual geodesics is linked to the coordinate system used. Let's take, for example, the torus T2. Its metric is usually given on the form:

(31) $$ds^2 = {r_0}^2\, d\theta^2 + (R + r_0 \cos\theta)^2\, d\varphi^2$$

The topology $S^1 \oplus S^1$ then appears . The corresponding signature is (+ +).

By considering the paths at $\theta = Cst$ and then at $\varphi = cst$, we immediately highlight the non-contractility of this surface. Let's now introduce the change of variable:

(32) $$\theta = arc\cos \frac{r-R}{r_0}$$

The line element becomes :

(33) $$ds^2 = r^2 d\varphi^2 + \frac{dr^2}{-R^2 + {r_0}^2 - r^2 + 2\, r\, R}$$

It is immediately apparent that the signature (+ +) will only be maintained if:

(34) $$R - r \le r \le R + r_0$$

Otherwise, we are outside the torus. The length s is expressed as follows:

(35) $$s = \int \sqrt{r^2 \dot{\varphi}^2 + \frac{\dot{r}^2}{-R^2 + {r_0}^2 - r^2 + 2\, r\, R}}\; ds$$

But we know that we will obtain the same system of Lagrange equations by doing:

(36) $$\delta \int \left(r^2 \dot{\varphi}^2 + \frac{\dot{r}^2}{-R^2 + {r_0}^2 - r^2 + 2\, r\, R} \right) ds = 0$$

Lagrange's equations give us the projection of the curves resulting from this variation calculation onto the plane (r , φ):

(37) $$d\varphi = \frac{\pm h\, dr}{r^2 \sqrt{(-R^2 + {r_0}^2 - r^2 + 2\, r\, R)\left(1 - \frac{h}{r^2}\right)}}$$

This allows us to plot the projection of a geodesic of the torus:

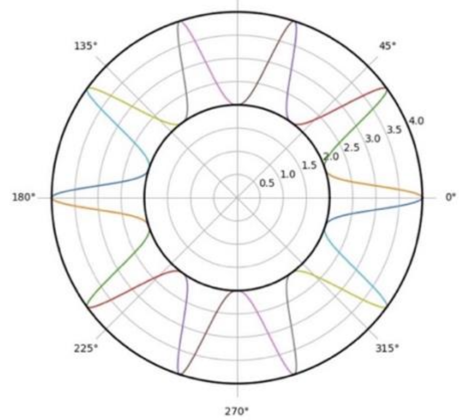

Fig.17 : Plane projection of the geodesics of the torus.

But this solution also generates real curves that are the projection of curves equipped with an imaginary length. Below are the planar projections of these virtual geodesics, "inside and outiside the torus":

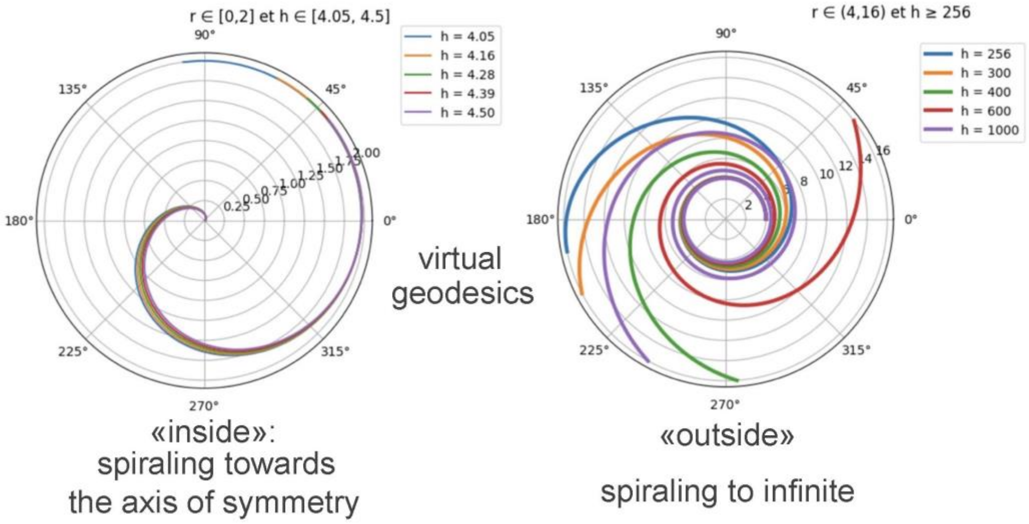

Fig.18 : Virtual geodesics of the torus

Things become clearer when we group these results into a perspective view:

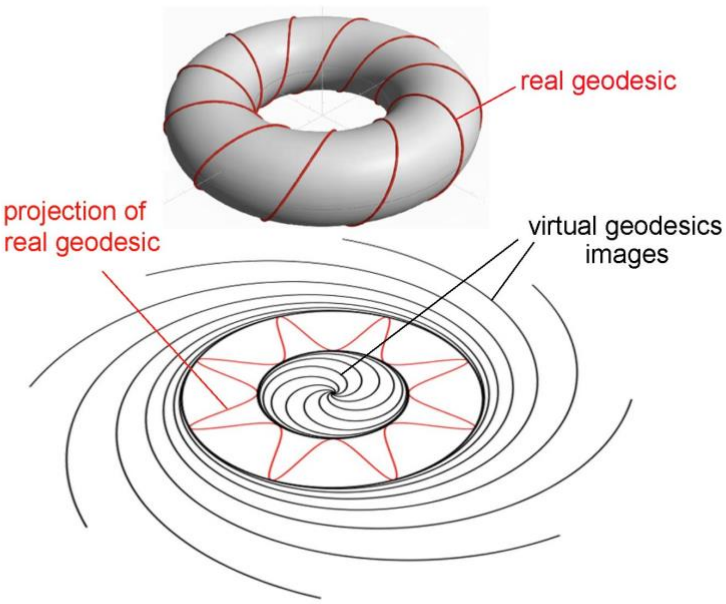

Fig.19 : The torus. Projections of real and virtual geodesics.

Let's now turn to the sphere. The standard form of its metric is:

(38) $$ds^2 = R^2 \, (d\theta^2 + sin^2\theta \, d\varphi^2)$$

Introduce the variable change :

(39) $$\theta = arc\ sin\ \frac{r}{R}$$

The line element becomes :

(40) $$ds^2 = r^2 d\varphi^2 + \frac{R^2\, dr^2}{R^2 - r^2}$$

The variation calculation leads to the equation giving the projection of the geodesics:

(41) $$d\,\varphi = \frac{h\,R}{r} \frac{dr}{\sqrt{(R^2 - r^2)(\,r^2 - h^2)}})$$

We get :

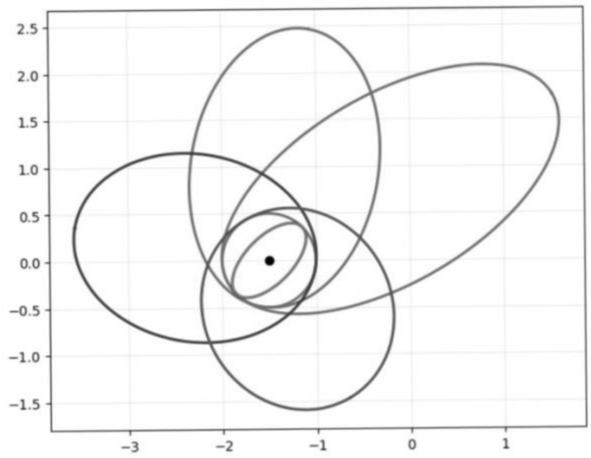

Fig.20 : Projections of geodesics and pseudo-geodesics of a sphere of radius 0.5

We have drawn, inside the circle, an ellipse, the projection of a geodesic of the sphere (these are "great circles"). But we have also drawn some of the projections of its "virtual geodesics". What are these curves, tangent to the sphere? Let's write:

(42) $$\frac{dr}{d\varphi} = \frac{r}{h\,R} \sqrt{(\,r^2 - h^2)\,(\,R^2 - r^2)}$$

Posing :

(43) $$r = \frac{1}{u}$$

(44) $$-\frac{d\varphi}{du} = \frac{1}{h\,R} \sqrt{(\,1 - h^2 u^2)\,(\,R^2 u^2 - \frac{11}{u^2}\,)}$$

(45) $$\frac{d^2 u}{d\varphi^2} + u = \frac{1}{h^2}$$

(46) $$u = \frac{1}{h^2} + A\,cos\,(\,\varphi - \varphi_0)$$

(47) $$r = \frac{h^2}{1 + e\,cos\,(\,\varphi - \varphi_0)}$$

We have just established the first theorem of imaginary geometry:

- *The projections of the virtual geodesics of the sphere are ellipses.*

Below, a perspective view:

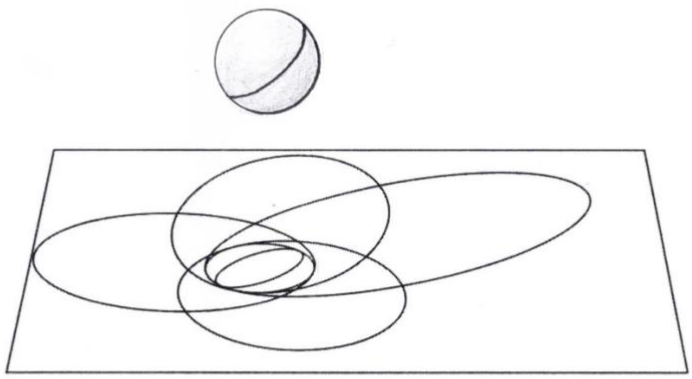

Fig.21 : The sphere and the plane projection of a geodesic and its elliptical virtual geodesics.

5 – Conclusion: Nature of the trajectory curves located "inside the Schwarzschild sphere".

The construction of trajectories of test-particles, according to Lagrange's equations, for $0 < R < \alpha$, represents an extension of real spacetime along a complex four-dimensional space, where the Schwarzschild sphere acts as a coboundary, in other words, it's a geometric analytic extension. Its cartographic extension is achieved via the Kruskal analytic extension [23]. This complex extension leads physicists to redefine the nature of the parameters by interchanging the roles of space and time variables. Projections onto the plane (R, φ) offer an extension with tangent continuity, along a curve that then appears to spiral indefinitely towards a central singularity. But in this region, we leave the real portion of spacetime and enter its complex extension. Note that the limiting axiomatic constraint $ds^2 \geq 0$, of Minkowski-Einstein, eliminates this phenomenon by reintroducing the non-contractivity of the solution surface.

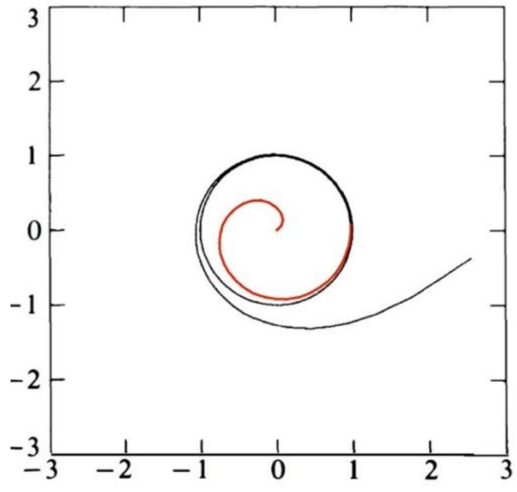

Fig.22 : Schwarzeschild geometry. Projection of geodesics on the plane (R , φ)[24] . The virtual part is in red color.

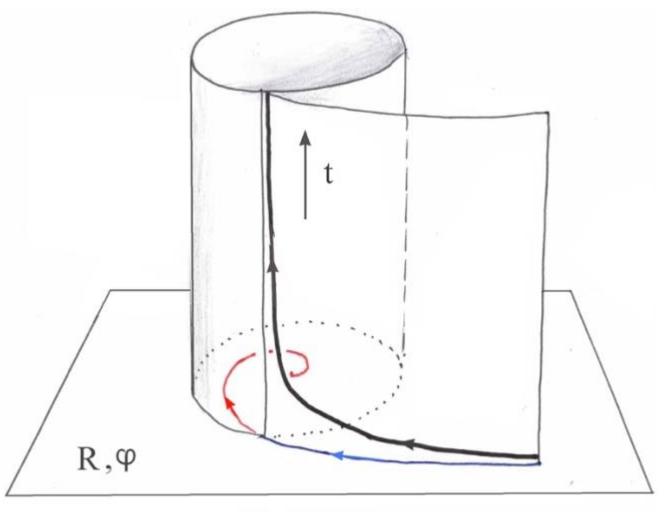

Fig.23 : Schwarzschild (R , φ , t) Spacetime geodesic. The projection of the real part on plane (R , φ) is in blue color. The projection of the virtual part in red.

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