Exploring CPT-Symmetry through the Action on the Torsors of the Electric Poincaré Group

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Abstract

In this article, we will examine the various symmetry groups as well as the action of the electric Poincaré group with charge symmetry on the elements of the dual of the Lie algebra, called torsors.

Keywords— Dynamic Groups, Pseudo-Orthogonal Groups, Pseudo-Euclidean Groups, Lorentz group, Poincaré group, Kaluza group, Electric Poincaré Group, torsors of a Lie group, Action on the torsors, CPT-symetry.

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Introduction

Groups and Lie algebras are mathematically significant objects that originate from the study of symmetries. These structures have widespread applications in various areas of mathematics, theoretical physics, and gauge theory. In this article, we will focus on the dynamic groups of Lorentz, Poincaré, and Poincaré group with charge symmetry.

The Poincaré group is an iconic example of a Lie group. It plays a central role in theoretical physics, particularly in special relativity. This symmetry group describes transformations that leave the laws of physics invariant under translations, rotations, and Lorentz transformations that mix spatial and temporal coordinates.

Poincaré's group, a fundamental concept in the realm of symmetries and physics, initially found its application in the study of gravitation. He played a crucial role in understanding the principles governing gravitational interactions, shedding light on the intricate interplay of spacetime geometry and massive objects.

Following the groundbreaking development of general relativity by Albert Einstein in 1915, the scientific community became captivated by the notion of achieving a grand unification of the fundamental forces of nature. Several distinguished minds sought to bridge the gap between gravitation and electromagnetism by introducing innovative ideas, and perhaps the most ingenious of these attempts revolved around the concept of an additional dimension (see [2], [3], [4], [7], [8]). In this audacious approach, theorists contemplated the existence of a fifth dimension, coiled and hidden from our ordinary perceptions. This notion opened up a vast landscape of possibilities, where the symmetries of spacetime extended beyond the conventional four dimensions.

Pioneers in this trailblazing field included Theodor Kaluza, who, in 1919, proposed a five-dimensional theory that merged gravitation and electromagnetism into a single elegant framework. His work laid the foundation for subsequent investigations into the unification of fundamental forces. Oskar Klein, building upon Kaluza's ideas, further refined the concept of compactification, suggesting that the fifth dimension was curled up and compact, escaping detection at the macroscopic scales we experience.

As the exploration of the fifth dimension unfolded, other notable figures joined the quest for a unified theory. Among them were Peter Bergmann and Salomon Barmgman, who independently contributed to the growing body of knowledge. Their collective efforts showcased the vibrancy of theoretical research during that era, as minds converged on a shared goal of unraveling the secrets of the cosmos.

Though some of their specific approaches might not have ultimately led to the grand unification they sought, their collective endeavors laid the groundwork for further advancements in theoretical physics.

However, as we delve deeper into their research, it becomes evident that their groundbreaking approach was not without its complexities. The five-dimensional theory, while displaying complete symmetry, presented a peculiar challenge that could not be easily dismissed. Einstein and Bergmann, despite their audaciousness in postulating novel concepts, decided to take a step back and modify the theory. The modification they introduced had far-reaching consequences, particularly with regards to the five-dimensional symmetry. For modern readers, their decision might seem contrived, but it must be viewed through the lens of the era in which they lived and worked. The theoretical landscape of the 1930s was vastly different from today's, and the scientific community operated within different constraints and paradigms.

This prediction, though intriguing, posed a dilemma for Einstein and Bergmann. On the one hand, it offered the tantalizing prospect of unification and new fundamental interactions. On the other, it raised questions about the experimental verifiability of such a field and the implications it might have for other well-established theories. As scientists seeking to push the boundaries of knowledge, they faced a difficult choice.

Ultimately, in 1938, Einstein and Bergmann made the pivotal decision not to embrace this prediction fully. The ramifications of this choice would reverberate through the years, prompting further investigations and theoretical developments. While the modified theory might have sacrificed some of its original symmetry, it still provided a remarkable stepping stone in the quest for a comprehensive understanding of the fundamental forces governing the cosmos.

In retrospect, the significance of their work becomes evident. Their willingness to explore new dimensions and their courage to revise their theories when faced with challenging prospects mark them as pioneers in the annals of astrophysics. Moreover, their cautionary stance reminds us of the delicate balance between theoretical exploration and empirical validation – a balance that continues to shape the scientific endeavors of today.

We begin the article with a generalization of the notion of symmetry. Then, we will explore the fundamental properties of Lorentz and Poincaré groups, their Lie algebras, and the dual of their Lie algebras. We conclude with the study of electric Poincaré group, which allow the study of five-dimensional spacetime with one dimension curled up. This curled-up dimension introduces an additional parameter which can be identified as a conserved scalar quantity, much like electric charge in theoretical physics.

This text is inspired by the work of Jean-Marie Souriau, which can be found in [9] and [10].

1 Generalities

In this section, we generalize the notion of symmetry. We conclude the section with the concept of matrix Lie groups and the associated affine group of a Lie group (see [6], [5], and [1]).

1.1 Standard Groups

We denote by $\mathcal{M}(n,\mathbb{R})$ the set of square matrices of size n with coefficients in $\mathbb{R}$.

For all $k,l \in \{1,\dots,n\}$, we denote the **elementary matrix** E_{kl} whose coefficients are all zero except for the coefficient in the k-th row and l-th column, which is equal to 1. Thus, the family

$$\mathcal{C}(\mathcal{M}(n,\mathbb{R})) := \{E_{kl},\ k,l \in \{1,\dots,n\}\}$$

is a basis for $\mathcal{M}(n,\mathbb{R})$ over $\mathbb{R}$.

The **general linear group of size n over $\mathbb{R}$** is given by:

$$\mathrm{GL}(n,\mathbb{R}) := \{B \in \mathcal{M}(n,\mathbb{R}),\ \det B \neq 0\}$$

1.2 Symmetry

Notation 1.1. (1) For $\alpha := (\alpha_1,\dots,\alpha_n) \in \{\pm 1\}^n$, we denote:

$$I_\alpha := \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

and the **signature of** α as:

$$s(\alpha) := (\text{card}\{i \in \{1, \dots, n\},\ \alpha_i = 1\},\ \text{card}\{i \in \{1, \dots, n\},\ \alpha_i = -1\})$$

Thus, we have:

$$I_\alpha^{-1} = I_\alpha.$$

There are two simple cases.

(a) If $\alpha := (1, \dots, 1, -1, \dots, -1)$ and $s(\alpha) := (p, n-p)$ $(0 \leq p \leq n)$, we use $s(\alpha)$ and α interchangeably in the notation. For example, we write:

$$I_{p,n-p} := I_{(p,n-p)} := I_{s(\alpha)} := \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix}.$$

(b) In the case where $\alpha := (1, \dots, 1)$, i.e., $s(\alpha) = (n, 0)$. We use α, $s(\alpha)$, and n interchangeably in the notation. For example, we write:

$$I_n = I_{n,0} = I_{(n,0)} = I_{s(\alpha)}.$$

(2) For any matrix $M \in \mathcal{M}(n, \mathbb{R})$, we denote $[M]_{kl}$ as the coefficient of M in row k and column l, and for any vector $V \in \mathbb{R}^n$, we denote $[V]_k$ as the k-th coordinate of V.

(3) We denote $\mathcal{C}(\mathbb{R}^n)$ as the canonical basis of $\mathbb{R}^n$:

$$e_{1,n} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ e_{2,n} := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix},\ \dots,\ e_{n,n} := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

If not specified, in the following we take $\alpha \in \{\pm 1\}^n$.

Definition 1.1. Transposes

Let $B \in \mathcal{M}(n, \mathbb{R})$.

(i) The **transpose** B^T **of** B is defined as:

$$[B^T]_{ij} := [B]_{ji}.$$

(ii) The α**-transpose** $\tau_\alpha(B)$ **of** B is defined as:

$$\tau_\alpha(B) := I_\alpha B^T I_\alpha.$$

We have for all $k, l \in \{1, \dots, n\}$:

$$[\tau_\alpha(B)]_{kl} = \alpha_k \alpha_l.$$

The mapping τ_α is an involutive $\mathbb{R}$-automorphism of $\mathcal{M}(n, \mathbb{R})$, thus we have $\tau_\alpha^{-1} = \tau_\alpha$.
Point (i) is a particular case of (ii), we have:

$$B^T = \tau_{n,0}(B) = \tau_{0,n}(B).$$

As $(BC)^T = C^T B^T$, we have directly:

$$\tau_\alpha(BC) = I_\alpha (BC)^T I_\alpha = I_\alpha C^T I_\alpha I_\alpha B^T I_\alpha = \tau_\alpha(C)\tau_\alpha(B).$$

Example 1.2. Let:

$$B := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \quad , \quad I_\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We have:

$$\tau_\alpha(B) = I_\alpha B^T I_\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -d & g \\ -b & e & -h \\ c & -f & j \end{pmatrix}$$

Definition 1.2. Let $k \in \{0, 1\}$. A matrix $B \in \mathcal{M}(n, \mathbb{R})$ is called **k-symmetric of size** α if:

$$B = (-1)^k \tau_\alpha(B).$$

We denote their set as $\mathcal{S}(k, \alpha)$.

(i) A 0-symmetric matrix of size α is also called a **symmetric matrix of size** α. We denote their set as $\mathcal{S}(\alpha)$, i.e., we have:

$$\mathcal{S}(\alpha) := \mathcal{S}(0, \alpha).$$

(ii) A 1-symmetric matrix of size α is also called an **antisymmetric matrix of size** α. We denote their set as $\mathcal{A}(\alpha)$, i.e., we have:

$$\mathcal{A}(\alpha) := \mathcal{S}(1, \alpha).$$

Noting:

$$\mathcal{S}(n) := \mathcal{S}((n, 0)) \quad , \quad \mathcal{A}(n) := \mathcal{S}((n, 0)).$$

We have the direct sum:

$$\mathcal{M}(n, \mathbb{R}) = \mathcal{S}(\alpha) \oplus \mathcal{A}(\alpha)$$

given by the decomposition for any $B \in \mathcal{M}(n, \mathbb{R})$:

$$B = \mathcal{S}_\alpha(B) + \mathcal{A}_\alpha(B)$$

with

$$\mathcal{S}_\alpha(B) := \frac{B + \tau_\alpha(B)}{2} \quad , \quad \mathcal{A}_\alpha(B) := \frac{B - \tau_\alpha(B)}{2}$$

respectively called the **symmetric and antisymmetric matrices of size** α associated with B.

Example 1.3. We use the notations from the previous example with:

$$B := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \quad , \quad I_\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We have:

$$\mathcal{S}_\alpha(B) = \frac{1}{2} \begin{pmatrix} 2a & b - d & c + g \\ -b + d & 2e & f - h \\ c + g & -f + h & 2j \end{pmatrix} \qquad \mathcal{A}_\alpha(B) = \frac{1}{2} \begin{pmatrix} 0 & b + d & c - g \\ b + d & 0 & f + h \\ -c + g & f + h & 0 \end{pmatrix}$$

For all $k, l \in \{1, \ldots, n\}$ such that $k < l$, we denote:

$$S(\alpha)_{kl} := E_{kl} + \tau_\alpha(E_{kl}) = E_{kl} + \alpha_k \alpha_l E_{lk}$$
$$A(\alpha)_{kl} := -E_{kl} + \tau_\alpha(E_{kl}) = -E_{kl} + \alpha_k \alpha_l E_{lk}$$

We then have:

$$\tau_\alpha\left(S(\alpha)_{kl}\right) = \tau_\alpha\left(E_{kl} + \alpha_k \alpha_l E_{lk}\right) = \alpha_k \alpha_l E_{lk} + E_{kl} = S(\alpha)_{kl}$$
$$\tau_\alpha\left(A(\alpha)_{kl}\right) = \tau_\alpha\left(-E_{kl} + \alpha_k \alpha_l E_{lk}\right) = -\alpha_k \alpha_l E_{lk} + E_{kl} = -A(\alpha)_{kl}$$

i.e., we have:

$$S(\alpha)_{kl} \in \mathcal{S}(\alpha) \quad , \quad A(\alpha)_{kl} \in \mathcal{A}(\alpha)$$

A basis of $\mathcal{S}(\alpha)$ over $\mathbb{R}$ is given by the family:

$$\mathcal{C}(\mathcal{S}(\alpha)) := \{S(\alpha)_{kl},\ k, l \in \{1, \ldots, n\}, k < l\} \cup \{E_{kk},\ k \in \{1, \ldots, n\}\}$$

thus:

$$\dim \mathcal{S}(\alpha) = \frac{n(n+1)}{2}.$$

And a basis of $\mathcal{A}(\alpha)$ over $\mathbb{R}$ is given by the family:

$$\mathcal{C}(\mathcal{A}(\alpha)) := \{A(\alpha)_{kl},\ k, l \in \{1, \ldots, n\}, k < l\}$$

thus:

$$\dim \mathcal{A}(\alpha) = \frac{n(n-1)}{2}.$$

Example 1.4. We resume the previous examples with:

$$\alpha := (-1, 1, -1)$$

(1) We have $\dim \mathcal{S}(\alpha) = 6$ and

$$\begin{aligned} \mathcal{S}(\alpha) &= \operatorname{Vect}\left(E_{11}, E_{22}, E_{33}, S(\alpha)_{12}, S(\alpha)_{13}, S(\alpha)_{23}\right) \\ &= \operatorname{Vect}\left(\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & -1 & 0\end{pmatrix}\right) \end{aligned}$$

(2) We have $\dim \mathcal{A}(\alpha) = 3$ and

$$\begin{aligned} \mathcal{A}(\alpha) &= \operatorname{Vect}\left(A(\alpha)_{12}, A(\alpha)_{13}, A(\alpha)_{23}\right) \\ &= \operatorname{Vect}\left(\begin{pmatrix}0 & -1 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & -1\\0 & 0 & 0\\1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{pmatrix}\right) \end{aligned}$$

The vector space of linear forms on $\mathcal{M}(n, \mathbb{R})$, i.e., the dual of $\mathcal{M}(n, \mathbb{R})$, is denoted by:

$$\mathcal{M}(n, \mathbb{R})^* := \mathcal{L}(\mathcal{M}(n, \mathbb{R}), \mathbb{R}).$$

There exists an isomorphism of vector spaces:

$$\begin{array}{rccc} \Phi: & \mathcal{M}(n, \mathbb{R}) & \longrightarrow & \mathcal{M}(n, \mathbb{R})^* \\ & M & \longmapsto & \Phi(M) : B \longmapsto \operatorname{Tr}(BM^T) \end{array}$$

Therefore, linear forms on $\mathcal{M}(n,\mathbb{R})$ can be characterized using the trace. A basis for $\mathcal{M}(n,\mathbb{R})^*$ is given by the family of linear forms:

$$E^{kl} : B \longmapsto \mathrm{Tr}(B\ E_{kl}^T) = \mathrm{Tr}(BE_{lk}).$$

(They are linearly independent since Φ is injective, so they form a basis for $\mathcal{M}(n,\mathbb{R})^*$.) Moreover, we have:

$$E^{kl}(E_{ij}) = \mathrm{Tr}(E_{ij}E_{lk}) = [E_{ij}]_{mn}[E_{lk}]_{nm} = \delta_{im}\delta_{jn}\delta_{ln}\delta_{km} = \delta_{ik}\delta_{jl}$$

Thus, the family

$$\mathcal{C}(\mathcal{M}(n,\mathbb{R}))^* := \left\{E^{kl},\ k,l \in \{1,\dots,n\}\right\}$$

is the dual basis associated with the canonical basis

$$\mathcal{C}(\mathcal{M}(n,\mathbb{R})) = \{E_{kl},\ k,l \in \{1,\dots,n\}\}.$$

We have the following result.

Proposition 1.5. Let $\beta \in \mathbb{R}^*$ and $\mathcal{B}$ be a subset of $\mathcal{C}(\mathcal{A}(\alpha))$. Let's define:

$$E := \mathrm{Vect}(\mathcal{B}) \subset \mathcal{A}(\alpha)$$

There exists an isomorphism of vector spaces:

$$\begin{array}{rccc} \Phi_\beta : & E & \longrightarrow & E^* \\ & M & \longmapsto & \Phi_\beta(M) : B \longmapsto \beta \mathrm{Tr}(BM) \end{array}$$

Proof. Since the trace is linear, Φ_β is clearly linear. It suffices to show that $\ker(\Phi_\beta) = \{0\}$. Let $M \in E$ such that $\Phi_\beta(M) = 0$, i.e., for all $B \in \mathcal{B}$:

$$\Phi_\beta(M)(B) = 0.$$

We will show that M is zero. It is already zero on the diagonal. Let's prove that it is zero off the diagonal. There exists a subset I of double indices:

$$I \subset \{(k,l) \in \{1,\dots,n\}^2,\ k < l\}$$

such that:

$$\mathcal{B} := \{A(\alpha)_{k,l},\ (k,l) \in I\}$$

Let's prove that $[M]_{kl} = 0$ for all $(k,l) \in I$. We have:

$$A(\alpha)_{kl} := -E_{kl} + \alpha_k\alpha_l E_{lk} \in \mathcal{A}(\alpha).$$

Thus, we have:

$$\begin{aligned} 0 &= \Phi_\beta(M)(A(\alpha)_{kl}) \\ &= \beta \mathrm{Tr}(A(\alpha)_{kl}M) \\ &= \beta[A(\alpha)_{kl}]_{ij}[M]_{ji} \\ &= \beta[-E_{kl} + \alpha_k\alpha_l E_{lk}]_{ij}[M]_{ji} \\ &= \beta\left(-[M]_{lk} + \alpha_k\alpha_l[M]_{kl}\right) \\ &= -2\beta[M]_{lk}. \end{aligned}$$

because $\alpha_k\alpha_l[M]_{kl} = -[M]_{lk}$. Thus, $[M]_{lk} = 0$. Therefore, $M = 0$. Hence, the result holds. □

Keeping the notation of the proof, we have for all $(i,j),(k,l) \in I$:

$$0 = \Phi_\beta(A(\alpha)_{ij})(A(\alpha)_{kl}) = -2\beta[A(\alpha)_{ij}]_{lk} = -2\beta\left(\delta_{il}\delta_{jk} - \alpha_i\alpha_j\delta_{ik}\delta_{jl}\right) = \pm 2\beta$$

It is then natural to take $\beta \in \{\pm 1/2\}$, so we choose:

$$\beta = -\frac{1}{2}.$$

Let's see why. We have:

$$\Phi_{-1/2}(A(\alpha)_{kl})(A(\alpha)_{kl}) = -2\frac{-1}{2} = 1.$$

Therefore, the family

$$\mathcal{C}(\mathcal{A}(\alpha))^* := \left\{\Phi_{-1/2}(A(\alpha)_{kl}),\ k,l \in \{1,\ldots,n\},\ k < l\right\}$$

is the dual basis associated with

$$\mathcal{C}(\mathcal{A}(\alpha)) = \{A(\alpha)_{kl},\ k,l \in \{1,\ldots,n\},\ k < l\}.$$

We then define the isomorphism:

$$\begin{array}{cccc} \bullet^\vee : & E & \longrightarrow & E^* \\ & M & \longmapsto & M^\vee : B \longmapsto -\frac{1}{2}\mathrm{Tr}(BM) \end{array} \tag{1}$$

We can therefore deduce the following proposition.

Proposition 1.6. The dual basis of $\mathcal{A}(\alpha)^*$ associated with

$$\mathcal{C}(\mathcal{A}(\alpha)) = \{A(\alpha)_{kl},\ k,l \in \{1,\ldots,n\},\ k < l\}$$

is the family:

$$\mathcal{C}(\mathcal{A}(\alpha))^* = \left\{A(\alpha)_{kl}^\vee,\ k,l \in \{1,\ldots,n\},\ k < l\right\}.$$

1.3 Generalities on Matrix Lie Groups

The linear group $\mathrm{GL}(n,\mathbb{R})$ is a Lie group, and its Lie algebra is denoted $\mathfrak{gl}(n,\mathbb{R})$, which is the vector space $\mathcal{M}(n,\mathbb{R})$ equipped with the Lie bracket:

$$\forall x,y \in \mathcal{M}(n,\mathbb{R}),\ [x,y] := xy - yx.$$

Definition 1.3. A **matrix Lie group** is a subgroup of $\mathrm{GL}(n,\mathbb{R})$ that is a submanifold of $\mathcal{M}(n,\mathbb{R})$.

According to Cartan's theorem, a subgroup of $\mathrm{GL}(n,\mathbb{R})$ is a matrix Lie group if and only if it is closed in $\mathrm{GL}(n,\mathbb{R})$.

The **special linear group of size n over $\mathbb{R}$** is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ defined by:

$$\mathrm{SL}(n,\mathbb{R}) := \{B \in \mathrm{GL}(n,\mathbb{R}),\ \det B = 1\}.$$

Definition 1.4. Let G be a subgroup of $\mathrm{GL}(n,\mathbb{R})$. The **special group associated with G** is:

$$\mathrm{S}G := G \cap \mathrm{SL}(n,\mathbb{R}).$$

The group $\mathrm{SL}(n,\mathbb{R})$ is a matrix Lie group and:

$$\mathfrak{sl}(n,\mathbb{R}) = T_{I_n}\mathrm{SL}(n,\mathbb{R}) = \{B \in \mathcal{M}(n,\mathbb{R}),\ \mathrm{Tr}(B) = 0\}.$$

We recall the usual definitions of the adjoint and coadjoint representations in the case of matrix Lie groups.

Definition 1.5. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$.

(i) The **adjoint representation of** G **on** $\mathfrak{g}$ is defined as:

$$\begin{array}{rccc} \mathrm{Ad}: & G & \longrightarrow & \mathrm{Aut}(\mathfrak{g}) \\ & a & \longmapsto & \mathrm{Ad}_a : Z \longmapsto aZa^{-1} \end{array}$$

(ii) The **coadjoint representation of** G **on** $\mathfrak{g}^*$ is defined as:

$$\begin{array}{rccc} \mathrm{Ad}^*: & G & \longrightarrow & \mathrm{Aut}(\mathfrak{g}^*) \\ & a & \longmapsto & \mathrm{Ad}^*_a : \psi \longmapsto (Z \longmapsto \psi(\mathrm{Ad}_{a^{-1}}(Z))) \end{array}$$

Suppose $G \subset \mathrm{GL}(n,\mathbb{R})$. We have, for example:

$$\mathrm{Ad}_{I_n} = \mathrm{Id}_{\mathfrak{g}} \qquad \mathrm{Ad}^*_{I_n} = \mathrm{Id}_{\mathfrak{g}^*} \qquad \mathrm{Ad}_a^{-1} = \mathrm{Ad}_{a^{-1}} \qquad (\mathrm{Ad}^*_a)^{-1} = \mathrm{Ad}^*_{a^{-1}}$$

We then have the following simple lemma.

Lemma 1.6.1. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$, and $\beta \in \mathbb{R}^*$. For any $a \in G$ and $M \in \mathfrak{g}$, we have:

$$\mathrm{Ad}^*_a(\Phi_\beta(M)) = \Phi_\beta(aMa^{-1}).$$

Proof. For any $Z \in \mathfrak{g}$, we have:

$$\mathrm{Ad}^*_a(\Phi_\beta(M))(Z) = \Phi_\beta(M)(a^{-1}Za) = \beta\mathrm{Tr}(a^{-1}ZaM) = \beta\mathrm{Tr}(ZaMa^{-1}) = \Phi_\beta(aMa^{-1})$$

□

Finally, we note that for any $B, B' \in \mathcal{A}(\alpha)$, we have:

$$\tau_\alpha([B,B']) = \tau_\alpha(BB' - B'B) = \tau_\alpha(BB') - \tau_\alpha(B'B) = \tau_\alpha(B'B) - \tau_\alpha(BB') = -\tau_\alpha([B',B])$$

This implies $[B,B'] \in \mathcal{A}(\alpha)$. We denote $\mathfrak{a}(\alpha)$ as the Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ with vector space $\mathcal{A}(\alpha)$.

Example 1.7. We start by noting that the usual cross product satisfies the Jacobi identity, making it a Lie bracket on $\mathbb{R}^3$:

$$\forall u,v,w \in \mathbb{R}^3,\ u \wedge (v \wedge w) + w \wedge (u \wedge v) + v \wedge (w \wedge u) = 0.$$

Therefore, $(\mathbb{R}^3, \wedge)$ is a Lie algebra.

Let's define the natural map:

$$\begin{array}{rccc} j: & \mathbb{R}^3 & \longrightarrow & \mathcal{A}(3) \\ & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \end{array}$$

where $\mathcal{A}(3) := \mathcal{A}(3,0) := \mathcal{A}((1,1,1))$. This map is clearly linear and injective. Since $\dim \mathcal{A}(3) = 3$, it is an isomorphism of vector spaces.

Furthermore, for any $u := u^i e_i, v := v^i e_i \in \mathbb{R}^3$, we have:

$$[j(u), j(v)] = j(u)j(v) - j(v)j(u) = \begin{pmatrix} 0 & v^1u^2 - u^1v^2 & v^1u^3 - v^3u^1 \\ u^1v^2 - u^2v^1 & 0 & v^2u^3 - v^3u^2 \\ u^1v^3 - u^3v^1 & u^2v^3 - u^3v^2 & 0 \end{pmatrix} = j(u \wedge v).$$

Thus, j extends to an isomorphism of Lie algebras (also denoted by j):

$$\begin{array}{rccc} j: & (\mathbb{R}^3, \wedge) & \longrightarrow & \mathfrak{a}(3) := (\mathcal{A}(3), []) \\ & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \end{array}$$

1.4 Lie Groups and Associated Affine Group

In this subsection, we consider a matrix Lie group G, which means that there exists $n \in \mathbb{N}_*$ such that G is a Lie subgroup of $\mathcal{M}(n, \mathbb{R})$.

Let $\mathfrak{g}$ be the Lie algebra of G, and we assume two things:

- There exists $\alpha \in \{\pm 1\}^n$.
- There exists a subset $\mathcal{B} := (Z_1, \ldots, Z_d)$ of $\mathcal{C}(\mathcal{A}(\alpha))$.

such that:

$$\mathfrak{g} = \mathrm{Vect}(\mathcal{B})$$

Therefore, $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{a}(\alpha)$ with dimension $d \leq n(n-1)/2$.

Definition 1.6. The **affine group associated with** G is defined as:

$$\mathrm{Aff}(G) := G \ltimes \mathbb{R}^n.$$

For any $(U, D), (U', D') \in \mathrm{Aff}(G)$, the composition law on $\mathrm{Aff}(G)$ is defined as:

$$(U, D)(U', D') := (UU', D + UD')$$

Lemma 1.7.1. Let's define the map:

$$\begin{array}{rccc} \mathrm{Paff}_G: & (\mathrm{Aff}(G), \cdot) & \longrightarrow & (\mathrm{GL}(n+1, \mathbb{R}), \times) \\ & (U, D) & \longmapsto & \begin{pmatrix} U & D \\ 0 & 1 \end{pmatrix} \end{array}$$

Then, Paff_G is an injective group homomorphism.

Proof. The map is clearly injective. For any $(U, D), (U', D') \in \mathrm{Aff}(G)$, we have:

$$\begin{aligned} \mathrm{Paff}_G(U, D)\mathrm{Paff}_G(U', D') &= \begin{pmatrix} U & D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U' & D' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} UU' & D + UD' \\ 0 & 1 \end{pmatrix} \\ &= \mathrm{Paff}_G(UU', D + UD') = \mathrm{Paff}_G((U, D) \cdot (U', D')) \end{aligned}$$

Therefore, the result follows. □

Thus, we can identify $(\mathfrak{aff}(G), [])$ as a subalgebra of Lie algebra $(\mathfrak{gl}(n+1, \mathbb{R}), [])$. This is what we will do from now on, i.e.:

$$\mathfrak{aff}(G) = \left\{ \begin{pmatrix} Z & v \\ 0 & 0 \end{pmatrix},\ Z \in \mathfrak{g} \ \wedge \ v \in \mathbb{R}^n \right\}. \tag{2}$$

Therefore, a basis of $\mathfrak{aff}(G)$ is given in this representation by:

$$\left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} Z_d & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & e_n \\ 0 & 0 \end{pmatrix} \right\}$$

Definition 1.7. The elements of $\mathfrak{aff}(G)^*$ are called **torsors**.

Before giving an explicit description of torsors, we notice that we have the natural isomorphism:

$$\begin{array}{cccc} \bullet^\vee : & \mathbb{R}^n & \longrightarrow & (\mathbb{R}^n)^* \\ & Q & \longmapsto & Q^\vee : R \longmapsto Q^T R \end{array} \tag{3}$$

Therefore, for all $i, j \in \{1, \dots, n\}$:

$$e_{j,n}^\vee(e_{i,n}) = \delta_{ij}.$$

The following proposition provides an explicit description of torsors.

Proposition 1.8. We have:

$$\mathfrak{aff}(G)^* = \left\{ \left\{ \ N \ \middle| \ Q \ \right\} : \begin{pmatrix} Z & v \\ 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(NZ) + Q^T v,\ N \in \mathfrak{g} \ \wedge \ Q \in \mathbb{R}^n \right\}.$$

Proof. Let's prove the equality of vector spaces by double inclusion:

$$\mathfrak{aff}(G)^* = \{(Z, v) \mapsto \phi(Z) + \psi(v),\ \phi \in \mathfrak{g}^* \ \wedge \ \psi \in (\mathbb{R}^n)^*\}.$$

The backward inclusion $\supset$ is clear. We only need to show the equality of dimensions. For every $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$, let's define:

$$\tilde{Z}_i : (Z, v) \mapsto Z_i^\vee(Z)$$
$$\tilde{e_{j,n}} : (Z, v) \mapsto e_{j,n}^\vee(Z)$$

and since:

$$\tilde{Z}_i(Z, v) = Z_i^\vee(Z) = Z_i^\vee(Z) + O_{\mathcal{L}(\mathbb{R}^n, \mathbb{R})}(v)$$
$$\tilde{e_{j,n}}(Z, v) = e_{j,n}^\vee(v) = 0_{\mathcal{L}(\mathfrak{g}, \mathbb{R})}(Z) + e_{j,n}^\vee(v)$$

we conclude that these $d + n$ functions are elements of the right vector space. Since $\dim \mathfrak{aff}(G) = d + n$, it suffices to show that these functions are linearly independent.

Let $\lambda^1, \dots, \lambda^d, \mu^1, \dots, \mu^n \in \mathbb{R}$ such that $\lambda^i \tilde{Z}_i + \mu^j \tilde{e_{j,n}} = 0$. Then, for all $i, k \in \{1, \dots, d\}$ and $j, l \in \{1, \dots, n\}$, we have:

$$0 = \lambda^i \tilde{Z}_i(Z_k, 0) + \mu^j \tilde{e_{j,n}}(Z_j, 0) = \lambda^k$$
$$0 = \lambda^i \tilde{Z}_i(0, e_{l,n}) + \mu^j \tilde{e_{j,n}}(0, e_{l,n}) = \mu^l$$

Thus, the vector spaces are equal.

According to the isomorphisms 1 and 3, for every $\phi \in \mathfrak{g}^*$ and $\psi \in (\mathbb{R}^n)^*$, there exist unique elements $N \in \mathfrak{g}$ and $Q \in \mathbb{R}^n$ such that $\phi = N^\vee$ and $\psi = Q^\vee$. Hence, we have the equalities:

$$\begin{aligned}\mathfrak{aff}(G)^* &= \left\{N^\vee + Q^\vee,\ N \in \mathfrak{g}\ \wedge\ Q \in \mathbb{R}^n\right\}\\ &= \left\{\left\{\ N \;\middle|\; Q\ \right\} : \begin{pmatrix} Z & v \\ 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(NZ) + Q^T v,\ N \in \mathfrak{g}\ \wedge\ Q \in \mathbb{R}^n\right\}\end{aligned}$$

□

Definition 1.8. The **action of the group** $\mathrm{Aff}(G)$ **on** $\mathfrak{aff}(G)^*$ is defined by the coadjoint representation, i.e., for every $a \in \mathrm{Aff}(G)$ and $\mu \in \mathfrak{aff}(G)^*$, we denote:

$$a \bullet \mu = \mathrm{Ad}_a^*(\mu).$$

Proposition 1.9. Let:

$$a := \begin{pmatrix} U & D \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}(G) \quad , \quad \left\{\ N \;\middle|\; Q\ \right\} \in \mathfrak{aff}(G)^*.$$

We have:

$$a \bullet \left\{\ N \;\middle|\; Q\ \right\} = \left\{\ UNU^{-1} - 2DQ^TU^{-1} \;\middle|\; \left(U^{-1}\right)^T Q\ \right\}.$$

Proof. We have:

$$\begin{aligned}\left(a \bullet \left\{\ N \;\middle|\; Q\ \right\}\right)\begin{pmatrix} Z & v \\ 0 & 0 \end{pmatrix} &= \left\{\ N \;\middle|\; Q\ \right\}\left(a^{-1}\begin{pmatrix} Z & v \\ 0 & 0 \end{pmatrix} a\right)\\ &= \left\{\ N \;\middle|\; Q\ \right\}\left(\begin{pmatrix} U^{-1} & -U^{-1}D \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Z & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} U & D \\ 0 & 1 \end{pmatrix}\right)\\ &= \left\{\ N \;\middle|\; Q\ \right\}\begin{pmatrix} U^{-1}ZU & U^{-1}ZD + U^{-1}v \\ 0 & 0 \end{pmatrix}\\ &= -\frac{1}{2}\mathrm{Tr}(NU^{-1}ZU) + Q^T(U^{-1}ZD + U^{-1}v)\\ &= -\frac{1}{2}\mathrm{Tr}(UNU^{-1}Z) + \mathrm{Tr}(Q^TU^{-1}ZD) + Q^TU^{-1}v \qquad \text{since } Q^TU^{-1}ZD \in \mathbb{R}\\ &= -\frac{1}{2}\mathrm{Tr}((UNU^{-1} - 2DQ^TU^{-1})Z) + Q^TU^{-1}v\\ &= \left\{\ UNU^{-1} - 2DQ^TU^{-1} \;\middle|\; \left(U^{-1}\right)^T Q\ \right\}\begin{pmatrix} Z & v \\ 0 & 0 \end{pmatrix}\end{aligned}$$

Thus, we obtain the result. □

2 Dynamic Groups

Let $\alpha \in \{\pm 1\}^n$ be fixed.

2.1 Pseudo-Orthogonal Groups

Definition 2.1. The **pseudo-orthogonal group of size** α **over** $\mathbb{R}$ is defined as:

$$\mathrm{O}(\alpha) := \{L \in \mathrm{GL}(n, \mathbb{R}),\ \tau_\alpha(L)L = I_n\}.$$

Thus, we have:

$$\mathrm{O}(\alpha) = \left\{L \in \mathrm{GL}(n, \mathbb{R}),\ L^T I_\alpha L = I_\alpha\right\}$$

and for all $L \in \mathrm{O}(\alpha)$, $L^{-1} = \tau_\alpha(L)$. We denote:

$$\mathrm{SO}(\alpha) := \mathrm{O}(\alpha) \cap \mathrm{SL}(n, \mathbb{R}) := \{L \in \mathrm{O}(\alpha),\ \det L = 1\}.$$

Proposition 2.1. $\mathrm{O}(\alpha)$ and $\mathrm{SO}(\alpha)$ are matrix Lie groups and their Lie algebras are equal and given by:

$$\mathfrak{o}(\alpha) = \mathfrak{so}(\alpha) = T_{I_n}\mathrm{O}(\alpha) = \mathfrak{a}(\alpha).$$

Proof. The group $\mathrm{O}(\alpha)$ is closed in $\mathrm{GL}(n, \mathbb{R})$ as it is the preimage of the singleton $\{I_\alpha\}$ under the continuous map:

$$\begin{array}{cccc} f: & \mathcal{M}(n, \mathbb{R}) & \longrightarrow & \mathcal{S}(n) \\ & x & \longmapsto & x^T I_\alpha x \end{array}$$

Since f is $\mathcal{C}^\infty$-class, it suffices to calculate the derivative of f at I_n in an arbitrary direction $H \in \mathcal{M}(n, \mathbb{R})$. Let $H \in \mathcal{M}(n, \mathbb{R})$ be a matrix. For every real number t in a neighborhood of 0:

$$f(I_n + tH) = I_\alpha + t\left(H^T I_\alpha + I_\alpha H\right) + O(t^2).$$

Since the map $H \mapsto H^T I_\alpha + I_\alpha H$ is linear and surjective (for every $y \in \mathcal{S}(n)$, let $x := 1/2 I_\alpha y$, then $y = x^T I_\alpha + I_\alpha x$), we have:

$$\mathrm{d}_{I_n} f : H \mapsto \tau_\alpha(H) I_\alpha + I_\alpha H.$$

and:

$$\mathfrak{o}(\alpha) = T_{I_n}\mathrm{O}(\alpha) = \mathrm{Ker}\ \mathrm{d}_{I_n} f.$$

Finally, since the map $\det : \mathrm{O}(\alpha) \longrightarrow \{\pm 1\}$ is continuous and $\{1\}$ is an open set in $\{\pm 1\}$, we conclude that $\mathrm{SO}(\alpha)$ is an open subset of $\mathrm{O}(\alpha)$. Thus, we obtain the result. □

Therefore, we have:

$$\dim \mathfrak{o}(\alpha) = \dim \mathfrak{so}(\alpha) = \frac{n(n-1)}{2}.$$

and a basis is given by $\mathcal{C}(\mathcal{A}(\alpha))$. Thus, we are in the particular case of the previous section with $d := n(n-1)/2$.

2.2 Pseudo-Euclidean Groups

Definition 2.2. The **pseudo-Euclidean group of size α over $\mathbb{R}$** is the affine group associated with the pseudo-orthogonal group $\mathrm{O}(\alpha)$, i.e.:

$$\mathrm{Euc}(\alpha) := \mathrm{Aff}(\mathrm{O}(\alpha)).$$

By Lemma 1.7.1, the pseudo-Euclidean group can be seen as a subgroup of $\mathrm{GL}(n+1, \mathbb{R})$ given by:

$$\mathrm{Euc}(\alpha) := \left\{ \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix},\ L \in \mathrm{O}(\alpha)\ \wedge\ C \in \mathbb{R}^\alpha \right\}$$

Since:

$$d + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

we can deduce from subsection 1.4 the following result.

Proposition 2.2. The group $\mathrm{Euc}(\alpha)$ is a Lie group of dimension $n(n+1)/2$, and its Lie algebra is given by:

$$\mathfrak{euc}(\alpha) := \left\{ \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix},\ \Lambda \in \mathfrak{a}(\alpha)\ \wedge\ \Gamma \in \mathbb{R}^\alpha \right\}$$

Therefore, a basis of $\mathfrak{euc}(\alpha)$ in this representation is given by:

$$\left\{\begin{pmatrix} A(\alpha)_{kl} & 0 \\ 0 & 0 \end{pmatrix},\ k,l \in \{1,\ldots,n\},\ k<l\right\} \cup \left\{\begin{pmatrix} 0 & e_{1,n} \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & e_{n,n} \\ 0 & 0 \end{pmatrix}\right\}$$

Moreover, we have that $\mathfrak{euc}(\alpha)$ satisfies the assumptions of $\mathfrak{g}$ in subsection 1.4, and thus, according to Proposition 1.8, we have:

$$\begin{aligned} \mathfrak{euc}(\alpha)^* &= \mathcal{L}(\mathfrak{euc}(\alpha), \mathbb{R}) \\ &= \left\{ \left\{ \ M \ \middle| \ P \ \right\} : \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \mapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) + P^T\Gamma,\ M \in \mathfrak{a}(\alpha)\ \wedge\ P \in \mathbb{R}^n \right\}. \end{aligned} \tag{4}$$

The action of the group $\mathrm{Euc}(\alpha)$ on $\mathfrak{poin}^*$ is defined by the coadjoint representation, i.e., for any $a \in \mathrm{Euc}(\alpha)$ and $\mu \in \mathfrak{euc}(\alpha)^*$, we denote:

$$a \bullet \mu := \mathrm{Ad}^*_a(\mu).$$

For all

$$a := \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \in \mathrm{Euc}(\alpha) \quad , \quad \left\{ \ M \ \middle| \ P \ \right\} \in \mathfrak{aff}(G)^*$$

and since $L^{-1} = \tau_\alpha(L)$, by Proposition 1.9 we have:

$$\begin{aligned} \left(a \bullet \left\{ \ M \ \middle| \ P \ \right\}\right)\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} &= \left\{ \ LM\tau_\alpha(L) - 2CP^T\tau_\alpha(L) \ \middle| \ \tau_\alpha(L)^T P \ \right\}\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \\ &= -\frac{1}{2}\mathrm{Tr}((LM\tau_\alpha(L) - 2CP^T\tau_\alpha(L))\Lambda) + P^T\tau_\alpha(L)\Gamma \end{aligned} \tag{5}$$

3 Applications to Relativity

3.1 The Lorentz Group

Definition 3.1. The **Lorentz group in dimension 4** is defined as:

$$\mathcal{L} := \mathrm{O}(1,3) = \left\{L \in \mathrm{GL}(4,\mathbb{R}),\ L^T I_{1,3} L = I_{1,3}\right\}.$$

We have $\mathcal{L} = \{L \in \mathrm{GL}(4,\mathbb{R}),\ \tau_{1,3}(L)L = I_4\}$. We have the following simple lemma.

Lemma 3.0.1. Let

$$L := \begin{pmatrix} e & b^T \\ c & d \end{pmatrix} \in \mathcal{L}$$

with $e := [L]_{00} \in \mathbb{R}$, $b, c \in \mathbb{R}^3$, and $d \in \mathcal{M}(3,\mathbb{R})$.

(i) We have $\det(L) = \pm 1$ and $e^2 = [L]_{00}^2 \geq 1$.

(ii) We have $b^T b = c^T c = e^2 - 1$.

Proof. Since $I_{1,3}^T = I_{1,3}$, we have $L^T I_{1,3} L = I_{1,3}$ and $T I_{1,3} L = I_{1,3}$ which implies:

$$\begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix} = \begin{pmatrix} e & b^T \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}\begin{pmatrix} e & c^T \\ b & d^T \end{pmatrix} = \begin{pmatrix} e^2 - b^T b & * \\ * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix} = \begin{pmatrix} e & c^T \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}\begin{pmatrix} e & b^T \\ c & d^T \end{pmatrix} = \begin{pmatrix} e^2 - c^T c & * \\ * & * \end{pmatrix}$$

Therefore, we have $e^2 - c^T c = e^2 - b^T b = 1$ and moreover as $c^T c = (c^i)^2 \geq 0$, we only have $e^2 \geq 1$. $\square$

Let's define:

$$T_4 := \begin{pmatrix} -1 & 0 \\ 0 & I_3 \end{pmatrix} \quad , \quad P_4 := \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}.$$

We define the eight fundamental subsets of $\mathcal{L}$.

Definition 3.2. (i) (a) The **neutral component** is:

$$\mathcal{L}_n := \{L \in \mathcal{L},\ \det(L) = 1 \ \wedge\ [L]_{00} \geq 1\}.$$

(b) The P**-symmetric component** is:

$$\mathcal{L}_p := \{L \in \mathcal{L},\ \det(L) = -1 \ \wedge\ [L]_{00} \geq 1\}.$$

(c) The T**-symmetric component** is:

$$\mathcal{L}_t := \{L \in \mathcal{L},\ \det(L) = -1 \ \wedge\ [L]_{00} \leq -1\}.$$

(d) The PT**-symmetric component** is:

$$\mathcal{L}_{pt} := \{L \in \mathcal{L},\ \det(L) = 1 \ \wedge\ [L]_{00} \leq -1\}.$$

(ii) (a) The **orthochronous component** is:

$$\mathcal{L}_o := \mathcal{L}_n \sqcup \mathcal{L}_p = \{L \in \mathcal{L},\ [L]_{00} \geq 1\}.$$

(b) The **antichronous component** is:

$$\mathcal{L}_a := \mathcal{L}_t \sqcup \mathcal{L}_{pt} = \{L \in \mathcal{L},\ [L]_{00} \leq -1\}.$$

(iii) (a) The **special component**[1] is:

$$\mathcal{L}_s := \mathcal{L}_n \sqcup \mathcal{L}_{pt} = \{L \in \mathcal{L},\ \det(L) = 1\}.$$

(b) The **improper component** is:

$$\mathcal{L}_i := \mathcal{L}_p \sqcup \mathcal{L}_t = \{L \in \mathcal{L},\ \det(L) = -1\}$$

Among these components, three are subgroups of $\mathcal{L}$.

Lemma 3.0.2. The components $\mathcal{L}_n$, $\mathcal{L}_o$, and $\mathcal{L}_s$ are subgroups of $\mathcal{L}$. They are respectively called:

(i) $\mathcal{L}_n$: **restricted Lorentz subgroup** or **proper orthochronous**, also denoted as $\mathrm{SO}_0(1,3)$;

(ii) $\mathcal{L}_o$: **orthochronous subgroup**;

(iii) $\mathcal{L}_s$: **special subgroup**, also denoted as $\mathrm{SO}(1,3)$.

Proof. We have $\mathcal{L}_s = \mathrm{SO}(1,3) = \mathcal{L} \cap \mathrm{SL}(4,\mathbb{R})$ so $\mathcal{L}_s$ is a subgroup of $\mathcal{L}$.

[1] In the literature, the special component is also called the **proper component**, so as not to confuse it with the $\mathcal{L}_p$ component, we choose to call it the special component and denote it "$\mathcal{L}_s$".

Let's show that $\mathcal{L}_o$ is a subgroup of $\mathcal{L}$. Let

$$L := \begin{pmatrix} e & b^T \\ c & d \end{pmatrix},\ L' := \begin{pmatrix} e' & b'^T \\ c' & d' \end{pmatrix} \in \mathcal{L}_o.$$

We define $L'' := \begin{pmatrix} e'' & b''^T \\ c'' & d'' \end{pmatrix} := LL'$. Then we have:

$$\begin{pmatrix} e'' & b''^T \\ c'' & d'' \end{pmatrix} = L'' = LL' = \begin{pmatrix} e & b^T \\ c & d \end{pmatrix} \begin{pmatrix} e' & b'^T \\ c' & d' \end{pmatrix} = \begin{pmatrix} ee' + b^T c' & * \\ * & * \end{pmatrix}$$

thus we have $b^T c' = e'' - ee'$. Therefore, using Cauchy-Schwartz and point (ii) of the previous lemma, we have:

$$|e'' - ee'|^2 = |b^T c'|^2 \leq (b^T b)(c'^T c') = (e^2 - 1)(e'^2 - 1).$$

Since $e, e' \geq 1$, we have $ee' \geq 1$ and thus:

$$e'' \geq ee' - \sqrt{(e^2 - 1)(e'^2 - 1)}.$$

To show that $e'' \geq 1$, it suffices to show that $e'' \geq 0$ because $|e''| \geq 1$. Therefore, it is enough to show by squaring that $(ee')^2 \geq (e^2 - 1)(e'^2 - 1)$. We have:

$$(ee')^2 - (e^2 - 1)(e'^2 - 1) = e_1^2 + e_2^2 - 1 \geq 0.$$

Thus we have $L'' \in \mathcal{L}_o$. Consequently, $\mathcal{L}_o$ is a subgroup of $\mathcal{L}$.

Since $\mathcal{L}_n := \mathrm{SO}_0(1,3) = \mathcal{L}_o \cap \mathcal{L}_s$ then $\mathcal{L}_n$ is a subgroup of $\mathcal{L}$. □

We have the usual decomposition into connected components:

$$\mathcal{L} = \mathcal{L}_n \sqcup \mathcal{L}_s \sqcup \mathcal{L}_t \sqcup \mathcal{L}_{pt} = \mathcal{L}_n \sqcup P_4\mathcal{L}_n \sqcup T_4\mathcal{L}_n \sqcup P_4T_4\mathcal{L}_n.$$

Since $\det(-L) = \det(L)$ for all $L \in \mathcal{L}$, we have:

$$-\mathcal{L}_t = \{-L,\ L \in \mathcal{L}_t\} = \{L \in \mathcal{L},\ \det(L) = -1 \ \wedge\ [L]_{00} \leq -1\} = \mathcal{L}_p$$
$$-\mathcal{L}_{pt} = \{-L,\ L \in \mathcal{L}_{pt}\} = \{L \in \mathcal{L},\ \det(L) = 1 \ \wedge\ [L]_{00} \geq 1\} = \mathcal{L}_n$$

and since $T_4^2 = I_4$, we have:

$$T_4\mathcal{L}_p = \{T_4L,\ L \in \mathcal{L}_p\} = \{T_4P_4L,\ L \in \mathcal{L}_n\} = \mathcal{L}_{pt}$$
$$T_4\mathcal{L}_t = \{T_4L,\ L \in \mathcal{L}_t\} = \{T_4^2L,\ L \in \mathcal{L}_n\} = \mathcal{L}_n$$

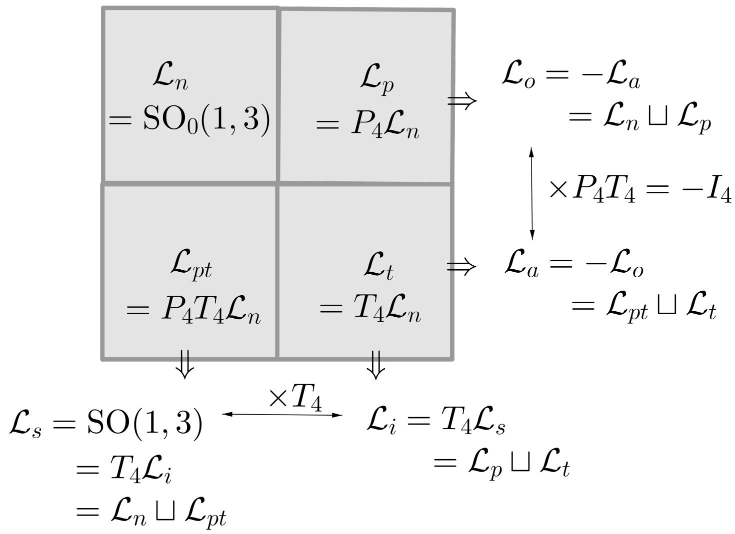
Therefore, we have the following decompositions:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_o \sqcup \mathcal{L}_a \\ &= \mathcal{L}_o \sqcup (-\mathcal{L}_o) \\ &= (\mathcal{L}_n \sqcup \mathcal{L}_s) \sqcup (\mathcal{L}_t \sqcup \mathcal{L}_{pt}) \\ &= (\mathcal{L}_n \sqcup P_4\mathcal{L}_n) \sqcup (T_4\mathcal{L}_n \sqcup P_4T_4\mathcal{L}_n) \end{aligned} \qquad \begin{aligned} \mathcal{L} &= \mathcal{L}_s \sqcup \mathcal{L}_i \\ &= \mathcal{L}_s \sqcup (T_4\mathcal{L}_s) \\ &= (\mathcal{L}_n \sqcup \mathcal{L}_{pt}) \sqcup (\mathcal{L}_t \sqcup \mathcal{L}_s) \\ &= (\mathcal{L}_n \sqcup P_4T_4\mathcal{L}_n) \sqcup (T_4\mathcal{L}_n \sqcup P_4L_n) \end{aligned}$$

In the rest of this section, we will particularly focus on the decomposition:

$$\mathcal{L} = \mathcal{L}_o \sqcup \mathcal{L}_a = \{\lambda L_o,\ L_o \in \mathcal{L}_o \ \wedge\ \lambda \in \{\pm 1\}\}.$$

These decompositions are summarized in the following diagram:



It is a Lie group of dimension 6. The Lie algebra of the Lorentz group is given by:

$$\begin{aligned}
\mathfrak{lor} :=&\, \mathfrak{o}(1,3) = \mathfrak{a}(1,3) \\
=&\mathrm{Vect}_{\mathbb{R}}\left(K_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \right. \\
&\left. K_4 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_5 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, K_6 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right)
\end{aligned}$$

We can use the isomorphism j from Example 1.7, and we have a more precise expression for the Lie algebra:

$$\mathfrak{lor} = \left\{ \begin{pmatrix} 0 & \beta^T \\ \beta & A \end{pmatrix},\ A \in \mathfrak{a}(3) \ \wedge\ \beta \in \mathbb{R}^3 \right\} = \left\{ \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix},\ \beta, w \in \mathbb{R}^3 \right\}$$

3.2 The Poincaré Group

In this subsection, we extend the results from the previous subsection on the Lorentz groups. The Poincaré group is the affine group of the Lorentz group. We refer to chapters 13 and 14 of [9] for the results.

Definition 3.3. The **Poincaré group** is defined as:

$$\mathcal{P} := \mathrm{Euc}(1,3).$$

It has four connected components, just like the Lorentz group. Let us denote:

$$\forall x \in \{n, p, t, pt, s, i, o, a\},\ \mathcal{P}_x := \left\{ \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix},\ L \in \mathcal{L}_x \ \wedge\ C \in \mathbb{R}^4 \right\}$$

and

$$T_5 := \begin{pmatrix} T_4 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & I_4 \\ 0 & 1 \end{pmatrix} \quad , \quad P_5 := \begin{pmatrix} P_4 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -I_4 \end{pmatrix}$$

As with the Lorentz group, we have:

$$\begin{aligned}\mathcal{P} &= \mathcal{P}_n \sqcup \mathcal{P}_p \sqcup \mathcal{P}_t \sqcup \mathcal{P}_{pt} = \mathcal{P}_n \sqcup P_5\mathcal{P}_n \sqcup T_5\mathcal{P}_n \sqcup P_5T_5\mathcal{P}_n \\ &= \mathcal{P}_o \sqcup \mathcal{P}_a = \mathcal{P}_o \sqcup (-P_5T_5\mathcal{P}_o) \\ &= \mathcal{P}_s \sqcup \mathcal{P}_i = \mathcal{P}_s \sqcup (T_5\mathcal{P}_s)\end{aligned}$$

It has the following representation:

$$\begin{aligned}\mathcal{P} &= \left\{ \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix},\ L \in \mathcal{L} \ \wedge\ C \in \mathbb{R}^{1,3} \right\} \\ &= \left\{ \begin{pmatrix} \lambda L_o & C \\ 0 & 1 \end{pmatrix},\ L_o \in \mathcal{L}_o \ \wedge\ C \in \mathbb{R}^{1,3} \ \wedge\ \lambda \in \{\pm 1\} \right\}\end{aligned}$$

It is a Lie group of dimension 10. We have an expression for the Lie algebra:

$$\begin{aligned}\mathfrak{poin} &= \left\{ \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix},\ \Lambda \in \mathfrak{a}(1,3) \ \wedge\ \Gamma \in \mathbb{R}^{1,3} \right\} = \left\{ \begin{pmatrix} 0 & \beta^T & \nu \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix},\ \beta, w, \gamma \in \mathbb{R}^3 \ \wedge\ \nu \in \mathbb{R} \right\} \\ &= \mathrm{Vect}_{\mathbb{R}} \left(\left\{ \tilde{K}_i := \begin{pmatrix} K_i & 0 \\ 0 & 0 \end{pmatrix},\ i \in \{1, \dots, 6\} \right\} \bigcup \left\{ J_i := \begin{pmatrix} 0 & e_{i,4} \\ 0 & 0 \end{pmatrix},\ i \in \{1, \dots, 4\} \right\} \right)\end{aligned}$$

The dual of $\mathfrak{poin}$ is given by equation (4):

$$\mathfrak{poin}^* = \mathcal{L}(\mathcal{P}, \mathbb{R}) = \left\{ \left\{ \ M \ \middle| \ P \ \right\} : \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) + P^T\Gamma,\ M \in \mathfrak{a}(1,3),\ \wedge\ P \in \mathbb{R}^{1,3} \right\}$$

For any $M \in \mathcal{A}(1,3)$, there exist $\ell, g \in \mathbb{R}^3$ such that:

$$M = \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix}.$$

Hence the following definition.

Definition 3.4. Let

$$\mu := \left\{ \ M \ \middle| \ P \ \right\} := \left\{ \ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \ \middle| \ \begin{pmatrix} E \\ p \end{pmatrix} \ \right\} \in \mathfrak{poin}^*.$$

(i) The matrix M is called the **moment matrix associated with** μ.

(ii) The vector $P \in \mathbb{R}^{1,3}$ is called the **energy-momentum vector associated with** μ.

(iii) The vector $p \in \mathbb{R}^3$ is called the **momentum vector**, and the scalar $E \in \mathbb{R}$ is called the **energy**.

(iv) The vector $\ell \in \mathbb{R}^3$ is called the **angular momentum of** M.

Therefore, we have:

$$\left\{ \ M \ \middle| \ P \ \right\} \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} = \left\{ \ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \ \middle| \ \begin{pmatrix} E \\ p \end{pmatrix} \ \right\} \begin{pmatrix} 0 & \beta^T & \nu \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
&= -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} \right) + \begin{pmatrix} E & p^T \end{pmatrix} \begin{pmatrix} \nu \\ \gamma \end{pmatrix} \\
&= -\frac{1}{2}\mathrm{Tr}\begin{pmatrix} g^T\beta & * \\ * & \beta^T g + j(\ell)j(w) \end{pmatrix} + \begin{pmatrix} E & p^T \end{pmatrix} \begin{pmatrix} \nu \\ \gamma \end{pmatrix} \\
&= -\frac{1}{2}\mathrm{Tr}(j(\ell)j(w)) - \frac{1}{2}\mathrm{Tr}(g^T\beta + g\beta^T) + p^T\gamma + E\nu \\
&= (\ell^1 w^1 + \ell^2 w^2 + \ell^3 w^3) - g^T\beta + p^T\gamma + E\nu \qquad \text{because } g^T\beta = g\beta^T \in \mathbb{R} \\
&= \ell^\vee(w) - g^\vee(\beta) + p^\vee(\gamma) + E\nu
\end{aligned}$$

We denote this last equality as:

$$\left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} \begin{pmatrix} 0 & \beta^T & \nu \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we have:

$$\mathfrak{poin}^* = \left\{ \left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} : \begin{pmatrix} 0 & \beta^T & \nu \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \ell^\vee(w) - g^\vee(\beta) + p^\vee(\gamma) + E\nu, \ \ell, g, p \in \mathbb{R}^3 \ \wedge \ E \in \mathbb{R} \right\}$$

The action of the group $\mathcal{P}$ on $\mathfrak{poin}^*$ is defined by the coadjoint representation *i.e.* for any $a \in \mathcal{P}$ and $\mu \in \mathfrak{poin}^*$, we denote:

$$a \bullet \mu := \mathrm{Ad}_a^*(\mu).$$

For all

$$a := \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \in \mathcal{P} \quad , \quad \left\{ \ M \ \middle| \ P \ \right\} \in \mathfrak{poin}^*$$

we have, from Proposition 1.9:

$$\begin{aligned}
\left(a \bullet \left\{ \ M \ \middle| \ P \ \right\}\right) \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} &= \left\{ \ LM\tau_{1,3}(L) - 2CP^T\tau_{1,3}(L) \ \middle| \ \tau_{1,3}(L)^T P \ \right\} \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \\
&= -\frac{1}{2}\mathrm{Tr}((LM\tau_{1,3}(L) - 2CP^T\tau_{1,3}(L))\Lambda) + P^T\tau_{1,3}(L)\Gamma
\end{aligned}$$

We can then deduce the action of the three fundamental matrices T_5 and P_5 on $\mathfrak{poin}^*$.

Proposition 3.1. Let:

$$\left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} \in \mathfrak{poin}^*.$$

We have:

$$\begin{aligned}
T_5 \bullet \left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} &= \left\{ \ \ell \ \middle| \ -g \ \middle| \ p \ \middle| \ -E \ \right\} \\
P_5 \bullet \left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} &= \left\{ \ \ell \ \middle| \ -g \ \middle| \ -p \ \middle| \ E \ \right\}
\end{aligned}$$

Therefore, for all $i, j \in \mathbb{N}$:

$$(T_5^i P_5^j) \bullet \left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} = \left\{ \ \ell \ \middle| \ (-1)^{i+j} g \ \middle| \ (-1)^j p \ \middle| \ (-1)^i E \ \right\}.$$

Proof. We have:

$$\begin{aligned} T_5 \bullet \left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} &= \left\{ \ T_4 \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} T_4 \ \middle| \ T_4 \begin{pmatrix} E \\ p \end{pmatrix} \ \right\} \\ &= \left\{ \ \begin{pmatrix} 0 & -g^T \\ -g & j(\ell) \end{pmatrix} \ \middle| \ \begin{pmatrix} -E \\ p \end{pmatrix} \ \right\} \\ &= \left\{ \ \ell \ \middle| \ -g \ \middle| \ p \ \middle| \ -E \ \right\} \end{aligned}$$

and we have:

$$\begin{aligned} P_5 \bullet \left\{ \ \ell \ \middle| \ g \ \middle| \ p \ \middle| \ E \ \right\} &= \left\{ \ P_4 \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} P_4 \ \middle| \ P_4 \begin{pmatrix} E \\ p \end{pmatrix} \ \right\} \\ &= \left\{ \ \begin{pmatrix} 0 & -g^T \\ -g & j(\ell) \end{pmatrix} \ \middle| \ \begin{pmatrix} E \\ -p \end{pmatrix} \ \right\} \\ &= \left\{ \ \ell \ \middle| \ -g \ \middle| \ -p \ \middle| \ E \ \right\} \end{aligned}$$

□

3.3 The electric Poincaré group

In this subsection, we extend the results from the previous subsection. We introduce the concept of electric Poincaré groups.

Definition 3.5. (i) The **electric Poincaré groups** is the subgroup of $\mathrm{GL}(6, \mathbb{R})$ given by:

$$\mathcal{K} := \left\{ \begin{pmatrix} e & 0 & \phi \\ 0 & L & C \\ 0 & 0 & 1 \end{pmatrix}, \ e \in \{\pm 1\} \ \wedge \ \phi \in \mathbb{R} \ \wedge \ L \in \mathcal{L} \ \wedge \ C \in \mathbb{R}^{1,3} \right\}$$

(ii) The **restricted electric Poincaré groups** is the subgroup of $\mathrm{GL}(6, \mathbb{R})$ given by:

$$\mathcal{K}_n := \left\{ \begin{pmatrix} 1 & 0 & \phi \\ 0 & L_n & C \\ 0 & 0 & 1 \end{pmatrix}, \ \phi \in \mathbb{R} \ \wedge \ L_n \in \mathcal{L}_n \ \wedge \ C \in \mathbb{R}^{1,3} \right\}$$

It has eight connected components. To define them, we introduce some notations. Let:

$$T_6 := \begin{pmatrix} 1 & 0 \\ 0 & T_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & I_4 \end{pmatrix} \quad , \quad P_6 := \begin{pmatrix} 1 & 0 \\ 0 & P_5 \end{pmatrix} = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & -I_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad C_6 := \begin{pmatrix} -1 & 0 \\ 0 & I_5 \end{pmatrix}.$$

Thus, we have:

$$\forall \lambda, \mu, \nu \in \{0, 1\}, \ T_6^\lambda P_6^\nu C_6^\mu \begin{pmatrix} 1 & 0 & \phi \\ 0 & L_n & C \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (-1)^\mu & 0 & \phi \\ 0 & T_4^\lambda P_4^\nu L_n & C \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore:

$$\mathcal{K} = \left\{ \begin{pmatrix} (-1)^\mu & 0 & \phi \\ 0 & T_4^\lambda P_4^\nu L_n & C \\ 0 & 0 & 1 \end{pmatrix},\ \lambda,\mu,\nu \in \{0,1\} \ \wedge\ \phi \in \mathbb{R} \ \wedge\ L \in \mathcal{L} \ \wedge\ C \in \mathbb{R}^{1,3} \right\}.$$

Let us define:

$$\forall \lambda,\mu,\nu \in \{0,1\},\ \mathcal{K}_{\lambda,\mu,\nu} := T_6^\lambda P_6^\nu C_6^\mu \mathcal{K}_n = \left\{ \begin{pmatrix} (-1)^\mu & 0 & \phi \\ 0 & T_4^\lambda P_4^\nu L_n & C \\ 0 & 0 & 1 \end{pmatrix},\ \phi \in \mathbb{R} \ \wedge\ L_n \in \mathcal{L}_n \ \wedge\ C \in \mathbb{R}^{1,3} \right\}$$

and $\mathcal{K}_n := \mathcal{K}_{0,0,0}$. And so we have:

$$\mathcal{K} = \bigsqcup_{\lambda,\mu,\nu \in \{0,1\}} \mathcal{K}_{\lambda,\mu,\nu} = \bigsqcup_{\lambda,\mu,\nu \in \{0,1\}} T_6^\lambda P_6^\nu C_6^\mu \mathcal{K}_n.$$

The group $\mathcal{K}$ is a Lie group of dimension 11, and its Lie algebra is given by the equation (2):

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \Lambda & \Gamma \\ 0 & 0 & 0 \end{pmatrix},\ \varepsilon \in \mathbb{R} \ \wedge\ \Lambda \in \mathfrak{a}(1,3) \ \wedge\ \Gamma \in \mathbb{R}^{1,3} \right\}$$

we are in the case:

$$Z := \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} \quad , \quad v = \begin{pmatrix} \varepsilon \\ \Gamma \end{pmatrix}.$$

A basis of $\mathfrak{k}$ is given in this representation by:

$$\left\{ \begin{pmatrix} 0 & 0 \\ \tilde{K}_i & 0 \end{pmatrix},\ i \in \{1,\dots,6\} \right\} \bigcup \left\{ \begin{pmatrix} 0 & 0 \\ J_i & 0 \end{pmatrix},\ i \in \{1,\dots,4\} \right\} \bigcup \left\{ \begin{pmatrix} 0 & 1 \\ 0_5 & 0 \end{pmatrix} \right\}$$

We are indeed under the assumptions of subsection 1.4. The Lie algebra $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{a}(\alpha')$ with $\alpha' := (-1,1,-1,-1,-1)$. We deduce the following proposition where the action of the group $\mathcal{K}$ on $\mathfrak{k}^*$ is defined by the coadjoint representation *i.e.*, for any $a \in \mathcal{K}$ and any $\mu \in \mathfrak{k}^*$, we denote:

$$a \bullet \mu := \mathrm{Ad}_a^*(\mu).$$

Proposition 3.2. (i) We have:

$$\mathfrak{k}^* = \left\{ \left\{ \ M \ \middle|\ P \ \middle|\ q \ \right\} : \begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \Lambda & \Gamma \\ 0 & 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) + P^T\Gamma + q\varepsilon,\ M \in \mathfrak{a}(1,3) \ \wedge\ P \in \mathbb{R}^{1,3} \ \wedge\ q \in \mathbb{R} \right\}.$$

(ii) We have:

$$a \bullet \left\{ \ M \ \middle|\ P \ \middle|\ q \ \right\} = \left\{ \ T_4^\lambda P_4^\nu L_n M \tau_{1,3}(L_n) P_4^\nu T_4^\lambda - 2CP^T\tau_{1,3}(L_n)P_4^\nu T_4^\lambda \ \middle|\ T_4^\lambda P_4^\nu \tau_{1,3}(L_n)^T P \ \middle|\ (-1)^\mu q \ \right\}.$$

Proof. Let $M \in \mathfrak{a}(1,3)$, $P \in \mathbb{R}^{1,3}$, and $q \in \mathbb{R}$. Let's define:

$$Q := \begin{pmatrix} q \\ P \end{pmatrix} \quad , \quad N := \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}.$$

(i) We have:

$$Q^T v = \begin{pmatrix} q & P \end{pmatrix} \begin{pmatrix} \varepsilon \\ \Gamma \end{pmatrix} = P^T\Gamma + q\varepsilon$$

and:

$$\mathrm{Tr}(NZ) = \mathrm{Tr}\left(\begin{pmatrix}0 & 0\\ 0 & M\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \Lambda\end{pmatrix}\right) = \mathrm{Tr}(M\Lambda).$$

By Proposition 1.8, the dual of $\mathfrak{k}$ is thus:

$$\mathfrak{k}^* = \left\{\left\{\ N \mid Q\ \right\} : \begin{pmatrix}Z & v\\ 0 & 0\end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(NZ) + Q^T v,\ N \in \mathfrak{g} \ \wedge\ Q \in \mathbb{R}^n\right\}$$

$$= \left\{\left\{\ M \mid P \mid q\ \right\} : \begin{pmatrix}0 & 0 & \varepsilon\\ 0 & \Lambda & \Gamma\\ 0 & 0 & 0\end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) + P^T\Gamma + q\varepsilon,\ M \in \mathfrak{a}(1,3)\ \wedge\ P \in \mathbb{R}^{1,3}\ \wedge\ q \in \mathbb{R}\right\}.$$

(ii) Let:

$$a := \begin{pmatrix}(-1)^\mu & 0 & \phi\\ 0 & T_4^\lambda P_4^\nu L_n & C\\ 0 & 0 & 1\end{pmatrix} \in \mathcal{K}.$$

Let's define:

$$U := \begin{pmatrix}(-1)^\mu & 0\\ 0 & T_4^\lambda P_4^\nu L_n\end{pmatrix} \quad , \quad D := \begin{pmatrix}\phi\\ C\end{pmatrix}.$$

We have:

$$U^{-1} = \begin{pmatrix}(-1)^\mu & 0\\ 0 & \tau_{1,3}(L_n)P_4^\nu T_4^\lambda\end{pmatrix}$$

$$UNU^{-1} = \begin{pmatrix}(-1)^\mu & 0\\ 0 & T_4^\lambda P_4^\nu L_n\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & M\end{pmatrix}\begin{pmatrix}(-1)^\mu & 0\\ 0 & \tau_{1,3}(L_n)P_4^\nu T_4^\lambda\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & T_4^\lambda P_4^\nu L_n M \tau_{1,3}(L_n)P_4^\nu T_4^\lambda\end{pmatrix}$$

$$DQ^TU^{-1} = \begin{pmatrix}\phi\\ C\end{pmatrix}\begin{pmatrix}q & P^T\end{pmatrix}\begin{pmatrix}(-1)^\mu & 0\\ 0 & \tau_{1,3}(L_n)P_4^\nu T_4^\lambda\end{pmatrix} = \begin{pmatrix}(-1)^\mu\phi q & \phi P^T\tau_{1,3}(L_n)P_4^\nu T_4^\lambda\\ (-1)^\mu qC & CP^T\tau_{1,3}(L_n)P_4^\nu T_4^\lambda\end{pmatrix}$$

$$Q^TU^{-1} = \begin{pmatrix}q & P^T\end{pmatrix}\begin{pmatrix}(-1)^\mu & 0\\ 0 & \tau_{1,3}(L_n)P_4^\nu T_4^\lambda\end{pmatrix} = \begin{pmatrix}(-1)^\mu q & P^T\tau_{1,3}(L_n)P_4^\nu T_4^\lambda\end{pmatrix}$$

By Proposition 1.9, we have:

$$a \bullet \left\{\ M \mid P \mid q\ \right\} = a \bullet \left\{\ N \mid Q\ \right\}$$

$$= \left\{\ UNU^{-1} - 2DQ^TU^{-1} \mid \left(U^{-1}\right)^T Q\ \right\}$$

$$= \left\{\ T_4^\lambda P_4^\nu L_n M\tau_{1,3}(L_n)P_4^\nu T_4^\lambda - 2CP^T\tau_{1,3}(L_n)P_4^\nu T_4^\lambda \mid T_4^\lambda P_4^\nu \tau_{1,3}(L_n)^T P \mid (-1)^\mu q\ \right\}$$

□

We keep the same notations as in the subsection on the Poincaré group. Therefore, we have:

$$\left\{\ M \mid P \mid q\ \right\}\begin{pmatrix}0 & 0 & \varepsilon\\ 0 & \Lambda & \Gamma\\ 0 & 0 & 0\end{pmatrix} = \left\{\ \begin{pmatrix}0 & g^T\\ g & j(\ell)\end{pmatrix} \middle| \begin{pmatrix}E\\ p\end{pmatrix} \middle| q\ \right\}\begin{pmatrix}0 & 0 & 0 & \varepsilon\\ 0 & 0 & \beta^T & \nu\\ 0 & \beta & j(w) & \gamma\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$= -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix}0 & g^T\\ g & j(\ell)\end{pmatrix}\begin{pmatrix}0 & \beta^T\\ \beta & j(w)\end{pmatrix}\right) + \begin{pmatrix}E & p^T\end{pmatrix}\begin{pmatrix}\nu\\ \gamma\end{pmatrix} + q\varepsilon$$

$$= (\ell^1 w^1 + \ell^2 w^2 + \ell^3 w^3) - g^T\beta + p^T\gamma + E\nu + q\varepsilon$$

$$= \ell^\vee(w) - g^\vee(\beta) + p^\vee(\gamma) + E\nu + q\varepsilon$$

We denote this last equality as:

$$\left\{\ \ell \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ q\ \right\}\begin{pmatrix}0&0&0&\varepsilon\\0&0&\beta^T&\nu\\0&\beta&j(w)&\gamma\\0&0&0&0\end{pmatrix}.$$

Therefore, we obtain the following result.

Proposition 3.3. The dual $\mathfrak{k}^*$ has the following description:

$$\left\{\left\{\ \ell \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ q\ \right\}:\begin{pmatrix}0&0&0&\varepsilon\\0&0&\beta^T&\nu\\0&\beta&j(w)&\gamma\\0&0&0&0\end{pmatrix}\longmapsto \ell^\vee(w)-g^\vee(\beta)+p^\vee(\gamma)+E\nu+q\varepsilon,\ \ \ell,g,p\in\mathbb{R}^3\ \wedge\ E,\varepsilon\in\mathbb{R}\right\}.$$

We deduce the following corollary.

Corollary 3.4. Let:

$$\left\{\ l \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ q\ \right\}\in\mathfrak{k}^*.$$

We have:

$$\begin{aligned}
P_6\bullet\left\{\ l \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ q\ \right\}&=\left\{\ l \ \middle|\ -g \ \middle|\ -p \ \middle|\ E \ \middle|\ q\ \right\}\\
T_6\bullet\left\{\ l \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ q\ \right\}&=\left\{\ l \ \middle|\ -g \ \middle|\ p \ \middle|\ -E \ \middle|\ q\ \right\}\\
C_6\bullet\left\{\ l \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ q\ \right\}&=\left\{\ l \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ -q\ \right\}.
\end{aligned}$$

Thus, for all $\lambda,\mu,\nu\in\{0,1\}$:

$$(T_6^\lambda P_6^\nu C_6^\mu)\bullet\left\{\ l \ \middle|\ g \ \middle|\ p \ \middle|\ E \ \middle|\ q\ \right\}=\left\{\ l \ \middle|\ (-1)^{\lambda+\nu}g \ \middle|\ (-1)^\nu p \ \middle|\ (-1)^\lambda E \ \middle|\ (-1)^\mu q\ \right\}.$$

Proof. We apply point (ii) of Proposition 3.2:

$$a\bullet\left\{\ M \ \middle|\ P \ \middle|\ q\ \right\}=\left\{\ T_4^\lambda P_4^\nu L_n M\tau_{1,3}(L_n)P_4^\nu T_4^\lambda-2CP^T\tau_{1,3}(L_n)P_4^\nu T_4^\lambda \ \middle|\ T_4^\lambda P_4^\nu\tau_{1,3}(L_n)^T P \ \middle|\ (-1)^\mu q\ \right\}.$$

(1) Case $a:=P_6$. We are in the case:

$$C:=0\ ,\ \lambda:=0\ ,\ \mu:=0\ ,\ \ \nu:=1\ ,\ \ L_n:=I_4.$$

We have:

$$\begin{aligned}
P_6\bullet\left\{\ l \ \middle|\ -g \ \middle|\ -p \ \middle|\ E \ \middle|\ q\ \right\}&=P_6\bullet\left\{\ \begin{pmatrix}0&g^T\\g&j(\ell)\end{pmatrix} \ \middle|\ \begin{pmatrix}E\\p\end{pmatrix} \ \middle|\ q\ \right\}\\
&=\left\{\ P_4\begin{pmatrix}0&g^T\\g&j(\ell)\end{pmatrix}P_4 \ \middle|\ P_4\begin{pmatrix}E\\p\end{pmatrix} \ \middle|\ q\ \right\}\\
&=\left\{\ \begin{pmatrix}0&-g^T\\-g&j(\ell)\end{pmatrix} \ \middle|\ \begin{pmatrix}-E\\p\end{pmatrix} \ \middle|\ q\ \right\}\\
&=\left\{\ l \ \middle|\ -g \ \middle|\ p \ \middle|\ -E \ \middle|\ q\ \right\}
\end{aligned}$$

(2) Case $a := T_6$. We are in the case:

$$C := 0\ ,\ \lambda := 1\ ,\ \mu := 0\ ,\ \nu := 0\ ,\ L_n := I_4.$$

We have:

$$\begin{aligned}
T_6 \bullet \left\{\ l \;\middle|\; -g \;\middle|\; -p \;\middle|\; E \;\middle|\; q\ \right\} &= T_6 \bullet \left\{\ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \;\middle|\; \begin{pmatrix} E \\ p \end{pmatrix} \;\middle|\; q\ \right\} \\
&= \left\{\ T_4 \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} T_4 \;\middle|\; T_4 \begin{pmatrix} E \\ p \end{pmatrix} \;\middle|\; q\ \right\} \\
&= \left\{\ \begin{pmatrix} 0 & -g^T \\ -g & j(\ell) \end{pmatrix} \;\middle|\; \begin{pmatrix} E \\ -p \end{pmatrix} \;\middle|\; q\ \right\} \\
&= \left\{\ l \;\middle|\; -g \;\middle|\; -p \;\middle|\; E \;\middle|\; q\ \right\}
\end{aligned}$$

(3) Case $a := C_6$. We are in the case:

$$C := 0\ ,\ \lambda := 0\ ,\ \mu := 1\ ,\ \nu := 0\ ,\ L_n := I_4.$$

We have:

$$\begin{aligned}
C_6 \bullet \left\{\ l \;\middle|\; -g \;\middle|\; -p \;\middle|\; E \;\middle|\; q\ \right\} &= C_6 \bullet \left\{\ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \;\middle|\; \begin{pmatrix} E \\ p \end{pmatrix} \;\middle|\; q\ \right\} \\
&= \left\{\ I_4 \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} I_4 \;\middle|\; I_4 \begin{pmatrix} E \\ p \end{pmatrix} \;\middle|\; -q\ \right\} \\
&= \left\{\ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \;\middle|\; \begin{pmatrix} E \\ p \end{pmatrix} \;\middle|\; -q\ \right\} \\
&= \left\{\ l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; -q\ \right\}
\end{aligned}$$

□

We can summarize the proposition with the following table using:

$$(T_6^\lambda P_6^\nu C_6^\mu) \bullet \left\{\ l \;\middle|\; g \;\middle|\; p \;\middle|\; E \;\middle|\; q\ \right\} = \left\{\ l \;\middle|\; (-1)^{\lambda+\nu} g \;\middle|\; (-1)^\nu p \;\middle|\; (-1)^\lambda E \;\middle|\; (-1)^\mu q\ \right\}.$$

		$\lambda = 0$	$\lambda = 1$
$\mu = 0$	$\nu = 0$	• neutral symetry • $a = I_6$ • $\mu' = \{\ell, g, p, E, q\}$	• $T-$ symetry • $a = T_6$ • $\mu' = \{\ell, -g, -p, E, -q\}$
	$\nu = 1$	• $P-$ symetry • $a = P_6$ • $\mu' = \{\ell, -g, -p, E, q\}$	• $PT-$ symetry • $a = T_6 P_6$ • $\mu' = \{\ell, g, -p, -E, q\}$
$\mu = 1$	$\nu = 0$	• $C-$ symetry • $a = C_6$ • $\mu' = \{\ell, g, p, E, -q\}$	• $CT-$ symetry • $a = T_6 C_6$ • $\mu' = \{\ell, -g, p, -E, -q\}$
	$\nu = 1$	• $CP-$ symetry • $a = P_6 C_6$ • $\mu' = \{\ell, -g, -p, E, -q\}$	• $CPT-$ symetry • $a = T_6 P_6 C_6$ • $\mu' = \{\ell, g, -p, -E, -q\}$

Conclusion

In physics, the fecundity of the use of groups as well as the invariance and symmetry relations attached to them no longer needs to be demonstrated. Thus the restricted Lorentz group, limited to its two connected orthochronous components, translates the different facets of special relativity. Composed with the group of 3D translations, it then becomes a sub-group of the isometry group of Minkowki space, itself defined by its metric, the Lorentz metric. By treating this group as the dynamic group of a flat space-time, the action on the dual of its Lie algebra makes it possible to obtain, according to the methods initiated by the mathematician J.M. Souriau ([10]) the components of its moment, which then characterize the classes of movements, carried out according to the geodesics of this space. Geodesics of zero length then refer to the motion of photons, those of non-zero length to masses.

In unfolding these different movements the P-symmetry, which transforms a "right" photon into a "left" photon, refers to the physical phenomenon of the polarization of light. In 1970, this technique enabled Souriau to identify the geometric nature of spin (not quantified). By adding a translation along a fifth dimension, this endows the moment with an additional scalar which can then be identified with the electrical charge, positive, negative or zero, with which the masses are then equipped. We can then give this extension of the Poincaré group the name "Kaluza group".

By adding an additional symmetry, inverting the fifth dimension, we double the number of connected components of the group. The action on the moment then links this new symmetry to the inversion of the electric charge. Here we obtain a geometric translation of C-symmetry, or charge conjugation, or matter-antimatter symmetry. We suggest designating this extension of the "Kaluza group", "Dirac group".

From this angle, the addition of a new symmetry challenges the physicist, the question being whether it can receive the status of a physical phenomenon. This is not how Dirac was led to propose the existence of what was later called antimatter. This idea aroused strong skepticism among theoreticians, in particular dear Niels Bohr. But fortunately the anti-electron was observed in cosmic rays, barely two years later.

By taking up this approach to reality through dynamic groups, the phenomenon of the polarization of light could have been suggested through the use of P-symmetry, translated by the second connected component of the orthochronous Poincaré group. Similarly the existence of antiparticles could have been suggested by adding a new symmetry to the "Kaluza group". Today, and this is the conclusion of this article, we must note the following path. Lorentz's group had come just in time to take on the properties of relativistic particles. But, immediately, one thing stands out. It has two orthochronous components and two antichronous components, which reverse time and energy. Fortunately, in 1970 J.M. Souriau provided a first clarifying answer. This inversion of the time coordinate, and not of the proper time, only translates geometrically the inversion of the energy, and of the mass of the elements, when they are endowed with it. The question must then be put another way: Does this new symmetry make physical sense? Can the universe contain in addition to its particles and antiparticles of positive mass, new particles and antiparticles, of negative mass, capable of emitting and capturing photons of negative energy? If so, what would be the phenomena that might emerge from all of this? Would these phenomena agree or contradict the observations? The question deserves examination. But, to do this, the particles must be bound by gravitational and possibly antigravitational forces. A model where spacetime is described by a Lorentz metric, whose dynamic group, its isometry group, the Poincaré group, is a flat space, without curvature. The gravitational force is absent from it, as well as the electromagnetic force. No force exists in this space, in fact. This Minkowskian space is not the theater of phenomena, of interactions, it only describes the conservation of the quantities composing the moment. To create these interactions, generators of phenomena, it is necessary to introduce the gravitational fields created by these masses, positive and negative, then to make a hypothesis concerning the interactions which link them. Finally see what could emerge from such a model. This will be the subject of the following article, where the metrics will be Riemanian and no longer Lorenzian and where these will be linked by coupled field equations, based on an action, constructed with the Ricci tensors of each of between them, and deriving from an action. If the approach turned out to be fruitful, if this model was likely to produce a better interpretation of observations, or even to predict new ones that would be verified, then this would mean that taking into consideration, through a purely mathematical approach to reality , driven by group theory, of a new symmetry, T-symmetry, would bear fruit. Then mathematics would play the role of a guide in relation to physics, illuminating the path to follow, rather than the path already taken.

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