

Mathematical and geometrical Inconsistency of the black hole model. Part II

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Abstract : Complementing a previous article, this second part shows that the spiral curves at the center of black holes are merely virtual geodesics, devoid of any physical reality. As an example, these virtual geodesics are constructed for the torus and the sphere. It is suggested that the generalization of the signature (- + + +), exempt from any ontological justification in cosmology, stems from Hilbert's erroneous interpretation of Schwarzschild's solution in his 1916 article.

1 - Introduction.

The evolution of the geometric context of physics, astrophysics, and cosmology was driven by the changing understanding of the nature of light among scientists. In 1690, the Dutchman Christiaan Huygens was the first to attempt to identify light as a wave, without relying on experimental confirmation. He suggested that this wave propagated in a kind of universal, rigid medium: the ether, which would fill all of space and remain constant. In 1704, the Englishman Isaac Newton [2] introduced the idea of a corpuscular nature of light, which dominated for a century, until 1802, when his compatriot Thomas Young, with his slit experiment [3], demonstrated an interference phenomenon, which immediately revived a wave interpretation. In 1814, the German Joseph Fraunhofer demonstrated [4]. The fact that light emitted by heated bodies passing through a prism is distributed according to a spectrum made up of lines. In 1819, the Frenchman Augustin Fresnel published a complete theory of interference and diffraction [5]. In 1838, using the parallax method, the German Friedrich Bessel obtained the first estimate of the distance to a star [6]. The measurement of the distance to light was then added to that of light sources. In 1842, the Austrian Christian Doppler introduced [7] the idea that the color of light could depend on the radial component of the source's velocity relative to the observer. In 1845, the Dutchman Christophorus Buys Ballot set up an experiment to confirm Doppler's idea in air [8]. Musicians were placed aboard a train traveling at 40 km/h. The pitch of the sound from their instrument is compared to that of a tuning fork, manipulated by stationary observers. When the train moves away, the sound is lower. When it approaches, the sound is higher. In 1848, the Frenchman Hippolyte Fizeau gave substance to this idea in a theoretical form [9]. Subsequently, in 1865, the Scotsman James Clerk Maxwell identified light as an electromagnetic wave [8]. In 1868, the Englishman William Huggins demonstrated the relative motion of stars by comparing two spectra of the same type [10]. From this, he deduced the value of the difference in the relative velocities of these stars with respect to the Earth. Astronomers thus acquired, after photometric and distance measurements, a third element of measurement: that of the relative radial velocity.

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At this stage, it doesn't occur to anyone that the universe could evolve. Light is therefore a wave that affects the ether, fundamentally underlying, invisible, assumed to be homogeneous, isotropic, but inaccessible to observation. Measurements are taken from the Earth, which moves around the Sun. As for the speed of light, it was first estimated in 1676 by the Dane Ole Rømer, then refined in 1728 by the Englishman James Bradley, in 1849 by the Frenchman Hippolyte Fizeau, and finally in 1862 by the Frenchman Léon Foucault. In 1881, the American Abraham Michelson conceived an experiment [12] intended to measure the speed of the Earth relative to the ether, a priori fundamentally immobile, within which all objects in the cosmos are supposed to move. But the sensitivity remained insufficient to draw a conclusion, and it was by repeating the experiment in 1887 with the American Edward Morley that the result was obtained [12]. And then, a dramatic turn of events: this speed maintained a constant value regardless of the direction considered.

In 1894, the Dutchman Hendrik Antoon Lorentz considered a solution to save both the ether and Maxwell's equations, which contain two parameters, ϵ_0 and μ_0 , from which the speed of light, identified as an electromagnetic wave, is deduced at the speed $c = 1/\sqrt{\epsilon_0\mu_0}$. He considered that motion alters the functioning of the measuring instrument. He began by considering a contraction of lengths [14]. Then, in 1895, an alteration of time [15]. These two elements were combined in 1904 in an article where Lorentz presented his famous transformations [16]. Poincaré immediately identified a group structure in these transformations, an idea he presented in 1905 in a lecture given in Palermo [17]. From the outset he associated spatio-temporal translations with this Lorentz group, which gave birth to the group that would bear his name.

In 1905, Einstein's interpretation was conceptually different [18]. He showed that recourse to the ether was no longer necessary. Instead of proposing transformations to acrobatically salvage both Maxwell's equations and the concept of the ether, which disappeared, he manipulated "inertial frames of reference" and presented a kinematic vision based on two principles: relativity and the invariance of the speed of light. But while Einstein's vision accounted for the measurements, he did not, at this stage, introduce a clear geometric vision.

At that time, mathematics and geometry were also present, but the link with physics was not yet established. With Bernhard Riemann [19], the concept of the manifold emerged. Only then did people begin to consider the existence of spaces that escape our mental imagery, spaces that deviate significantly from the Euclidean. These objects exist, for mathematicians, who know how to equip them with metrics, associate them with isometry groups, and link them to invariants. The vertiginous leap that Minkowski dared to make was to understand, initially somewhat confusedly, that this approach was not simply a game for mathematicians, but that it touched upon the geometry of the real world.

The Norwegian mathematician Sophus Lie gradually developed the group theory tools between 1870 and 1874. A summary of his monumental works was published with Friedrich Engel between 1888 and 1893 [20]. The pieces are there, scattered. All that remains is to assemble them, to understand that they constitute the building blocks of a new physics and represent the conceptual leap of refounding the geometric structure in which we operate. But the mental attachment to the Euclidean model remains very strong. In 1907, Hermann Minkowski published a first article [21] in which the idea of integrating the time dimension into a four-dimensional space (x_1, x_2, x_3, x_4) is outlined. But the reference to the ether remains, and the Lorentzian character of space is managed by taking up the idea suggested by Poincaré in 1905

by introducing a purely imaginary time dimension: $x_4 = i t$. The following year [22] he declared in a conference:

Die Vorstellung, dass Raum und Zeit unabhängig voneinander sein können, ist nichts als eine eitle Illusion. Wir werden eine Vereinigung der beiden in Betracht ziehen müssen.

The idea that space and time can be independent is nothing but a vain illusion. We will have to consider a union of the two.

This is the birth of the entity that is space-time. Further on:

Das Welt wird die Menge aller denkbaren Werte von x, y, z, t sein.

The universe will be the set of all imaginable values of x, y, z, t .

And :

Das Zeitelement dt entspricht den Variationen dx, dy, dz, dt der Koordinaten des Punktes im Raum.

The time element dt will correspond to the variations dx, dy, dz, dt of the coordinates of the point in space.

Further on, Minkowski explicitly gives the metric of his spacetime:

Wir betrachten eine Progression dx, dy, dz, dt entlang einer Weltgeraden und die Größe:

We will consider a progression dx, dy, dz, dt along a world line and the quantity:

$$d\tau = \frac{1}{c} \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}$$

We couldn't put it more clearly, and this introduces a purely geometric definition.

Das Integral $\tau = \int d\tau$ dieser Größe entlang dieser Weltlinie, von einem beliebigen Ursprung P_0 bis zu einem variablen Endpunkt P , wird als Eigenzeit τ (Eingezeit) des substanziellen Punktes bei P bezeichnet.

The integral $\tau = \int d\tau$ of this quantity taken along this world line, from any origin P_0 , to a variable extremity P , will be called the proper time τ (Eingezeit) of the substantial point at P .

This leads to a purely geometric definition of the photon. The fact that particles cannot travel along these world lines at speeds greater than c follows from the fact that s and τ are real.

The photon is therefore the particle that, in spacetime, follows a world line of zero length ($ds = d\tau = 0$). Incidentally:

$$(1) \quad c^2 dt^2 - dx^2 - dy^2 - dz^2 \geq 0 \quad \text{or } c \geq \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

In other words, dit $v \leq c$ and that says it all. Minkowski's universe is a space where coordinates are real, where time is measured in meters, where length, assimilated to proper time, is real.

If $ds^2 < 0$ we are simply stepping outside the world of physics.

Until now, physicists had never considered any space other than Euclidean space, with its metric:

$$(2) \quad ds^2 = dx^2 + dy^2 + dz^2$$

It is understandable that this new geometry had difficulty taking root in the minds of men in general, and scientists in particular.

It was the Englishman Sylvester who introduced the concept of signature in 1852 [23], but by linking it to quadratic forms, and not systematically to a metric. In 1883, the Dane Jørg Gram [24] introduced a matrix presentation of bilinear forms. With X being a column vector (a column matrix) and tX a row vector (the transpose of this row matrix), a bilinear form can be written as:

$$(3) \quad {}^tX G X$$

G being the Gram matrix:

$$(4) \quad G = \begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & \pm 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \pm 1 \end{pmatrix}$$

In this context, the Lorentz metric is written:

$$(5) \quad ds^2 = {}^t dX G dX$$

And the corresponding Gram matrix is then:

$$(6) \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This matrix notation then allows us, in just a few lines of calculation, to obtain the elegant axiomatic definition of the element L of the Lorentz group:

$$(7) \quad {}^t L G L = G$$

If we add the space-time translation column vector

$$(8) \quad C = \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

We immediately obtain the representation, in the form of (5,5) matrix structures, of the Poincaré group:

$$(9) \quad \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

If we switch to a Euclidean space, its metric is of the form (5) where its Gramma matrix becomes the identity matrix I . Its isometry subgroup is the group of matrices defined by:

$$(10) \quad {}^t a I a = {}^t a a = I$$

That is, the orthogonal matrices and the isometry group become the Euclid group :

$$(10) \quad \begin{pmatrix} a & C \\ 0 & 1 \end{pmatrix}$$

Noether's theorem tells us that every subgroup is associated with the conservation of a quantity. For the Euclid group, we have the rotation subgroup $SO(3)$, which conserves the radius. For the Lorentz group, a subgroup of the Poincaré group, we have conservation of the quantity:

$$(11) \quad c^2 t^2 - x^2 - y^2 - z^2$$

2 – Regarding the phenomenon of metric signature inversion.

Today, in all treatises on theoretical physics and cosmology, one reads:

$$(12) \quad ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

This "modern signature" (- + + +) gradually became widespread after the Second World War. Since the coordinates are real, defining world lines equipped with real proper time then requires a purely imaginary length element ds :

$$(13) \quad d\tau = \sqrt{-ds^2}$$

Given that the solutions to Einstein's field equation are metrics, it would be logical to interpret these geometries as hypersurfaces whose geodesics of non-zero length are the paths followed by masses, with this very simple guiding principle:

- Where the length s is real: we are in the hypersurface, therefore in physical reality
- Where it is purely imaginary: we are outside the hypersurface, therefore we leave the world of physics.

By modifying the signature, we abandon this criterion and give rise to two families of trajectories:

- "Time-like" trajectories, with $ds^2 > 0$

- "Space-like" trajectories, with $ds^2 < 0$, which are actually located outside the hypersurface, are real, and represent an extension of spacetime into complex numbers.

3 – The birth of the field equation.

This Lorentz metric describes geometry in a vacuum. Ten years later, Einstein set out to describe the geometry created by the presence of masses, both outside and inside them. Initially, the great mathematician David Hilbert did not believe that this world of high mathematics in which he moved, and where he produced groundbreaking work, had any connection whatsoever with that of physics. During numerous meetings in 1915, Einstein managed to change his mind. It is important to remember that at that time, the dynamics of the universe were understood to be limited to the action of the electromagnetic force and the force of gravity. Hilbert then envisioned the possibility of carrying out an immensely ambitious project, constituting "the first theory of everything." When we read his 1915 memoir, entitled "The Foundations of Physics," we discover a space whose points are located by four coordinates $\{w_1, w_2, w_3, w_4\}$, time being the fourth. Let us quote him:

(14)

Es seien w_s ($s = 1, 2, 3, 4$) irgendwelch die Weltpunkte wesentlich eindentig benennende Koordinaten, die sogenannten Weltparameter (allgemeinste Rau-Zeit-Koordinaten)

Traduction :

Let w_s ($s = 1, 2, 3, 4$) be any coordinates labeling the world's points essentially uniquely—the so-called world parameters (most general spacetime coordinates).

Below is the complete action integral proposed by Hilbert:

(15)

und zwar muß die Variation des Integrals

$$\int H \sqrt{g} \, d\omega \quad (g = |g_{\mu\nu}|, \, d\omega = dw_1 dw_2 dw_3 dw_4)$$

für jedes der 14 Potentiale $g_{\mu\nu}, q_s$ verschwinden.

Traduction :

where the variation of the integral

$$\int H \sqrt{g} \, d\omega \quad (g = |g_{\mu\nu}|, \, d\omega = dw_1 dw_2 dw_3 dw_4)$$

must vanish for each of the fourteen potentials $g_{\mu\nu}, q_s$.

Here we recognize its Lagrange function H and the four-dimensional hypervolume:

$$(16) \quad \sqrt{g} \, d\omega = \sqrt{g} \, dw_1 dw_2 dw_3 dw_4$$

Hilbert represents by $|g_{\mu\nu}|$, the absolute value of the determinant constructed from the metric potentials $g_{\mu\nu}$. He will explicitly give the form of his Lagrange function:

$$(17) \quad H = K + L$$

Where K is the Ricci scalar and L the Lagrangian of matter. What is crucial, when reading his first article from 1915, is to understand that Hilbert performs all his calculations with the coordinates w_s ($s = 1, 2, 3, 4$) and that at no point does either the signature of the metric or the metric itself appear in the form ds^2 . For the mathematician David Hilbert, the solution to his field equation is not a metric, defining an element of length ds , but a bilinear differential form $g_{\mu\nu} dw_\mu dw_\nu$, whose sign, a priori, is of little importance to him. Given the historical significance of this event, here is Hilbert's field equation, as it appears in his 1915 article:

Unter Verwendung der vorhin eingeführten Bezeichnungsweise für die Variationsableitungen bezüglich der $g^{\mu\nu}$ erhalten die Gravitationsgleichungen wegen (20) die Gestalt

$$(21) \quad [\sqrt{g} K]_{\mu\nu} + \frac{\partial \sqrt{g} L}{\partial g^{\mu\nu}} = 0.$$

Das erste Glied linker Hand wird

$$[\sqrt{g} K]_{\mu\nu} = \sqrt{g} (K_{\mu\nu} - \frac{1}{2} K g_{\mu\nu}),$$

Fig.1 : Hilbert's field equation (November 20, 1915) [26]

Traduction :

Using the notation introduced earlier for the variational derivatives with respect to the $g^{\mu\nu}$, the gravitational equations, because of (20), take the form

$$[\sqrt{g} K]_{\mu\nu} + \frac{\partial \sqrt{g} L}{\partial g^{\mu\nu}} = 0. \quad (21)$$

The first term on the left hand side becomes

$$[\sqrt{g} K]_{\mu\nu} = \sqrt{g} (K_{\mu\nu} - \frac{1}{2} K g_{\mu\nu}),$$

Fig.2 : English translation of the passage in Figure 1 [27].

What we can write:

$$(18) \quad K_{\mu\nu} - \frac{1}{2} K g_{\mu\nu} = - \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} L}{\partial g_{\mu\nu}}$$

Simply replacing the letter K with the letter R (for Ricci) reveals that this equation is equivalent to the one published by Einstein five days later [28]! Hilbert's left-hand side results from his

choice to include the Ricci scalar, which he calls K , in the definition of its action. It's worth noting that Einstein published his equation in a form equivalent to the one that would later be considered standard, which is thus closer to the one published by Hilbert.

Ist in dem betrachteten Raume »Materie« vorhanden, so tritt deren Energietensor auf der rechten Seite von (2) bzw. (3) auf. Wir setzen

$$G_{im} = -\kappa \left(T_{im} - \frac{1}{2} g_{im} T \right), \quad (2a)$$

wobei

$$\sum_{\rho\sigma} g^{\rho\sigma} T_{\rho\sigma} = \sum_{\sigma} T_{\sigma}^{\sigma} = T \quad (5)$$

gesetzt ist; T ist der Skalar des Energietensors der »Materie«, die rechte Seite von (2a) ein Tensor. Spezialisieren wir wieder das Koordinatensystem in der gewohnten Weise, so erhalten wir an Stelle von (2a) die äquivalenten Gleichungen

$$R_{im} = \sum_l \frac{\partial \Gamma_{im}^l}{\partial x_l} + \sum_{\rho l} \Gamma_{i\rho}^l \Gamma_{ml}^{\rho} = -\kappa \left(T_{im} - \frac{1}{2} g_{im} T \right) \quad (6)$$

$$\sqrt{-g} = 1 \quad (3a)$$

Fig.3 : Einstein's field equation [28]

Traduction :

When there is "matter" in the space under consideration, its energy tensor occurs on the right-hand sides of (2) and (3), respectively. We set

$$G_{im} = -\kappa \left(T_{im} - \frac{1}{2} g_{im} T \right), \quad (2a)$$

where

$$\sum_{\rho\sigma} g^{\rho\sigma} T_{\rho\sigma} = \sum_{\sigma} T_{\sigma}^{\sigma} = T. \quad (5)$$

T is the scalar of the energy tensor of "matter," and the right-hand side of (2a) is a tensor. If we specialize the coordinate system again in the familiar manner, we get in place of (2a) the equivalent equations

$$R_{im} = \sum_l \frac{\partial \Gamma_{im}^l}{\partial x_l} + \sum_{\rho l} \Gamma_{i\rho}^l \Gamma_{ml}^{\rho} = -\kappa \left(T_{im} - \frac{1}{2} g_{im} T \right) \quad (6)$$

$$\sqrt{-g} = 1. \quad (3a)$$

Fig4 : Translation of the previous figure.

In this construction of the field equation including gravitation, Hilbert preceded Einstein by five days. This fact placed Einstein in a delicate position. In June-July 1915, Einstein was invited by Hilbert, a request supported by Felix Klein, to give lectures at the University of Groningen where he endeavored to show that his physical ideas required profound mathematical tools, and that these were not arbitrary, but dictated by physics itself. During the summer and autumn of 1915, Einstein maintained a dense correspondence with Hilbert [29,30]. Einstein sensed that Hilbert was undertaking to tackle the problem from a variational and mathematical perspective. He insisted, sometimes nervously, that physics preceded form (what we would call modeling today), that equations were not an autonomous aesthetic construct. Their correspondence reached a peak in intensity in the last 6 to 8 weeks before their publication. Einstein sought not only to convince Hilbert that his mathematics was correct, but also that general covariance was not a formal luxury, that mathematical structure was imposed by the physical principle of equivalence. Hilbert, conversely, was fascinated by the possibility that the variational approach could encompass all of physics. Two converging approaches, two temperaments, and a few months during which theoretical physics was written at the pace of letters and lectures, with no finalized articles. Two approaches, pursued differently, at breakneck speed, but leading to the same result. On November 18, Einstein, without revealing his equation to Hilbert, informed him that his approach had succeeded and wrote:

“Ich bin sehr gespannt, ob Sie mit Ihrer Lösung ebenso zufrieden sein werden wie ich mit der meinen.“

Translation:

"I am very curious to see if you will be as satisfied with your solution as I am with mine."

Their approaches were so similar, so parallel, that a joint publication could have been considered. But the stakes remained enormous, and they knew it. Especially since, for Hilbert, it was the equation of physics on which everything might depend: gravitation and electromagnetism. Nevertheless, Hilbert made the first move, sending his result for publication without informing Einstein, who immediately sent him a chilling letter on November 25, 1915 [29]. Einstein wrote, in particular:

« Die von Ihnen gegebene Lösung stimmt, soweit ich sehe, genau mit dem überein, was ich in meiner letzten Abhandlung gegeben habe. »

Translation:

"The solution you provided coincides, as far as I can tell, exactly with the one I presented in my last work."

Their relationship nearly broke down. But the two managed to preserve their friendship and mutual respect. Hilbert made the first move on December 15, 1915, in a letter congratulating Einstein on what he considered his achievement. He noted the convergence of independent efforts but stated that he claimed no priority. It was a magnanimous gesture to which Einstein replied on December 21, 1915:

“Ich freue mich, dass zwischen uns keine Verstimmung entstanden ist, und hoffe, dass wir uns auch künftig als Freunde betrachten können.“

Translation:

"I am pleased that no awkwardness has arisen between us, and I hope that we can continue to consider each other as friends."

At the time, Einstein was already a member of the Royal Prussian Academy of Sciences, a professor at the University of Berlin (without a regular teaching obligation), and director of the Kaiser Wilhelm Institute for Physics (then being established). His academic position was therefore firmly established. He and Hilbert knew that this breakthrough was too important to be tainted by a question of paternity, and opted for an amicable solution.

4 – La géométrisation de la physique.

The French mathematician Jean-Marie Souriau, who is an exceptional example combining high mathematics [31] and a keen sense of physics, used to say:

"A little mathematics takes you away from reality, a lot brings you back to it."

Cosmology is a perfect example, provided one knows how to use mathematical tools correctly. From what has been said above, we see that what guided Einstein was his extraordinary physical intuition. Didn't he say that in creating his special relativity, he strove to imagine how an observer "sitting on a photon" would perceive the universe? A few years later, Minkowski geometrized this vision, and everything became incredibly simple, expressed in just a few lines. Poincaré added to this the group tool and the group that bears his name (10), which became the isometry group of Minkowski space. The Lorentz metric, thanks to the use of the matrix tool, is reduced to a given Gram matrix (6) and emerges from an extremely elegant axiomatic system (7), which evokes the extension of rotations to this 4D space. This is where we realize that this geometrization of reality, while impeccably accounting for the physical phenomena of the moment, provides a context that suggests looking beyond the world that gave birth to it, through the hidden symmetries it reveals.

Let us take Euclidean space as an example. Being in this space allows us to construct the operations of rotations and symmetries in three dimensions. By combining the two, we can then imagine the object that allows us to extract a cork from a bottle. By geometrizing these operations, we create the Euclid group, where rotations (subgroup $SO(3)$) are translated into the form of matrices a . These are then defined axiomatically (10) from a Gram matrix, which becomes the identity matrix. We then realize that the complete group possesses an operation that had not been considered: "mirror symmetry." The shaping of objects, using our machine tools, represents the implementation of group actions. This is how we can manufacture "straight" corkscrews. The group suggests an inversion, a P-symmetry. So our machine tools produce "face-and-grab corkscrews", "left-handed" corkscrews, which screw in the other direction.

Geometrizing special relativity, as Minkowski and Poincaré did, gives us a group structure that accounts for the two symmetries it comprises: 4D rotations, thanks to the Lorentz subgroup L , and spacetime translations. This subgroup contains the inversion of the orientation of space, the P-symmetry. So far, so good. This symmetry exists in our physics, even in the movements of the elementary objects that compose it. The phenomenon of polarization allows us to add helicity to the photon's motion. And so, there exist two photons of the same energy with

opposite helicities. But the complete Poincaré group possesses antichronous components, which reverse time. At this point, the geometrizing approach represented by taking motion into account in Minkowski space is no longer sufficient. An additional act of geometrization is required, represented by symplectic geometry [31]. And here our spatial intuition and our effort at analogical extrapolation completely fail us. While this geometric extension, of pure mathematics, was introduced by mathematicians such as the American Bertram Kostant [32, 33] and the Russian Alexander Kirilov [34, 35], it is Souriau [31], as the third member of the trio, who highlights the impact of the tool in the world of physics, which is expressed in the following sentence:

To consider that there is a movement of a particle which is inscribed along a geodesic which is deduced from the "normal movement" by inversion of the time coordinate, is equivalent to considering that this particle carries a negative energy and, when it has a mass, that this mass is negative.

Assimilating special relativity meant admitting that particles—photons—could exist in spacetime, as presented by Minkowski, traveling along geodesics of zero length. This already required a considerable effort of mental representation. In a comic strip [36], I mention that the photon's death certificate is printed on the back of its birth certificate, a way of evoking the extreme brevity of its life, in its proper time. But imagining a spacetime resulting from the inversion of the temporal coordinate is beyond human imaginative possibilities—mine, at least. Yet it is this more advanced geometrization process, stemming from the implementation of all the symmetries of the Poincaré group [33], that gave rise to the Janus model [34]. And it is this new geometric conception of the universe that allows us to decode observations such as the acceleration of cosmic expansion, the large-scale, incomplete structure of the universe, the confinement of galaxies and the emergence of their spiral structure, the early birth of galaxies and stars, and the nature of the supermassive objects M87* and Sgr A* [39]. And, once again, it is up to Nature to decide. To better understand the topological aspects of the Janus model, refer to my comic strip *Topo the World* [40].

What the correspondence between Hilbert and Einstein reveals is the latter's skepticism towards this approach to pure mathematics, based on a variational model, and his fear of seeing this tool stray from reality. To Einstein's surprise, it enabled Hilbert to be the first to construct the tensorial field equation, geometrizing gravitation. Regarding the Janus Cosmological Model (JCM), while the topological aspects were described as early as 1994 [41], only the variational approach [38] allows this geometry to be equipped with its system of coupled field equations:

$$(19) \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi \left(T_{\mu\nu} + \sqrt{\frac{\bar{g}}{g}} K_{\mu\nu} \right)$$

$$(20) \quad \bar{R}_{\mu\nu} - \frac{1}{2} \bar{R} \bar{g}_{\mu\nu} = -\chi \left(\bar{T}_{\mu\nu} + \sqrt{\frac{g}{\bar{g}}} \bar{K}_{\mu\nu} \right)$$

If this resonates with the reader's intuition, we can attempt to summarize the approach taken in the Janus Cosmological Model as follows. In their embedding in \mathbb{R}^3 , every surface, a two-dimensional object, locally possesses a front and a back. The solutions to Einstein's equation on four-dimensional hypersurfaces. The extension of general relativity to JCM amounts to considering that the 4D hypersurface "has a front and a back" (which follows from the fact that it is orientable), with positive masses and photons of positive energy traveling on the "front," and negative masses traveling on the "back." What is then difficult to imagine and marks the

limits of this creation of a mental image is the fact that at two points A and B of this hypersurface there exist two different geodesics, one inscribed on the front and the other on the back, which are by no means each the image of the other and that the distances traveled and the travel times are different depending on whether one travels on the front or the back of the hypersurface.

5 – The Schwarzschild exterior metric solution.

It is November 1915. Hilbert and Einstein have just developed the "field equation." At the same time, Einstein, a physicist first and foremost, has published his own version and produced an approximate, linearized solution [42] that accounts for the advance of the perihelion of Mercury's orbit. As early as January 1916, Karl Schwarzschild provided an exact, nonlinear solution to the field equation in a vacuum and with spherical symmetry [43]. This remarkable work answered the question raised as early as 1846 by Le Verrier [44, 45, 46]. Nowadays, faced with the impossibility of observing this new planet, which Le Verrier had named Vulcan, our modern theorists might have called it a dark planet. Nevertheless, a month later Schwarzschild completed his work by publishing a second paper describing the geometry inside the mass [47]. Before analyzing Hilbert's article, we will point out that at the time, all the scientists involved in these developments of relativity opted, without exception, for a signature (+ - - -). Below is an excerpt from Einstein's article on the advance of Mercury's perihelion [42]:

Wir gehen nun in solcher Weise vor. Die $g_{\mu\nu}$ seien in »nullter Näherung« durch folgendes, der ursprünglichen Relativitätstheorie entsprechende Schema gegeben

$$\left. \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{matrix} \right\}, \quad (4)$$

Fig.5 : The Gram matrix used by Einstein [42]

Unambiguously, if we decide to focus on the concept of signature, Einstein's choice is clearly (- - - +) or (+ - - -), which amounts to exactly the same thing. The same options were used by contemporaries such as Karl Schwarzschild, Hermann Weyl [49], Ludwig Flamm [50], J. Droste, [51], etc. The only exception is Hilbert, in 1916 [48]. We need to understand why. As it appears explicitly in his first article of January 1916, the concept of geodesic is perfectly defined by Schwarzschild [43]:

Ein Punkt bewege sich gemäß der Forderung

$$\left. \begin{matrix} \delta \int ds = 0, \\ ds = \sqrt{\sum g_{\mu\nu} dx_\mu dx_\nu} \quad \mu, \nu = 1, 2, 3, 4 \end{matrix} \right\} \quad (1)$$

Fig.6: Geodesics, as defined by Schwarzschild [43].

In simpler terms, these are variationally defined curves, along which the length can only be real. The solution constructed by Schwarzschild is a metric, that is, a formula for calculating this real length element along a curve. He starts with a set of coordinates $\{t, r, \theta, \varphi\}$ accounting for spherical symmetry, where $r = \sqrt{x^2 + y^2 + z^2}$ is automatically non-negative. Taking into account spherical symmetry and stationarity, this metric becomes the following bilinear form (its equation (6)):

$$(21) \quad ds^2 = F dt^2 - (G + H) dr^2 - Gr^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

All that remains is for him to express, using this metric, the components of the Ricci tensor in order to inject them into Einstein's homogeneous equation:

$$(22) \quad R_{\mu\nu} = 0$$

To obtain the functions $F(r)$, $G(r)$, and $H(r)$, which, expressed using these coordinates and a positive integration constant α , gives

$$(23) \quad ds^2 = \frac{(r^3 + \alpha^3)^{1/3} - \alpha}{(r^3 + \alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4 dr^2}{(r^3 + \alpha^3)[(r^3 + \alpha^3)^{1/3} - \alpha]} - (r^3 + \alpha^3)^{2/3} (d\theta^2 + \sin^2\theta d\varphi^2)$$

More precisely :

$$(24) \quad ds = \sqrt{\frac{(r^3 + \alpha^3)^{1/3} - \alpha}{(r^3 + \alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4 dr^2}{(r^3 + \alpha^3)[(r^3 + \alpha^3)^{1/3} - \alpha]} - (r^3 + \alpha^3)^{2/3} (d\theta^2 + \sin^2\theta d\varphi^2)}$$

With this choice of coordinates, ds is always real. What cosmologists haven't yet understood is that a metric solution also contains "topology" information. The only intrinsic quantity is the length s . From this metric, we can calculate the surface area of a sphere corresponding to a given value of the variable r and with $dt = 0$. The metric then gives:

$$(25) \quad d\sigma^2 = (r^3 + \alpha^3)^{2/3} (d\theta^2 + \sin^2\theta d\varphi^2)$$

The determinant is then:

$$(26) \quad \gamma = (r^3 + \alpha^3)^{4/3} \sin^2\theta$$

The area of this sphere is:

$$(27) \quad A(r) = \iint \sqrt{\gamma} d\theta d\varphi = 4\pi (r^3 + \alpha^3)^{2/3}$$

This coordinate r is not a "radius". This area has a minimum value for the value $r = 0$, which is:

$$(28) \quad A_{min} = 4\pi \alpha^2$$

This means that the geometric object described by this metric solution is not contractible. We can, as Schwarzschild does, present this solution with a different set of coordinates by introducing the change of variable:

$$(27) \quad r = (R^3 - \alpha^3)^{1/3} \quad \text{ou} \quad R = (r^3 + \alpha^3)^{1/3}$$

The metric then becomes:

$$(29) \quad ds^2 = \left(1 - \frac{\alpha}{R}\right) c^2 dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad R = (r^3 + \alpha^3)^{1/3}$$

And it was in this form that Schwarzschild presented his solution in January 1916. But it then becomes absurd to consider studying the hypersurface for values $R < \alpha$ since, under these conditions, we leave the domain of definition with imaginary values of r . We will not dwell on this, as this issue has already been raised in [52].

6 - Calculation of geodesics using Lagrange's equations.

As indicated by Schwarzschild, see figure 6, geodesics result from the variation:

$$(30) \quad \delta \int \sqrt{\sum g_{\mu\nu} dx_\mu dx_\nu} = \delta \int \sqrt{\sum g_{\mu\nu} \frac{dx_\mu}{dp} \frac{dx_\nu}{dp}} dp$$

Posing :

$$\frac{dx_\mu}{dp} = \dot{x}_\mu \quad \frac{dx_\nu}{dp} = \dot{x}_\nu$$

$$(31) \quad \delta \int \sqrt{\sum g_{\mu\nu} \dot{x}_\mu \dot{x}_\nu} dp = \delta \int F(x_i, \dot{x}_i) dp = 0$$

Which leads to Lagrange's equations:

$$(32) \quad \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}_i} \right) = \frac{\partial F}{\partial x_i}$$

However, in all articles, treatises, and books, the Lagrangian is presented in the following form (example: [52], page 98):

**19. The geodesics in the Schwarzschild space-time:
the time-like geodesics**

We have shown in Chapter 1 (§6(a), equation (203)) that the equations governing the geodesics in a space-time with the line element,

$$ds^2 = g_{ij} dx^i dx^j, \quad (78)$$

can be derived from the Lagrangian

$$2\mathcal{L} = g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}, \quad (79)$$

where τ is some affine parameter along the geodesic. For time-like geodesics, τ may be identified with the proper time, s , of the particle describing the geodesic.

For the Schwarzschild space-time, the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \left[\left(1 - \frac{2M}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - 2M/r} - r^2 \dot{\theta}^2 - (r^2 \sin^2 \theta) \dot{\phi}^2 \right], \quad (80)$$

where the dot denotes differentiation with respect to τ .

Fig.7 : The Lagrangian, after Chandrasekhar [52].

And it is from this that we construct the Lagrange equations. This requires explanation. By writing:

(33)

$$\delta \int (g_{\mu\nu} \dot{x}_\mu \dot{x}_\nu) dp = 0$$

It seems that we are basing our calculations not on the Lagrange function derived from the expression for the length, but on its square, and that we are actually writing:

(34)

$$\delta \int [F(x_i, \dot{x}_i)]^2 dp = 0$$

This then leads to the following equations:

(35)

$$\frac{1}{L} \left(\frac{dL}{dt} \right) \left(\frac{\partial L}{\partial \dot{x}_i} \right) + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = \frac{\partial L}{\partial x_i}$$

So we have an extra term. How do we get rid of it? By choosing that:

(36)

$$\frac{dL}{dt} = 0$$

That is, by making the function L equal to a constant of arbitrary value. If we give it the value $L = 1$, we obtain (we don't see where this coefficient $\frac{1}{2}$ that Chandrasekhar adds comes from, see figure 7), we get this:

$$(37) \quad 1 = \left(1 - \frac{2M}{r}\right) \dot{t}^2 - \frac{1}{1 - \frac{2M}{r}} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

Or, under these conditions:

$$(38) \quad ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

We can therefore see, by constructing a system of Lagrange equations, and linking it to a metric solution (38), that if we start from the actions:

$$(39) \quad A = \int \sqrt{\left(1 - \frac{2M}{r}\right) \dot{t}^2 - \frac{1}{1 - \frac{2M}{r}} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)} ds$$

Or from :

$$(40) \quad A' = \int \left(\left(1 - \frac{2M}{r}\right) \dot{t}^2 - \frac{1}{1 - \frac{2M}{r}} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right) ds$$

We arrive at the *same* system of Lagrange equations! And the same would be true, with any n, for the action:

$$(41) \quad A' = \int \left(\left(1 - \frac{2M}{r}\right) \dot{t}^2 - \frac{1}{1 - \frac{2M}{r}} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right)^n ds$$

But (39) requires that the quantity under the radical remain positive, whereas (40) and (41) do not. By opting for (40), see Figure 7, we then construct real curves that "extend inside the sphere" of radius $R = 2M = \alpha$, the Schwarzschild sphere, but where the portions of the curves located "inside" are then equipped with purely imaginary lengths. Without realizing it, postwar theorists achieved an extension into the world of complex numbers.

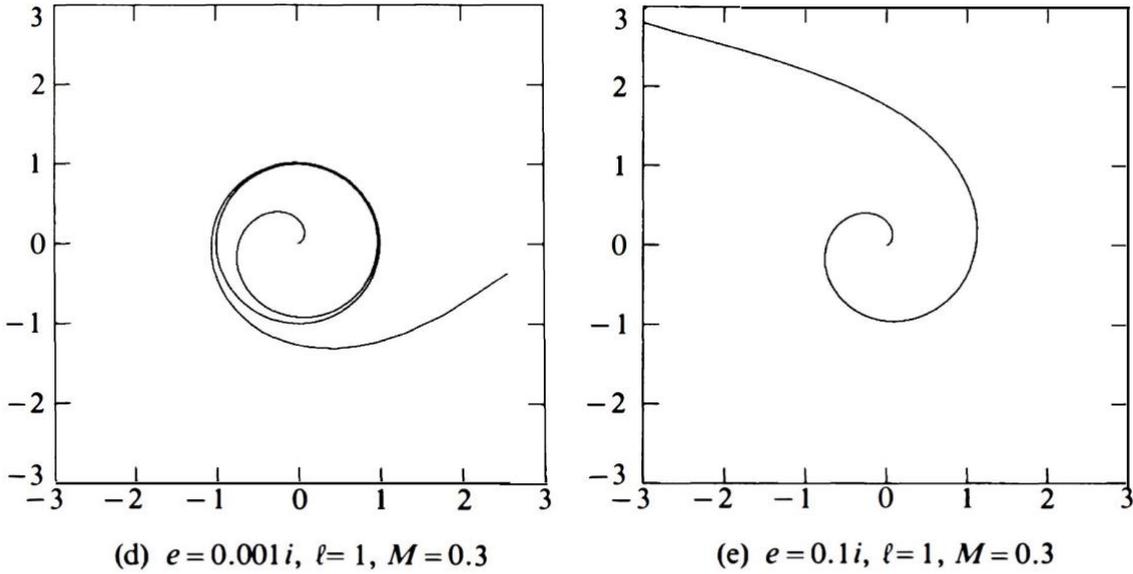


FIG. 7b. Various classes of time-like geodesics described by a test particle with $E^2 > 1$: (a), (b), (c): orbits of the first and the second kind with eccentricity $e = 3/2$ and latera recta, 4.5, 2.5, and 1.94 respectively ($M = 3/14$ in the scale along the coordinate axes); (d), (e): unbound orbits with $l = 1$ and with imaginary eccentricities $e = 0.001i$ and $0.1i$ ($M = 0.3$ in the scale along the coordinate axes).

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Fig.8 : Chandrasekhar's "plunging geodesic" [52], page 121.

It should be noted that these curves are merely the projections of the geodesics onto the plane $\theta = \text{constant}$ where they lie. This distinction is important, as we will see later. Nevertheless, when we use the differential equation that gives these geodesics (equation (6.90) from reference [53]), derived from the system of Lagrange's equations, we do indeed find these projections, which evoke the crossing of the Schwarzschild sphere with superb continuity of the tangent to these curves (in projection!). Below is this equation with a Schwarzschild radius.

$$(42) \quad R_s = 2M = \alpha = \frac{2Gm}{c^2} = 1$$

$$(43) \quad \varphi = \varphi_0 + \int \frac{dr}{r^2 \sqrt{\frac{l^2 c^2}{h^2} + \frac{1}{r h^2} - \frac{1}{r^2} + \frac{1}{r^3}}}$$

About thirty years ago, in an attempt to get to the bottom of this, we recalculated everything until we arrived at this equation. Plotting the curves immediately revealed these quasi-elliptical geodesics, with the phenomenon of perihelion precession. See below, in Chandrasekhar ([52] figure 7a):

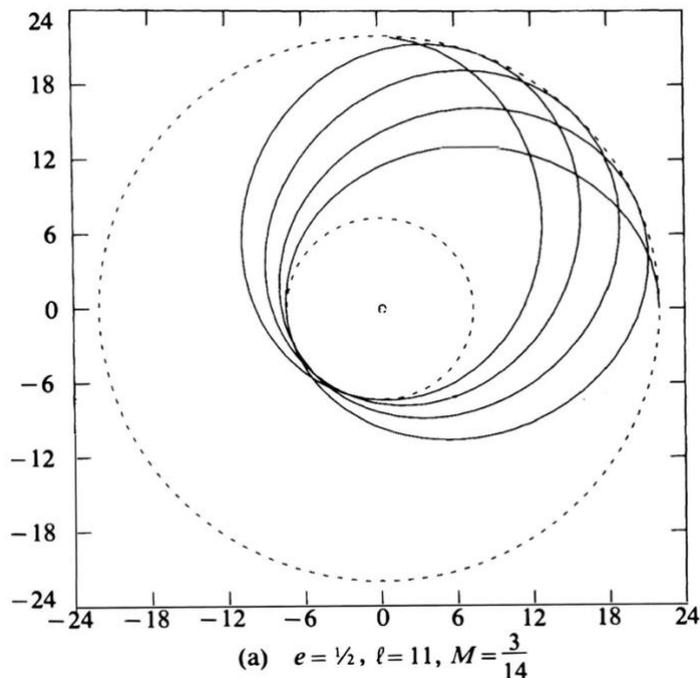


Fig.9 : Advance of perihelion ([52] figure 7a):

In the previous figure, the small circle with the dashed line is not the Schwarzschild circle but the circle along which the perihelion moves. Below is the curve I myself obtained, not without surprise:

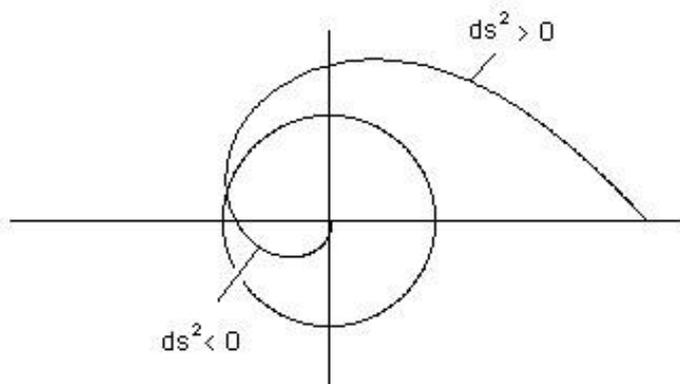


Fig.10: Real extension, in projection, of a curve equipped with an *imaginary length*.

The deceptive nature of this equation arises from the fact that, when one imagines extending the curve "inside the Schwarzschild sphere" using this equation, one obtains a perfect continuity of the tangent on the circle, which represents the projection of the sphere onto the plane $\theta = Cst$. In fact, as we will see later in a 3D representation (t, r, φ) and a perspective view, the geodesic never reaches the sphere! From this point, switching to the Kerr metric, which also has its own horizon, changes nothing. This has not prevented many authors from specifying "the structure of the "central singularity", *which exists only in their imagination* !

7- The geodesics of the Flamm surface are not those of the Schwarzschild hypersurface!

As early as 1916, the young Austrian mathematician Ludwig Flamm precisely analyzed [54] the structure of the solution hypersurface constructed by Schwarzschild. This constituted a brilliant demonstration of mastery of 4D geometry, a skill absent today among black hole specialists, who have not integrated the topological aspects of the solution. Since this solution does not depend on the time coordinate, Flamm deduced that the object is a 3D hypersurface undergoing a time translation. The metric of this 3D object is then simply, still with the indicated constraint:

$$(44) \quad d\sigma^2 = \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad R = (r^3 + \alpha^3)^{1/3}$$

What is difficult to grasp is that our choice of coordinates is merely a *system of representation*. It so happens that this 3D hypersurface described by equation (44) and with $\alpha > 0$ is embeddable in \mathbb{R}^3 (which would not be the case for the hypersurface described by the same metric, equipped with $\alpha < 0$). Mentally, we therefore imagine this object embedded in the only representational space available to our minds: 3D Euclidean space. The Schwarzschild sphere then becomes a throat sphere, the point of junction of this structure, which then appears, in the mental image we are trying so hard to construct, as a point connecting two 3D Euclidean spaces constituting a covering. The geodesics resulting from the metric (44) lie in planes. Below, on the left, is a planar geodesic that grazes the throat surface without touching it, and is therefore confined to the same 3D sheet. On the right, this geodesic "crosses the throat sphere" and thus extends into the other 3D sheet (shown as a dotted line):

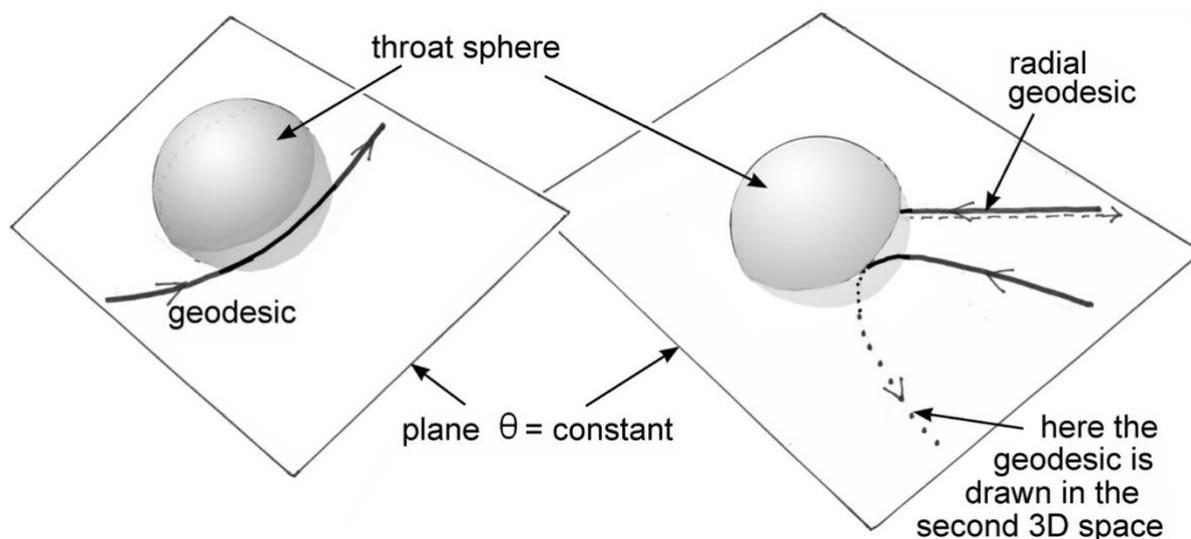


Fig.10: The hypersurface from equation (44) figured as a space bridge linking two 3D-spaces.

It is important to keep in mind that these are only mental images, derived from a representation in the specific coordinates (R, θ, φ) . If we return to the initial coordinates chosen by Schwarzschild, the 4D hypersurface, equation (23), is then the time translation of a 3D hypersurface defined by the metric:

$$(45) \quad d\sigma^2 = \frac{r^4 dr^2}{(r^3 + \alpha^3)[(r^3 + \alpha^3)^{1/3} - \alpha]} + (r^3 + \alpha^3)^{2/3} (d\theta^2 + \sin^2\theta d\varphi^2)$$

The throat sphere corresponds to the value $r = 0$. In other words, while its area is $4\pi\alpha^2$, in this representation it reduces to a point. This means that mental images have limits. But let's return to Flamm's analysis. If the hypersurface is three-dimensional, by setting, for example, $\theta = \pi/2$, we obtain a two-dimensional object, described by the metric:

$$(46) \quad d\sigma^2 = \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2 d\varphi^2 \quad R = (r^3 + \alpha^3)^{1/3}$$

Still with the same constraint $R \geq \alpha$. Flamm [54] observes that this object can be embedded in \mathbb{R}^3 and that it is then generated by the rotation of a horizontal parabola.

$$(47) \quad z = \pm 2\sqrt{\alpha(r - \alpha)}$$

The length of a portion of a curve corresponds to the integral:

$$(48) \quad \int \sqrt{\frac{r^2}{1 - \frac{\alpha}{r}} + r^2 \dot{\varphi}^2} ds$$

As we saw above, we obtain the Lagrange equations by basing the calculus of variations on:

$$(49) \quad \delta \int \left(\frac{r^2}{1 - \frac{\alpha}{r}} + r^2 \dot{\varphi}^2 \right) ds = 0$$

But this eliminates the constraint of non-negativity of the quantity under the radical. The equations giving the solution curves are the same:

$$(50) \quad \frac{d\varphi}{dr} = \pm \frac{h}{r^2} \frac{1}{\sqrt{\left(1 - \frac{\alpha}{r}\right)\left(1 - \frac{h^2}{r^2}\right)}}$$

We see that for $r > \alpha$, we obtain real curves for $|h| < \alpha$. But for $r < \alpha$, we also obtain real curves with $|h| > \alpha$. However, along these curves, which are spirals that start tangent to the circle and wind in a spiral, the length is purely imaginary. When we embed the Flamm surface in \mathbb{R}^3 , the absurdity becomes evident:

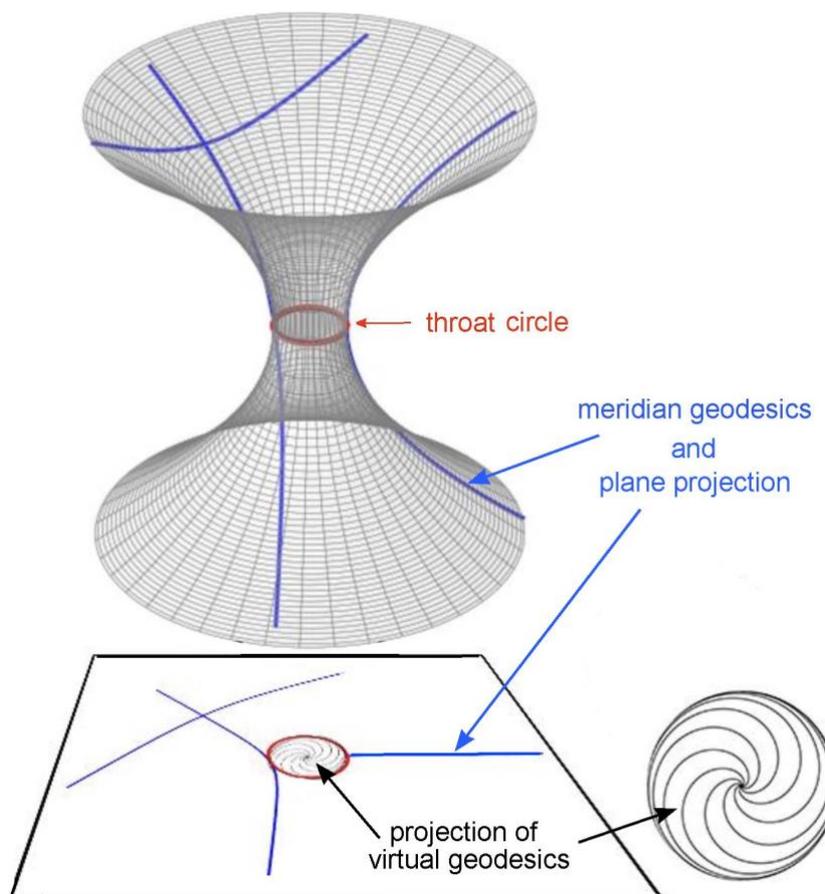


Fig. 11: Flamm surface, real and... virtual geodesics

But the geodesics of the Flamm surface are not those of a 4D hypersurface: the equations from which they are derived are different. Among other things, one cannot construct quasi-elliptic geodesics like those in Figure 9 on the Flamm surface. The virtual geodesics from (50) are tangent to the throat circle, while the curves from (43) intersect this circle at an angle. See Figures 8 and 10. But the idea is the same. Along these real curves, the length, and therefore proper time, is purely imaginary. It will probably be a long time before black hole specialists admit that these trajectories plunging into the horizon sphere and converging in a spiral toward the central singularity, like the singularity itself, have no physical reality. That being said, the Flamm surface remains a cross-section of Schwarzschild spacetime with constant θ . If we want to represent these geodesics in this embedding of the cross-section in \mathbb{R}^3 , we must lift the curves from the plane towards the Flamm surface, where they belong. But then they are not geodesics of this surface.

8 – The concept of "virtual geodesics".

It is easy to endow the torus and the sphere with these real curves, traversed by an imaginary length. What physicists have failed to grasp is that metrics like those derived from Einstein's equation possess not only the symmetries that preceded their construction, but also their own topology, revealed by an appropriate choice of coordinates. Let us illustrate this idea by taking the example of the torus, presented according to the metric:

$$(51) \quad ds^2 = r_0^2 d\theta^2 + (R + r_0 \cos\theta)^2 d\varphi^2$$

The signature is $(++)$. Considering the paths at $\theta = Cst$ and then at $\varphi = cst$, we immediately highlight the non-contractility of this surface. Let's now introduce the change of coordinate:

$$(52) \quad \theta = \arccos \frac{r-R}{r_0}$$

The line element becomes :

$$(53) \quad ds^2 = r^2 d\varphi^2 + \frac{dr^2}{-R^2 + r_0^2 - r^2 + 2rR}$$

It is immediately apparent that the signature $(++)$ will only be maintained if:

$$(54) \quad R - r \leq r \leq R + r_0$$

Otherwise, we are outside the torus. The length is expressed as follows:

$$(55) \quad s = \int \sqrt{r^2 \dot{\varphi}^2 + \frac{\dot{r}^2}{-R^2 + r_0^2 - r^2 + 2rR}} ds$$

But we know that we will obtain the same system of Lagrange equations by doing:

$$(56) \quad \delta \int \left(r^2 \dot{\varphi}^2 + \frac{\dot{r}^2}{-R^2 + r_0^2 - r^2 + 2rR} \right) ds = 0$$

Lagrange's equations give us the projection of the curves resulting from this variation calculation onto the plane (r, φ) :

$$(57) \quad d\varphi = \frac{\pm h dr}{r^2 \sqrt{(-R^2 + r_0^2 - r^2 + 2rR) \left(1 - \frac{h}{r^2}\right)}}$$

This allows us to plot the projection of a geodesic of the torus:

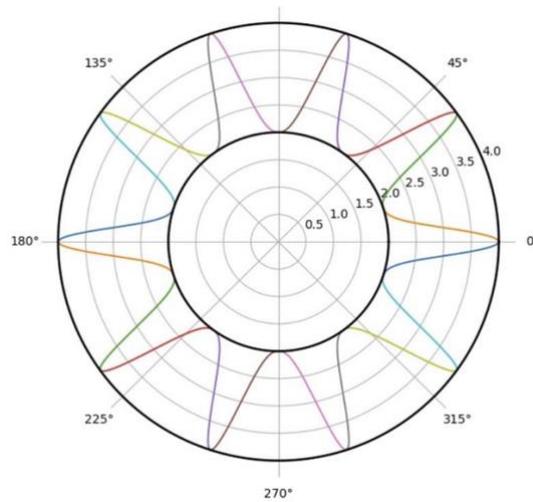


Fig.12: Projection of a geodesic of the torus.

Mais cette solution engendre également des courbes réelles qui sont la projection de courbes équipées d'une longueur imaginaire. Ci-après ces géodésiques virtuelles, à l'intérieur du tore:

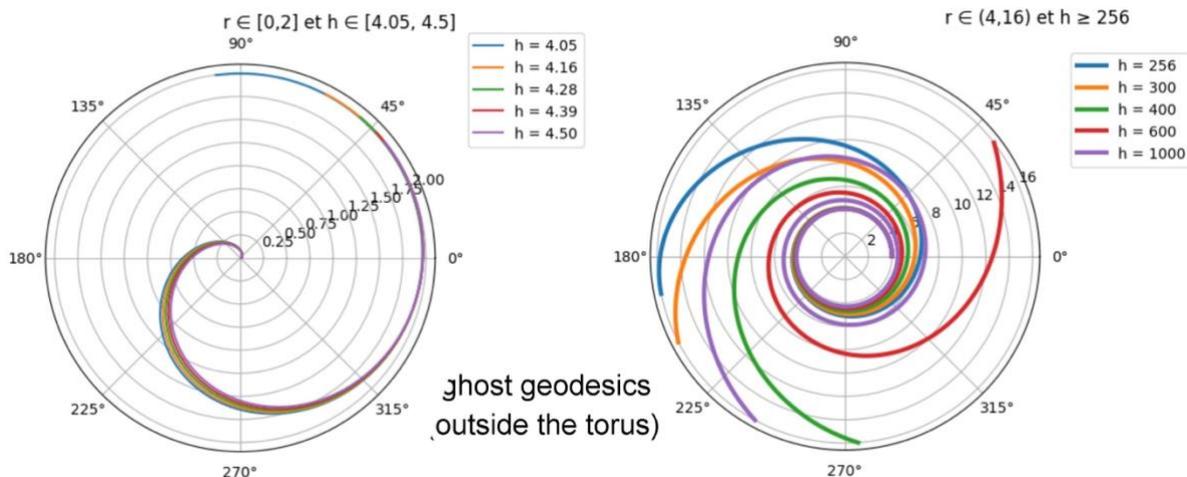


Fig. 13: Virtual geodesics of the torus "inside" its throat circle and "outside" its rim circle

A perspective view encompasses these results:

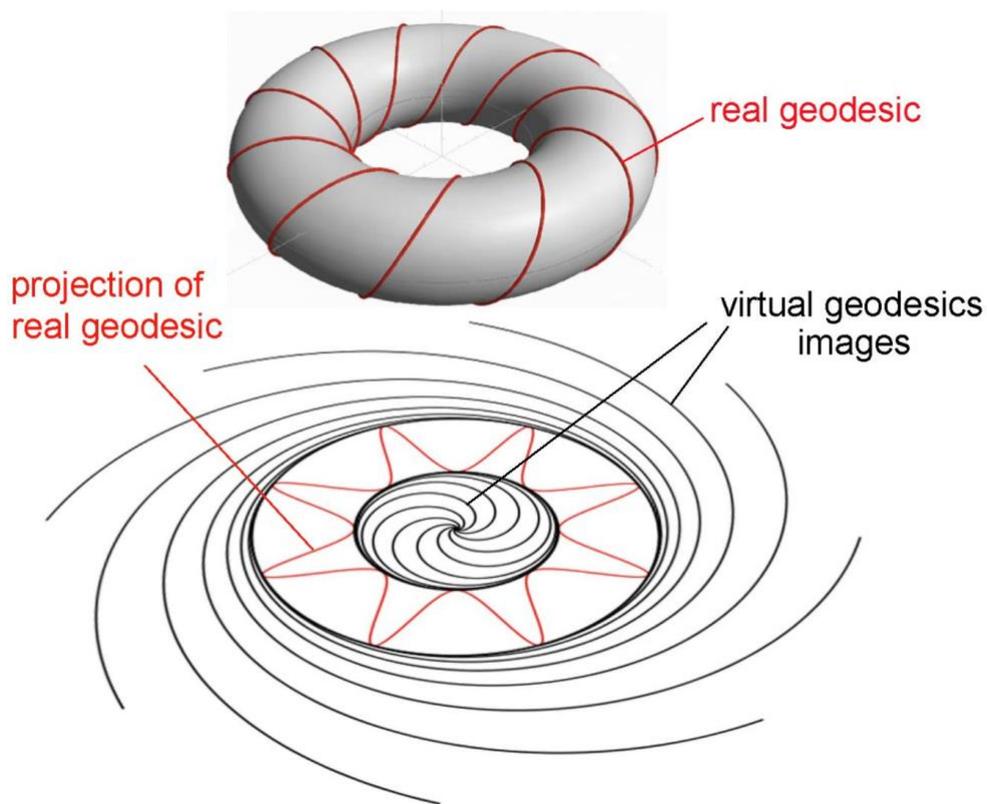


Fig.14: The torus, equipped with its (red) real and (black) virtual geodesics

Let's now turn to the sphere. The standard form of its metric is:

$$(58) \quad ds^2 = R^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

Let's make the following change of coordinate:

$$(59) \quad \theta = \arcsin \frac{r}{R}$$

The line element becomes :

$$(60) \quad ds^2 = r^2 d\varphi^2 + \frac{R^2 dr^2}{R^2 - r^2}$$

The variation calculation leads to the equation giving the projection of the geodesics:

$$(61) \quad d\varphi = \frac{hR}{r} \frac{dr}{\sqrt{(R^2 - r^2)(r^2 - h^2)}}$$

We get that :

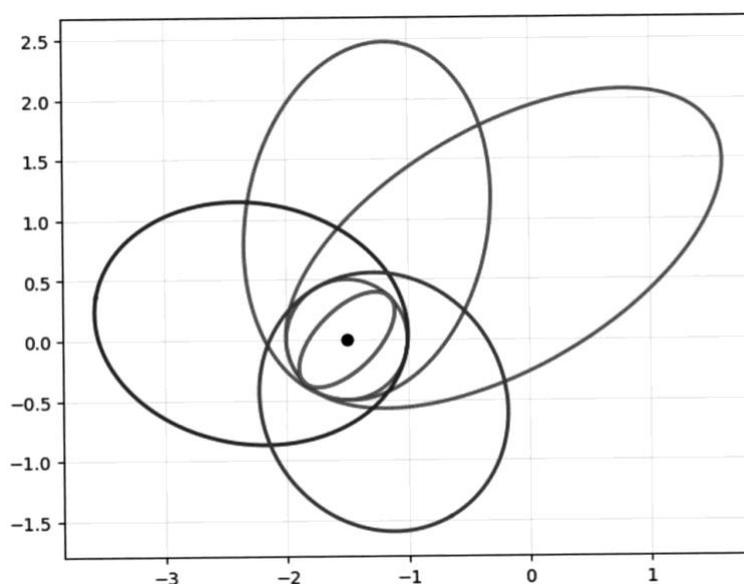


Fig.15 : Projections of geodesics and pseudo-geodesics of a sphere of radius 0.5

We have drawn, inside the circle, an ellipse, the projection of a geodesic of the sphere (these are "great circles"). But we have also drawn some of the projections of its "virtual geodesics". What are these curves, tangent to the sphere? Let's write:

$$(62) \quad \frac{dr}{d\varphi} = \frac{r}{hR} \sqrt{(r^2 - h^2)(R^2 - r^2)}$$

Posing :

$$(63) \quad r = \frac{1}{u}$$

$$(64) \quad - \frac{d\varphi}{du} = \frac{1}{hR} \sqrt{(1 - h^2 u^2)(R^2 u^2 - \frac{11}{u^2})}$$

$$(65) \quad \frac{d^2 u}{d\varphi^2} + u = \frac{1}{h^2}$$

$$(66) \quad u = \frac{1}{h^2} + A \cos(\varphi - \varphi_0)$$

The parameter R disappears.

$$(67) \quad r = \frac{h^2}{1 + e \cos(\varphi - \varphi_0)}$$

We have just established the first theorem of *imaginary geometry*:

The projections of the virtual geodesics of the sphere are ellipses!

Below, a perspective view:

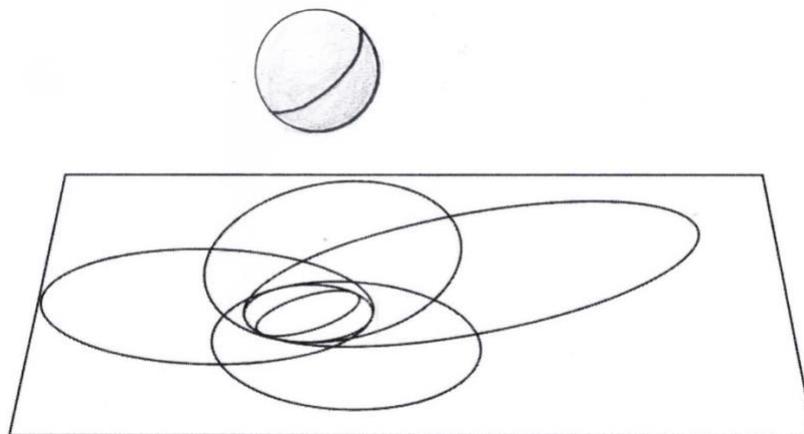


Fig.16 : Projection of geodesics and pseudo-geodesics of the sphere

Renowned scientists have produced some very elegant theorems about "the central singularity that black holes harbor at their center." Alas, these objects exist only in their imagination.

Using the Kerr metric would only complicate matters further. This object also possesses an event horizon, within which length, and therefore proper time, are purely imaginary. Thus, we would have the same phenomenon of extension through virtual geodesics, undoubtedly quite striking.

We can see, therefore, that the black hole model is a fantastic mathematical chimera, an artifact resulting from a flawed geometric interpretation. But one question remains: How is it possible to observe such continuity when, beyond the Schwarzschild circle, we construct the curve—real, but endowed with an imaginary length—that extends the geodesic?

9 – The geodesics of the Schwarzschild solution in 3D space (t, r, φ)

We will focus on the wormhole model, where two Schwarzschild solutions, in a form free of cross-terms in $drdt$, are associated to form a "space bridge". The curves then have the following schematic appearance:

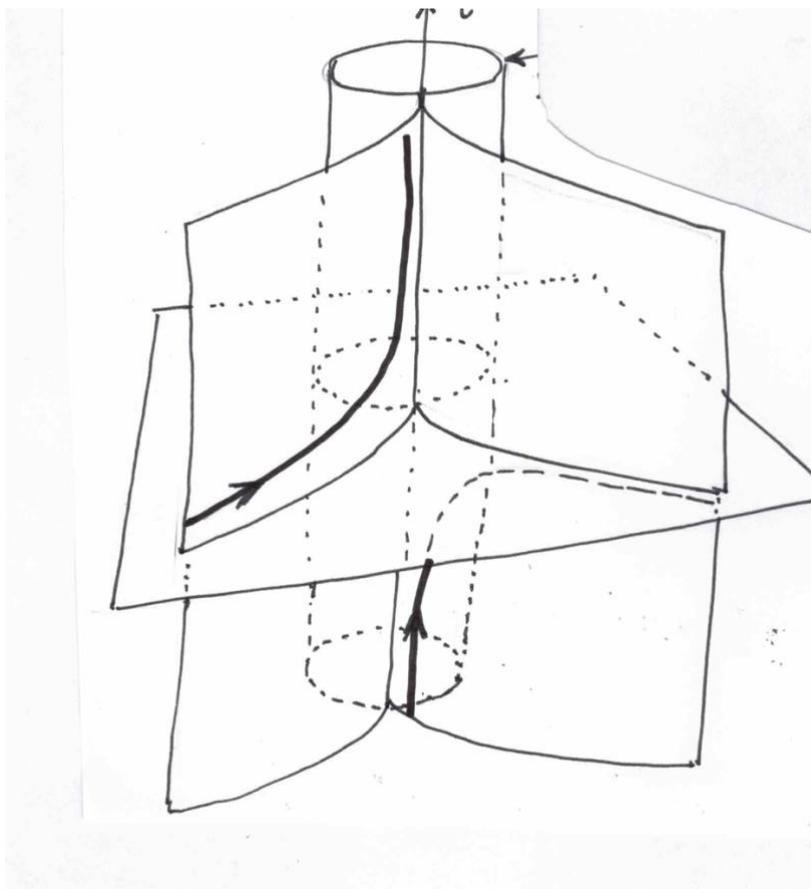


Fig.17 : Geodesics of the Schwarzschild wormhole in (t, r, φ)

We see that, evaluated using this coordinate t (called the "Schwarzschild time"), the free-fall and escape times are infinite. Crossing the throat sphere is then accompanied by a discontinuity in the geodesic. The figure shows the trace of these trajectories in their projection onto the plane. The break is therefore linked to the projection (r, φ) . This explains why theorists were encouraged to extend this geodesic along a curve spiraling towards the center. The continuity was only apparent. We can now move on to a Koiran wormhole [55,51] where two Schwarzschild solutions are conjugate, with crossed terms of opposite signs;

In our own old (attractive field):

$$(68) \quad ds^2 = \left(1 - \frac{\alpha}{r}\right) c^2 dt_E^2 - \left(1 + \frac{\alpha}{r}\right) dr^2 - \frac{2\alpha c}{r} dr dt_E - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

In the adjacent fold (repulsive field):

$$(69) \quad ds^2 = \left(1 + \frac{\alpha}{r}\right) c^2 dt_E^2 - \left(1 - \frac{\alpha}{r}\right) dr^2 + \frac{2\alpha c}{r} dr dt_E - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

Since we are trying to model the embedding in the wormhole and its emergence in the adjacent sheet, we have represented the direction of time (this time coordinate t_E , of Eddington), downwards:

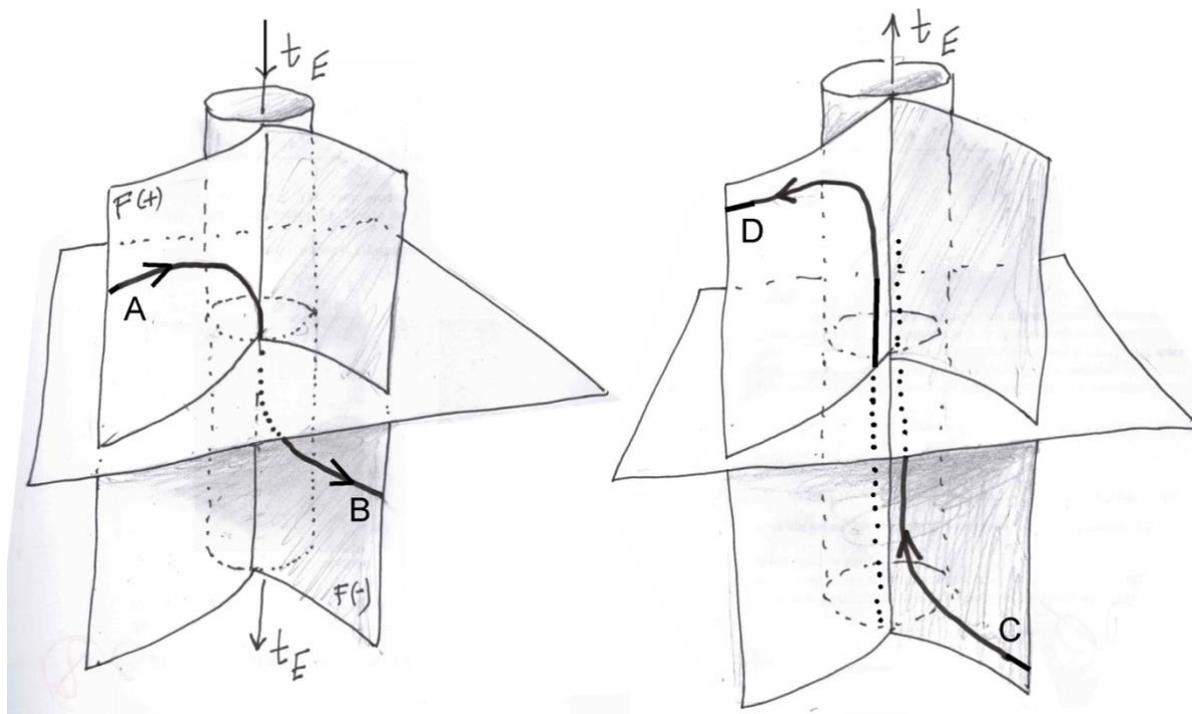


Fig.18 : Geodesics of the Koiran wormhole in (t_E, r, φ)

Wormholes do not belong to the realm of physics, as there, too, one would be considering describing a highly unsteady process using a steady-state solution. These images are simply intended to provide an interpretation of a geometric solution exhibiting continuity in geodesics. This interpretation of the solution illustrates the concept of a one-way membrane. Geodesics do not behave the same way depending on whether they are plunging or emerging.

On the left, at A, a control mass reaches the throat sphere in a finite time, and at B emerges from it into the second layer, also in a finite time. On the right, a control mass moving towards the throat sphere at C reaches it in an infinite time. At D, it emerges in an infinite time. Transit in the reverse direction is therefore impossible.

The figure on the left also shows the projection of the geodesics onto the plane (r, φ) , with their apparent break, which had led theorists to project these curves along a spiral converging towards the "center." The figure on the left, the only one depicting a physical phenomenon, shows a continuous geodesic, and it is its projection that produces this apparent break. To the cosmologists' credit, it must be admitted that this apparent continuity of the tangent in the plane (r, φ) , was entirely convincing enough to encourage them to consider such an extension.

10 – When the black hole model becomes, in the 21st century, the equivalent of the atom.

In 1989, seventy-three years after Hilbert's publication, the Canadian mathematician L. Abrams was the first to publish an article entitled "The black holes: the legation of Hilbert's error" [56]. In 2004, A. Loinger published a new article entitled "Black holes do not exist. Thus spoke Karl

Schwarzschild" [57]. In 2011, C. Corda attempted to refute these arguments [58]. The fact that these exchanges are located exclusively in the arXiv database, moreover in its "physics" section and not in the section where cosmologists publish their preprints: "general relativity and quantum gravity, abbreviated gr-qc," shows that no peer-reviewed journal has seen fit to submit to its reviewers an article questioning the existence of black holes. Thus, this journal is the first to address this question in an article [25] where we have simply reiterated and expanded upon arguments raised in previous articles, to general indifference. Conversely, the article [59] published in 2019 by Robbert Dijkgraaf, current director of the prestigious Institute for Advanced Study in Princeton, a veritable Mecca of world science, founded in 1930 and to which Einstein and Oppenheimer belonged, reveals the general adherence of current theorists to this model. It is 2019. The EHT has just obtained the first image of the supermassive object located at the center of the galaxy M87 [60]. Note the title of the article: The Shadow of the Supermassive Black Hole. It is a black hole... "because we cannot see what else it could be." Dijkgraaf, who is very active in science education, writes, and we quote [59]:

The iconic black hole image released by the Event Horizon Telescope collaboration has gone viral these days.... It is difficult to overestimate the importance of this discovery. There is only one moment in human history when we will have seen a black hole for the first time ... Black holes thus play a similar role as atoms did a hundred years ago ... The black hole is just such a paradox, the atom of the twenty-first century.

It is clear that it will prove very difficult to position this question within a scientific community that has made black holes the idols of a religion of dreary times, especially since an alternative interpretation of the available observational data : plugstars, has just emerged. [61]

11 – David Hilbert's stange conception of space-time.

We now return to 1916. Hilbert felt compelled to report, in an expanded version of his 1915 paper, on these solutions to the field equation, which had recently appeared in the literature. But physical applications remained quite new to him, as he had been navigating this world since July 1915 as if on another planet. When one reads his 1916 article [48], one realizes that he had not yet incorporated the concept of spacetime as defined by Minkowski in 1909, namely that the time coordinate t , modulo the multiplicative coefficient c , thus becoming the quantity ct , like x , y , and z , is expressed in meters. It's as if an ancient surveyor, after starting with land surveying and measuring length and width, suddenly realized that heights and altitudes can also be expressed in acres, miles, feet, and meters, and that by turning one's head 90 degrees, it's possible to consider things differently, from another angle. The same applies to relativity, except that it's the Lorentz group that allows us to perform these strange four-dimensional rotations.

In his first article [26], Hilbert constructed his field equation with his set of coordinates $\{w_1, w_2, w_3, w_4\}$, without ever needing to specify the nature of these quantities, where the three coordinates $\{w_1, w_2, w_3\}$ refer to space and w_4 to time. Strangely, Hilbert's article makes no mention of the second solution published by Schwarzschild, describing the geometry inside the masses [47]. He never attempts to incorporate into his paper anything other than Schwarzschild's first solution, which refers to the vacuum [43].

In his article, attention should be focused on certain key passages that reveal Hilbert's conception of the universe. It is essential to keep in mind that in 1916 the universe was not evolving and relativistic physics did not yet exist. The curvatures affecting it were merely minute perturbations unfolding within a decidedly Euclidean framework, associated with metric potentials $\delta_{\mu\nu}$. We read:

Im Falle der pseudo-Euklidischen Geometrie haben wir

$g_{\mu\nu} = \delta_{\mu\nu}$

worin

$\delta_{\mu\mu} = 1, \quad \delta_{\mu\nu} = 0 \quad (\mu \neq \nu)$

bedeutet. Für jede dieser pseudo-Euklidischen Geometrie benachbarte Maßbestimmung gilt der Ansatz

Fig. 19: Hilbert's "pseudo-Euclidean" view of spacetime [48].

Translation :

In the case of the pseudo-Euclidean geometry we have

$g_{\mu\nu} = \delta_{\mu\nu},$

where

$\delta_{\mu\mu} = 1, \quad \delta_{\mu\nu} = 0 \quad (\mu \neq \nu).$

For every metric in the neighborhood of this pseudo-Euclidean geometry the ansatz

$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} + \dots \tag{37}$

is valid, where ε is a quantity converging to zero, and $h_{\mu\nu}$ are functions of the w_s .

Fig. 20: Spacetime, according to Hilbert, is pseudo-Euclidean.

The matrix $\delta_{\mu\nu}$ is the identity matrix. The spacetime metric, according to Hilbert, is therefore:

$$(70) \quad ds^2 = g_{11}dw_1^2 + g_{22}dw_2^2 + g_{33}dw_3^2 + g_{44}dw_4^2$$

Then, the « signature » of his bilinear form is (+ + + +)

In Hilbert's theory, spacetime is not defined by a metric. *It is determined by a bilinear form.* The difference is crucial. If its points are located along a trajectory using a parameter p , he writes:

$$(71) \quad G\left(\frac{dw_1}{dp}, \frac{dw_2}{dp}, \frac{dw_3}{dp}, \frac{dw_4}{dp}\right) = g_{11} \left(\frac{dw_1}{dp}\right)^2 + g_{22} \left(\frac{dw_2}{dp}\right)^2 + g_{33} \left(\frac{dw_3}{dp}\right)^2 + g_{44} \left(\frac{dw_4}{dp}\right)^2$$

In the following figures, Hilbert, in his own way, attempts to integrate into the new version of his memoir "Foundation of Physics" [48] the exact solution that Karl Schwarzschild had just published in January 1916 [43], the one describing the geometry of the void surrounding a mass. In the plate below, you will find gathered all the errors made by Hilbert that would become the foundation of "modern cosmology." Let us begin with the German text:

Nach Schwarzschild ist die allgemeinste diesen Annahmen entsprechende Maßbestimmung in räumlichen Polarkoordinaten, wenn

$$\begin{aligned} w_1 &= r \cos \vartheta \\ w_2 &= r \sin \vartheta \cos \varphi \\ w_3 &= r \sin \vartheta \sin \varphi \\ \mathbf{a} \longrightarrow w_4 &= l \end{aligned}$$

gesetzt wird, durch den Ausdruck

$$(42) \quad F(r) dr^2 + G(r) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + H(r) dl^2$$

dargestellt, wo $F(r)$, $G(r)$, $H(r)$ noch willkürliche Funktionen von r sind. Setzen wir

$$\mathbf{b} \longrightarrow r^* = \sqrt{G(r)},$$

so sind wir in gleicher Weise berechtigt r^* , ϑ , φ als räumliche Polarkoordinaten zu deuten. Führen wir in (42) r^* anstatt r ein und lassen dann wieder das Zeichen $*$ weg, so entsteht der Ausdruck

$$(43) \quad M(r) dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 + W(r) dl^2,$$

wo $M(r)$, $W(r)$ die zwei wesentlichen willkürlichen Funktionen von r bedeuten. Die Frage ist, ob und wie diese auf die allgemeinste Weise zu bestimmen sind, damit den Differentialgleichungen (36) Genüge geschieht.

Zu dem Zwecke müssen die bekannten in meiner ersten Mitteilung angegebenen Ausdrücke $K_{\mu\nu}$, K berechnet werden. Der erste Schritt hierzu ist die Aufstellung der Differentialgleichungen der geodätischen Linien durch Variation des Integrals

$$\mathbf{c} \longrightarrow \int \left(M \left(\frac{dr}{dp} \right)^2 + r^2 \left(\frac{d\vartheta}{dp} \right)^2 + r^2 \sin^2 \vartheta \left(\frac{d\varphi}{dp} \right)^2 + W \left(\frac{dl}{dp} \right)^2 \right) dp.$$

Fig.21: The errors made by Hilbert, in their original version [48].

Translation :

The assumptions about the $g_{\mu\nu}$ are the following:

1. The metric is represented in a Gaussian coordinate system, except that g_{44} is left arbitrary, i.e. we have

$$g_{14} = 0, \quad g_{24} = 0, \quad g_{34} = 0.$$

2. The $g_{\mu\nu}$ are independent of the time coordinate x_4 .
3. The gravitation $g_{\mu\nu}$ is centrally symmetric with respect to the origin of coordinates.

According to Schwarzschild the most general metric conforming to these assumptions is represented in polar coordinates, where

$$\begin{aligned} w_1 &= r \cos \vartheta \\ w_2 &= r \sin \vartheta \cos \varphi \\ w_3 &= r \sin \vartheta \sin \varphi \end{aligned}$$

a \longrightarrow $w_4 = l,$

by the expression

$$F(r)dr^2 + G(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + H(r)dl^2 \quad (42)$$

where $F(r), G(r), H(r)$ are still arbitrary functions of r . If we put

b \longrightarrow $r^* = \sqrt{G(r)},$

then we are equally justified in interpreting r^*, ϑ, φ as spatial polar coordinates. If we introduce r^* in (42) instead of r and then eliminate the sign $*$, the result is the expression

$$M(r)dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + W(r)dl^2, \quad (43)$$

where $M(r), W(r)$ mean the two essential, arbitrary functions of r . The question is whether and how these can be determined in the most general way so that the differential equations (36) enjoy satisfaction. |

To this end the well-known expressions $K_{\mu\nu}, K$ given in my first communication must be calculated. The first step in this is the derivation of the differential equations for geodesic lines by variation of the integral

c \longrightarrow $\int \left(M \left(\frac{dr}{dp} \right)^2 + r^2 \left(\frac{d\vartheta}{dp} \right)^2 + r^2 \sin^2\vartheta \left(\frac{d\varphi}{dp} \right)^2 + W \left(\frac{dl}{dp} \right)^2 \right) dp.$

Fig. 22: Hilbert's multiple errors, all on the same plate.

\rightarrow As you can see is **a**, the calculations will be undertaken without the specific nature of this fourth coordinate being defined. Hilbert feels no need for it at this stage. In his equation (42), we find its bilinear form, with three unknown functions $F(r), G(r), H(r)$, to be determined. He knows that in polar coordinates, the bilinear form that gives substance to Euclidean space is:

$$(73) \quad dr^2 + r^2 d\theta^2 + \sin^2\theta d\varphi^2$$

→ In **b**, replacing the function $G(r)$ with it therefore seems to be a way to simplify the calculation by converging more quickly towards the expected result. Especially since this form must be identified with (73) at infinity. This is the error that L. Abrams pointed out in 1989, *seventy-three years* later [56].

→ In **c**, Hilbert defines its action. We disregard the length. We saw the consequences above. It remains for him to calculate the coefficients of his Ricci tensor, which he denotes by $K_{\mu\nu}$, and to calculate the Christoffels coefficients. And the field equation is then, in a vacuum, simply:

$$(74) \quad K_{\mu\nu} = 0$$

Having completed all calculations, here is its result, that is to say its bilinear form, still expressed using this fourth time coordinate l .

$$(75) \quad G(dr, d\theta, d\varphi, dl) = \frac{r}{r-\alpha} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 + \frac{r-\alpha}{r} dl^2$$

with signature (+ + + +)

Since it was necessary to recover Schwarzschild's result, Hilbert posited $l = it$, which shows that he had not at all incorporated the vision of Einstein and Minkowski and remained attached to the hypothesis initially proposed by Poincaré in 1902 [16], then by Minkowski in 1907 [21] (which the latter would abandon the following year). Despite eight months of intense dialogue with Einstein:

→ For Hilbert, Special Relativity boils down to the fact that time is purely imaginary

This leads him to reformulate his result according to:

$$(76) \quad G(dr, d\theta, d\varphi, dt) = \frac{r}{r-\alpha} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 - \frac{r-\alpha}{r} dt^2$$

→ Thus, in December 1916, we have the birth certificate of the "modern" signature:

$$(+ + + -)$$

Neglecting differences in sign, Hilbert concludes that this expression is, in his opinion, identical to the result published by Schwarzschild. He then mentions, in a footnote, this change of variable used by Schwarzschild:

$$(77) \quad R = (r^3 + \alpha^3)^{1/3}$$

He considers this a way to bring the singularity (at $r = \alpha$) back to the origin (at $r = 0$), specifying that "there are simpler ways to do this." This is without realizing that his simplification **b** transforms his initial variable r , which he considers "radial," into this "intermediate quantity" R of Schwarzschild.

He still needs to construct the geodetic trajectories. He then performs the variation according to:

Die Differentialgleichungen der geodätischen Linien für das zentrische Gravitationsfeld (45) entspringen aus dem Variationsproblem

$$\rightarrow \delta \int \left(\frac{r}{r-\alpha} \left(\frac{dr}{dp} \right)^2 + r^2 \left(\frac{d\vartheta}{dp} \right)^2 + r^2 \sin^2 \vartheta \left(\frac{d\varphi}{dp} \right)^2 - \frac{r-\alpha}{r} \left(\frac{dt}{dp} \right)^2 \right) dp = 0,$$

sie lauten nach bekanntem Verfahren:

$$(46) \quad \frac{r}{r-\alpha} \left(\frac{dr}{dp} \right)^2 + r^2 \left(\frac{d\vartheta}{dp} \right)^2 + r^2 \sin^2 \vartheta \left(\frac{d\varphi}{dp} \right)^2 - \frac{r-\alpha}{r} \left(\frac{dt}{dp} \right)^2 = A,$$

$$(47) \quad \frac{d}{dp} \left(r^2 \frac{d\vartheta}{dp} \right) - r^2 \sin \vartheta \cos \vartheta \left(\frac{d\varphi}{dp} \right)^2 = 0,$$

$$(48) \quad r^2 \sin^2 \vartheta \frac{d\varphi}{dp} = B,$$

$$(49) \quad \frac{r-\alpha}{r} \frac{dt}{dp} = C,$$

wo A, B, C Integrationskonstante bedeuten.

Fig.23: Hilbert's variational approach [48]

The differential equations of geodesic lines for the centrally symmetric gravitational field (45) arise from the variational problem

$$\rightarrow \delta \int \left(\frac{r}{r-\alpha} \left(\frac{dr}{dp} \right)^2 + r^2 \left(\frac{d\vartheta}{dp} \right)^2 + r^2 \sin^2 \vartheta \left(\frac{d\varphi}{dp} \right)^2 - \frac{r-\alpha}{r} \left(\frac{dt}{dp} \right)^2 \right) dp = 0,$$

and become, by well-known methods:

$$\frac{r}{r-\alpha} \left(\frac{dr}{dp} \right)^2 + r^2 \left(\frac{d\vartheta}{dp} \right)^2 + r^2 \sin^2 \vartheta \left(\frac{d\varphi}{dp} \right)^2 - \frac{r-\alpha}{r} \left(\frac{dt}{dp} \right)^2 = A, \quad (46)$$

$$\frac{d}{dp} \left(r^2 \frac{d\vartheta}{dp} \right) - r^2 \sin \vartheta \cos \vartheta \left(\frac{d\varphi}{dp} \right)^2 = 0, \quad (47)$$

$$r^2 \sin^2 \vartheta \frac{d\varphi}{dp} = B, \quad (48)$$

$$\frac{r-\alpha}{r} \frac{dt}{dp} = C, \quad (49)$$

where A, B, C denote constants of integration.

Fig. 24: Hilbert's variational approach (translation)

This part of his article [48] is perhaps the one that most deserves our attention. In 1915, Hilbert introduced [27] the variational technique into cosmology for the first time, which allowed him

to use it to construct the field equation. His genius lay in constructing the action based on the Ricci scalar, which he called K , and on the four-dimensional hypervolume $\sqrt{|g|} dx_1 dx_2 dx_3 dx_4$. The scientific community would later designate this approach as based on the "Hilbert-Einstein action." In these diagrams, the first red arrow shows that the action is constructed not on the elementary length, but using the bilinear form. Of the four equations (46), (47), (48), and (49) in Figures 23 and 24, the last three are the Lagrange equations, which are derived from the indicated variation. However, equation (46), which assumes the constancy of the bilinear form, is an additional assumption of Hilbert's, the very one that explains why his chosen action will indeed yield the geodesics, even though the action is not designed to minimize the length, but its square. See what was explained above. In the subsequent calculation, Hilbert concludes that this constant A must be negative.

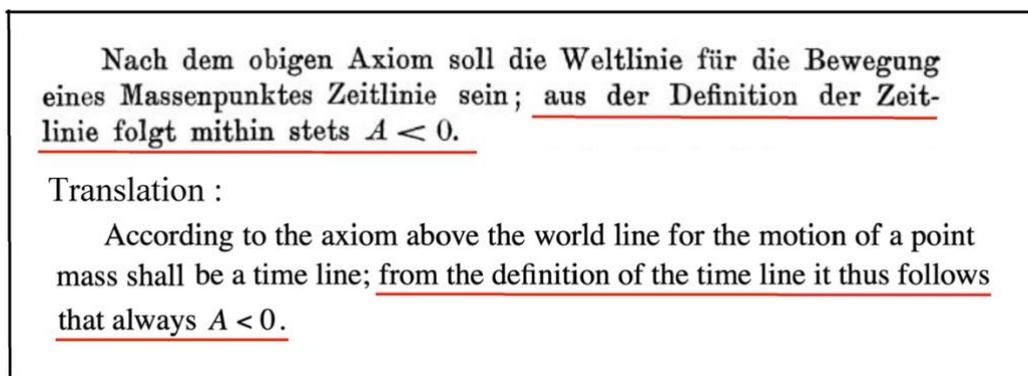


Fig.25: New implicit inversion of the signature of the bilinear form.

The solutions to these equations must be timelines. Now, in this same article, Hilbert defines the proper time associated with a timeline, adding in passing another "length λ " this time associated with trajectories that will later be described as "space-like":

Ist in dieser Geometrie eine Kurve

$$x_s = x_s(p) \quad (s = 1, 2, 3, 4)$$

gegeben, wo $x_s(p)$ irgend welche reelle Funktionen des Parameters p bedeuten, so kann diese in Teilstücke zerlegt werden, auf denen einzeln der Ausdruck

$$\mathbf{a} \longrightarrow G\left(\frac{dx_1}{dp}, \frac{dx_2}{dp}, \frac{dx_3}{dp}, \frac{dx_4}{dp}\right)$$

nicht sein Vorzeichen ändert: ein Kurvenstück, für welches

$$G\left(\frac{dx_s}{dp}\right) > 0$$

ausfällt, heiße eine *Strecke* und das längs dieses Kurvenstücks genommene Integral

$$\lambda = \int \sqrt{G\left(\frac{dx_s}{dp}\right)} dp$$

heiße die *Länge der Strecke*; ein Kurvenstück, für welches

$$\mathbf{b} \longrightarrow G\left(\frac{dx_s}{dp}\right) < 0$$

ausfällt, heiße eine Zeitlinie und das längs dieses Kurvenstückes genommene Integral

$$\mathbf{c} \longrightarrow \tau = \int \sqrt{-G\left(\frac{dx_s}{dp}\right)} dp$$

heiße die Eigenzeit der Zeitlinie; endlich heiße ein Kurvenstück, längs dessen

$$G\left(\frac{dx_s}{dp}\right) = 0$$

wird, eine *Nullinie*.

Fig.26: The two lengths introduced by Hilbert [48]

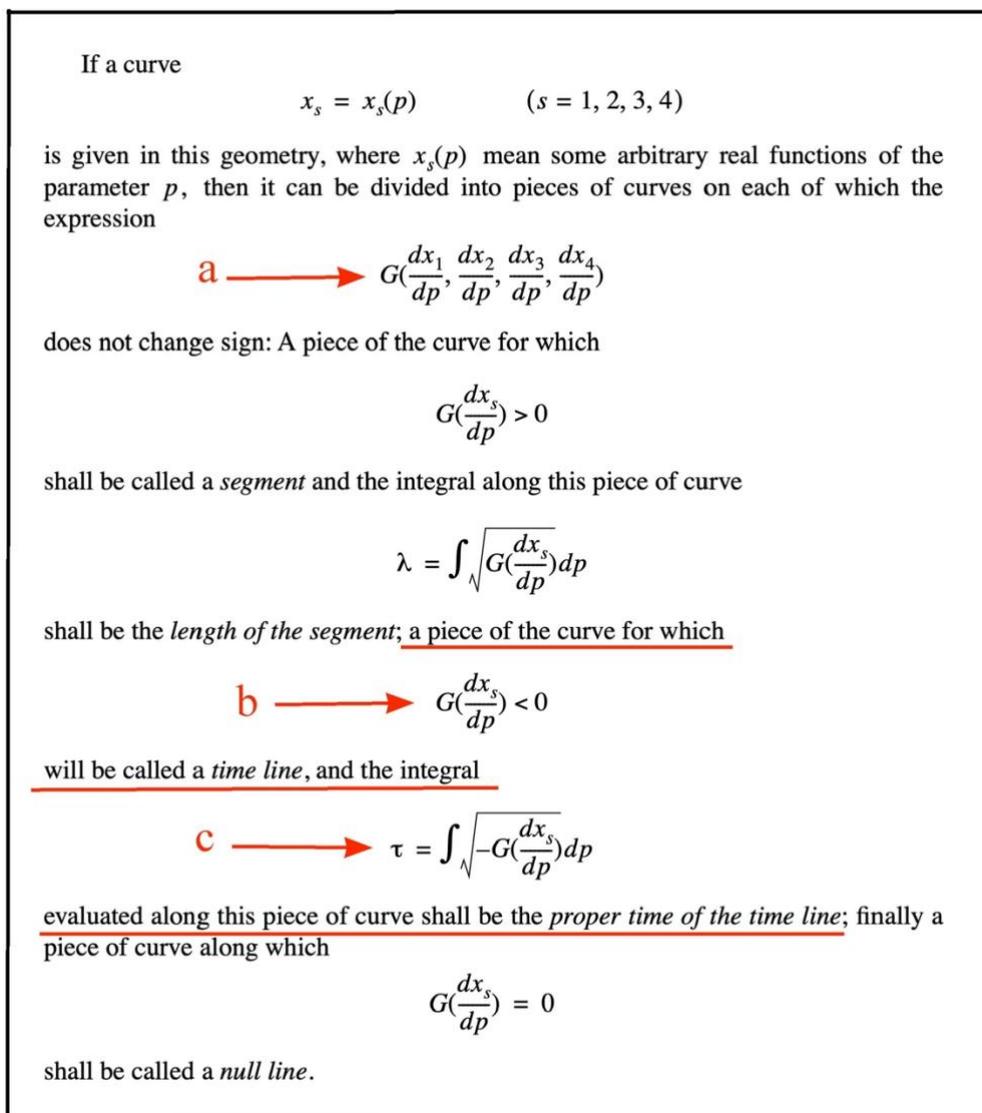


Fig.27 : Translation.

In **a** its bilinear form \mathbf{G} , which therefore corresponds to equation (76). Now, in Figure 25, he has just determined that, for these trajectories to have physical meaning, to be timelines, G is negative. We are therefore in case b. So, why introduce this catch-up c which allows us to obtain a real proper time? Wouldn't it be simpler and more logical, given this sign constraint, to return to the fundamentals introduced by Einstein, Schwarzschild, Wey, Droste, and all the others, and simply write:

$$(78) \quad G(dr, d\theta, d\varphi, dt) = ds^2 = \frac{r-\alpha}{r} dt^2 - \frac{r}{r-\alpha} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

with signature $(+ - - -)$

This would then align with Schwarzschild's result of January 1916 and, by taking a unit value for the speed of light as he did, would transform his expression **c** into:

$$(79) \quad \tau = s = \int \sqrt{G\left(\frac{dx_s}{dp}\right)} dp$$

We see in this section that several things came into play, initially stemming from Hilbert's vision of spacetime as Euclidean geometry, suddenly accompanied by a time coordinate that reveals its relativistic nature, that of a purely imaginary entity. Then, while retaining his signature (+++), to recover a real proper time, he had to change the sign of his bilinear form to a radical sign. These are subtleties of which we find little trace in works devoted to cosmology. Adding this question of virtual geodesics, everything was then in place to give birth to a chimera: the model of the black hole. The transition from the original signature (+ + + -), where the guiding thread, allowing exploration of the solution hypersurface, defined by its metric being to stick to a real length s , to a metric, called modern (+ - - -) has caused this thread to be lost, leading to expressions like $d\tau = \sqrt{-ds^2}$ et and, on the level of a physical interpretation to this surrealistic conclusion that "inside the black hole the coordinates of time and space exchange their respective roles".

11- Conclusion.

This second article complements what was presented in the first [25], where the mathematical and geometric inconsistency of the black hole model was outlined. It discusses the approach that gradually brought to light the true geometric context of special relativity, Minkowski space, with its signature (+ - - -). It recalls that the requirement for subluminal speeds boils down to the imperative of considering only trajectories of real length s , that is, with $ds^2 \geq 0$. It is shown that this principle guided the approaches of all cosmologists for half a century, until the appearance of the signature (- + + +) which has now become widespread, even though, at least in cosmology, this change is not based on any ontological argument, and no article advocates such an option.

We return to the creation of the field equation by Hilbert in 1915, using the variational method. We recall that this was done within a four-dimensional framework without mentioning the particular role played by the time coordinate w_4 .

Returning to the solution presented by Schwarzschild in January 1916, we recall that the spiral trajectories towards the geometric center are in fact merely an artifact resulting from an erroneous application of the solutions to Lagrange's equations, used to construct geodesics, and that these real portions of curves are endowed with an imaginary length (and therefore a proper time). We propose to designate them as virtual geodesics. To illustrate the concept, we construct the virtual geodesics of the torus and the sphere.

It is noted that while Schwarzschild spacetime geodesics are inscribed, in the planes $\theta = Cst$, on the Flamm surface, they differ from the geodesics of this surface. 3D images of spacetime geodesics are produced in a coordinate system (t, r, φ) .

Finally, we examine carefully how, in his December 1916 article [48], David Hilbert attempted to integrate Schwarzschild's result, focusing exclusively on the stationary solution of the homogeneous field equation [43] and completely neglecting its logical complement, the interior metric [47]. He then treats this solution of the homogeneous field equation for a bilinear form, not as a metric; at no point in either of his articles does an equation begin with $s^2 = etc$.

This amounts to him granting himself the freedom to choose the elements of the signature, in a stationary solution with spherical symmetry, expressed using coordinates (x, y, z, l) , and, considering that physical phenomena, whether special relativity or curvature effects, are merely small perturbations of a fundamentally Euclidean geometry, he opts for the signature $(++++)$. It is only after all calculations have been done that he translates his conception of relativity by giving his time coordinate l a purely imaginary nature it , thus adhering to an idea initially introduced in 1902 by Poincaré [62], taken up by Lorentz in 1904 [16] and then by Minkowski in 1907 [21]. The signature then becomes $(+++ -)$. Without attaching importance to the inversion of the signs, he concludes that he has thus rediscovered Schwarzschild's metric

To obtain the geodesics, he constructs the Lagrange equations using the bilinear form $G\left(\frac{dx_s}{dp}\right)$ as the Lagrange function. Since this form is constant (so that the geodesics can be derived from it and are time lines), and also negative, this observation should have led him to reverse the signs and opt for $(--- +)$. However, to obtain a real proper time, he chooses to add a minus sign under the radical:

$$\tau = \int \sqrt{-G\left(\frac{dx_s}{dp}\right)} dp$$

When, in 1916, Hilbert discovered that Einstein and Schwarzschild had produced linearized and non-linearized solutions to the field equation, he hastened to incorporate them by publishing, with the same title, a new version of his 1915 article, "Foundations of Physics." His choice to completely ignore the solution with a non-linear term suggests a certain haste on his part, inconsistent with his legendary rigor. It is worth noting that he sent this new article on December 23, 1916, Christmas Eve. The general impression is that, in this rather rough second article, two approaches clash. Clearly, Hilbert did not subscribe to this new view of the universe, as Minkowski had just specified it in 1909, which stemmed from the definition of length and was accompanied by a signature $(+ - - -)$. It then becomes clear that for him, it is not this definition of length that structures space, but a bilinear form. What also emerges is that Hilbert considers these new contributions, such as relativity and curvature, to be only very small perturbations to a space he conceives as fundamentally Euclidean. Indeed, a phenomenon like the advance of Mercury's perihelion represents only a tiny perturbation compared to Newtonian dynamics. At that time, no one could imagine that a relativistic physics would one day emerge, where objects could move at speeds so close to that of light, like "cosmic rays," to the point that relativistic corrections must be taken into account to interpret their apparent lifespan. In 1916, astrophysicists were well aware that the Schwarzschild radius was negligible compared to the radius of the Sun, and probably of stars in general. Who would have imagined at that time that hyperdense objects, such as neutron stars, or those located at the centers of galaxies, would one day take their place in the cosmic bestiary?

The general impression, after the interlude of the Second World War, is that something was lost at that time: a certain mastery of geometry (think of the analysis published by Flamm in 1916). As noted in our two articles, this chimera of the black hole stems from a complete disregard for topological aspects, the basis of the Janus Cosmological Model, which originates in the work of the mathematician Werner Boy [63], who, like Schwarzschild, enlisted at the outbreak of the First World War and died almost immediately in France on September 6, 1914. If these two had remained alive, the face of science would undoubtedly have been different. Everything suggests that after the Second World War, those who called themselves cosmologists simply reproduced

these different elements identically, including the inversion of the signature, as if they were simple words of the Gospel, completely missing "the train of topology".

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