SCALE INVARIANT COSMOLOGY II : EXTRA DIMENSIONS AND THE REDSHIFT OF DISTANT SUPERNOVAE

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Abstract

The cosmological problem is studied here starting from a metric with a variable scale factor for time and two different ones for space, noted R for ordinary space and S for the extra dimensions. Once R has been obtained from physical principles, one is left for S with a differential equation which can be solved through Lie group analysis and dynamical systems theory, and a frequent use of computer algebra has been made in the ensuing calculations. This leads to a good agreement with the recent observational results relating to distant supernovae; on the other hand, the solution obtained for S is particularly simple if spacetime has a dimension at least equal to ten, a result to be compared with those given by superstring or M theories.

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1. Introduction

In the recent years, theories of time varying speed of light and other physical constants (see for example [1 - 7]) have been proposed to solve the difficulties of modern cosmology, especially the horizon problem (in our previous paper however [6], although the coefficient g_{00} of the metric is time-dependent, the observable value of light velocity is constant). Just after the redaction of [6] had been completed, new observational results were published concerning the relation between the redshift and the luminosity distance of remote supernovae [8]. Since no adjustable parameter is available in [6], and as nothing could be gained from an introduction of the cosmological constant, we wondered whether the results we obtained for this relation (see line (5) of table 1) might be improved through the introduction of a specific scale parameter for the extra dimensions. This was a logical extension of [6] since, already there, one extra dimension at least had to be added to space in order to get a consistent theory.

In the following, we briefly show how the Einstein equations in more than four dimensions with zero on their right-hand side can be established for the present metric; then, as in [6], following an idea initially due to Einstein and recently reintroduced by P. Wesson and his co-workers [9, 10], we determine $R(t)$ by identifying these equations with the usual four-dimensional Einstein equations with matter-energy terms. Then we study the differential equation for $S(t)$ so as to be able to evaluate $c(t)$ and the luminosity distance.

2. The Einstein equations

The components of the Ricci tensor for a metric

$$ds^2 = -c^2(t)dt^2 + R^2(t)\sum_{i=1}^{n}\tilde{g}_{ii}^2(d\xi^i)^2 + S^2(t)\sum_{i=n+1}^{n+m}\tilde{g}_{ii}^2(d\xi^i)^2 \qquad (1)$$

can be obtained as in [6] replacing in [6 (7)] and [6 (8)] the non-zero components of the affine connexion by

$$\Gamma^0_{ii} = \begin{cases} RR'\tilde{g}_{ii} & (1 \le i \le n) \\ SS'\tilde{g}_{ii} & (n+1 \le i \le n+m) \end{cases}$$

and

$$\Gamma^i_{0i} = \Gamma^i_{i0} = \begin{cases} R'/R & (1 \le i \le n) \\ S'/S & (n+1 \le i \le n+m) \end{cases} ,$$

yielding for the spatial part (indices $i, j = 1, 2, \ldots, n+m$)

$$R_{ij} = -\Big[\frac{R}{c^2}(R'' - \frac{c'}{c}R') + (n-1)(\frac{R'^2}{c^2} + k) + m\frac{RR'S'}{Sc^2}\Big]\,\tilde{g}_{ij} \quad (1 \le i, j \le n) \qquad (2)$$

and

$$R_{ij} = -\Big[\frac{S}{c^2}(S'' - \frac{c'}{c}S') + (m-1)(\frac{S'^2}{c^2} + k) + n\frac{SS'R'}{Rc^2}\Big]\,\tilde{g}_{ij} \quad (n+1 \le i, j \le n+m) \quad , (3)$$

where the derivatives are taken with respect to cosmic time t, where k is the (common) curvature index of the subspaces with scale parameters R and S respectively, the dimensions of which are n and m, and where the g_{ij} are those of the spatial part of a Robertson-Walker metric [11], with

$$g_{ij} \;=\; R^2(t)\tilde{g}_{ij} \quad (1 \le i,j \le n) \quad \text{or} \quad g_{ij} \;=\; S^2(t)\tilde{g}_{ij} \quad (n+1 \le i,j \le n+m)$$

where

$$\tilde{g}_{ij} \;=\; 0 \quad \text{if} \quad i \neq j$$

and for the temporal part

$$R_{00} \;=\; \frac{n}{Rc^2}\,(R'' - \frac{c'}{c}R') + \frac{m}{Sc^2}\,(S'' - \frac{c'}{c}S') \quad . \tag{4}$$

The Einstein equations with zero as a second member follow immediately and can be written after a few simplifications:

$$\frac{1}{R}(R'' - \frac{c'}{c}R') + \frac{n-1}{R^2}(R'^2 + kc^2) + m\frac{R'S'}{RS} \;=\; 0 \tag{2'}$$

$$\frac{1}{S}(S'' - \frac{c'}{c}S') + \frac{m-1}{S^2}(S'^2 + kc^2) + n\frac{R'S'}{RS} \;=\; 0 \tag{3'}$$

$$\frac{n}{R}\,(R'' - \frac{c'}{c}R') + \frac{m}{S}\,(S'' - \frac{c'}{c}S') \;=\; 0 \quad , \tag{4'}$$

so that a coupling term appears in equations (2') and (3').

3. Determining $R(t)$

Noting as in [6]

$$R_{00}^{(F;1+3)} \;=\; \frac{3R''}{Rc^2} \tag{5}$$

and

$$R_{ij}^{(F;1+3)} \;=\; -\frac{1}{c^2}(RR'' + 2R'^2)\,\tilde{g}_{ij} \quad (i,j \;=\; 1,3) \tag{6}$$

the parts of the Ricci tensor components which correspond to the Friedmann model with zero curvature, the time-time and space-space components of the Einstein equations can be written:

$$R_{00} \;=\; R_{00}^{(F;1+3)} + (n-3)\frac{R''}{Rc^2} - n\frac{R'c'}{Rc^3} + \frac{m}{Sc^2}\,(S'' - \frac{c'}{c}S') \;=\; 0 \tag{7}$$

$$R_{ij} \;=\; R_{ij}^{(F;1+3)} + \Big[\frac{RR'c'}{c^3} - (n-3)\frac{R'^2}{c^2} - (n-1)k - m\frac{RR'S'}{Sc^2}\Big]\,\tilde{g}_{ij} \quad (i,j \;=\; 1,3) \quad , \tag{8}$$

hence, identifying (7) and (8) with the usual quadridimensional equations

$$R_{00}^{(F;1+3)} + \frac{8\pi G}{c^2} S_{00} = 0 \tag{7'}$$

and

$$R_{ij}^{(F;1+3)} + \frac{8\pi G}{c^2} S_{ij} = 0 \tag{8'}$$

with source terms

$$S_{00} = \frac{1}{2}(\rho + \frac{3p}{c^2}) \tag{9}$$

$$S_{i0} = 0 \tag{10}$$

$$S_{ij} = \frac{1}{2}(\rho - \frac{p}{c^2})R^2 \tilde{g}_{ij} \quad , \tag{11}$$

the equations

$$\frac{4\pi G}{c^2}(\rho + \frac{3p}{c^2}) = -\frac{3R''}{Rc^2} \tag{12}$$

$$\frac{4\pi G}{c^2}(\rho - \frac{p}{c^2}) = \frac{1}{c^2}(\frac{R''}{R} + 2\frac{R'^2}{R^2}) \tag{13}$$

which are naturally those of the zero curvature Friedmann model.

Of course, for this identification to be possible, the $(n+m)$-space has to contain an Euclidean three dimensional variety, which was the only consistent possibility found in [6]: an example of this situation has been given there for a negative curvature n-space and we shall take $k = -1$ here also; however we shall keep on using index k in order to discuss the consequences of attributing it other values when appropriate.

The resolution of (12) and (13) follows classically: for an equation of state

$$p = (2\delta - 1)\,\frac{\rho c^2}{3} \quad (\delta \text{ constant}) \tag{14}$$

equivalent to

$$\frac{\rho c^2 + 3p}{\rho c^2 - p} = \frac{3\delta}{2-\delta} \quad , \tag{15}$$

one gets

$$\frac{3\delta}{2-\delta} = -\frac{3RR''}{RR'' + 2R'^2} \tag{16}$$

or

$$\frac{RR''}{R'^2} = -\delta \tag{17}$$

the non-trivial solution of which satisfying

$$R(0) = 0 \tag{18}$$

and

$$R(t_0) = R_0 \tag{19}$$

is, as in [6],

$$R(t) \;=\; R_0\Big(\frac{t}{t_0}\Big)^{\frac{\gamma}{\gamma+1}} \tag{20}$$

with $\gamma \;=\; 1/\delta$.

This gives, for $\gamma \;=\; 1$ and 2 respectively, the solutions in $t^{1/2}$ and $t^{2/3}$ describing the radiation era and the dust universe in the zero curvature Friedmann model.

4. Determining $S(t)$

Combining (2'), (3') and (4') easily yields

$$n(n-1)(\frac{R'^2}{R^2}+k\frac{c^2}{R^2})+m(m-1)(\frac{S'^2}{S^2}+k\frac{c^2}{S^2})+2mn\frac{R'}{R}\frac{S'}{S} \;=\; 0 \tag{21}$$

hence

$$kc^2 \;=\; -\frac{m(m-1)\frac{S'^2}{S^2}+n(n-1)\frac{R'^2}{R^2}+2mn\frac{R'}{R}\frac{S'}{S}}{\frac{m(m-1)}{S^2}+\frac{n(n-1)}{R^2}} \quad . \tag{22}$$

One can see here that the hypothesis $k=0$ is not interesting: (22) would then give for S an expression of the form

$$\frac{S}{S_0} \;=\; \Big(\frac{t}{t_0}\Big)^{\alpha}$$

with α a root of

$$m(m-1)\alpha^2+\frac{4}{3}mn\alpha+\frac{4}{9}n(n-1) \;=0$$

and thus for c in (4') a power of t always different from $-1/3$, the value which allows one to avoid the occurrence of horizons [6].

Noting that in (2') c appears only in kc^2 and in

$$\frac{c'}{c} \;=\; \frac{1}{2c^2}(c^2)' \quad , \tag{23}$$

it can be seen that the differential equation which can be formed for S starting from (2'), (20) and (22) does not depend on the sign of k. Noting also that, still in (2'), the denominator of c^2/R^2 involves the expression

$$m(m-1)\frac{R^2}{S^2} \;=\; m(m-1)\frac{R_0^2}{t_0^{\frac{2\gamma}{\gamma+1}}}\frac{t^{\frac{2\gamma}{\gamma+1}}}{S(t)^2} \tag{24}$$

so that defining

$$S(t) \;=\; \Sigma_0 s(t) \tag{25}$$

with
$$\Sigma_0 = \frac{R_0}{t_0^{\frac{\gamma}{\gamma+1}}} \tag{26}$$

yields for $s(t)$ a differential equation where R_0 and t_0 never appear and which writes, with $\beta = \gamma/(\gamma+1)$:

$$\begin{aligned}
&(m-1)\beta[n(n-1)s^2 + m(m-1)t^{2\beta}]t^3 ss's'' + n\beta^2[n(n-1)s^2 + m(m-1)t^{2\beta}]t^2 s^2 s'' \\
&+ m(m-1)^2(n-1)t^4 ss'^4 + (m-1)\beta[n(n-1)(3m-1)s^2 - m^2(m-1)t^{2\beta}]t^3 s'^3 \\
&+ \beta\{n(n-1)[(3nm-2n-m+1)\beta + m-1]s^2 - m(m-1)[(3m-2)n\beta - m+1)]t^{2\beta}\}t^2 ss'^2 \\
&+ n\beta^2\{n(n-1)[(n-1)\beta+1]s^2 - (m-1)[(3nm-n-m+1)\beta - m]t^{2\beta}\}ts^2 s' \\
&- n^2(n-1)(m-1)\beta^4 t^{2\beta} s^3 = 0 \quad .
\end{aligned} \tag{27}$$

This equation can be solved through the Lie group method. As we shall see, one generator of the group of equation (27) is easily determined; however, knowing all the generators is important, and this may demand heavy computations – so we have used F. Schwarz's program SPDE [12] on the Computer Centre of the GMD at Sankt Augustin. As this program works on equations with rational arguments only, we have dealt with the cases $\beta = 2/3$ (dust universe) and $\beta = 1/2$ (radiation era) separately.

A) For the dust universe, changing the time variable to $u = t^{1/3}$ transforms (27) into the following equation, where s' now stands for ds/du :

$$\begin{aligned}
&2(m-1)[n(n-1)s^2 + m(m-1)u^4]u^2 ss's'' + 4n[n(n-1)s^2 + m(m-1)u^4]us^2 s'' \\
&+ m(m-1)^2(n-1)u^3 ss'^4 + 2(m-1)[n(n-1)(3m-1)s^2 - m^2(m-1)u^4]u^2 s'^3 \\
&+ 2(6nm-4n-m+1)[n(n-1)s^2 - m(m-1)u^4]uss'^2 \\
&+ 4n[n(n-1)(2n-1)s^2 - (m-1)(6nm-2n-3m+2)u^4]s^2 s' \\
&- 16n^2(n-1)(m-1)u^3 s^3 = 0 \quad .
\end{aligned} \tag{28}$$

As suggested by the occurrence of sums of terms in s^2 and u^4, this equation remains invariant in the inhomogeneous scale transformation $\bar{u} = \lambda^{1/2}u, \bar{s} = \lambda s$, the generator of which is
$$U = \frac{1}{2}u\frac{\partial}{\partial u} + s\frac{\partial}{\partial s} \tag{29}$$

as can be easily seen applying Taylor's theorem up to first order in ϵ to a differentiable function $f(\bar{u},\bar{s})$ with $\lambda = 1+\epsilon$, $\epsilon << 1$ and identifying with $(1+\epsilon U)f(u,s)$ [13] (turning to the time variable t, it may be noted that U is but the generator X of the inhomogeneous dilation associated with Kepler's third law [6, 13]). This generator is also the only one to be found using SPDE. So, equation (28) can be reduced in order by one and we have to determine the first extension $U^{(1)}$ of the generator U: starting from the operator of total differentiation with respect to u:

$$D = \frac{\partial}{\partial u} + s'\frac{\partial}{\partial s} + s''\frac{\partial}{\partial s'} + \ldots \tag{30}$$

one has to compute $D(\eta)$ and $D(\xi)$ where $\eta(u,s)$ and $\xi(u,s)$ are the respective coefficients of $\partial/\partial s$ and $\partial/\partial u$ in the expression of U, that is

$$\eta = s \quad \text{and} \quad \xi = \frac{u}{2} \tag{31}$$

hence

$$D(\eta) = s' \quad \text{and} \quad D(\xi) = 1/2 \quad . \tag{32}$$

The general definition of $U^{(1)}$ being:

$$U^{(1)} = \xi(u,s)\frac{\partial}{\partial u} + \eta(u,s)\frac{\partial}{\partial s} + \zeta^{(1)}(u,s,s')\frac{\partial}{\partial s'} \tag{33}$$

where

$$\zeta^{(1)}(u,s,s') = D(\eta) - s'D(\xi) \quad , \tag{34}$$

this yields here

$$\zeta^{(1)} = \frac{s'}{2} \tag{35}$$

and

$$U^{(1)} = \frac{u}{2}\frac{\partial}{\partial u} + s\frac{\partial}{\partial s} + \frac{s'}{2}\frac{\partial}{\partial s'} \quad . \tag{36}$$

This process allows one to define new variables p and q such that, in these variables, the original second order equation (28) will be transformed into a first order one: for this to be obtained, p and q have to be solutions of

$$Uq = 0 \quad \text{and} \quad U^{(1)}p = 0 \quad , \tag{37}$$

that is here

$$\frac{u}{2}\frac{\partial q}{\partial u} + s\frac{\partial q}{\partial s} = 0 \tag{38}$$

and

$$\frac{u}{2}\frac{\partial p}{\partial u} + s\frac{\partial p}{\partial s} + \frac{s'}{2}\frac{\partial p}{\partial s'} = 0 \quad . \tag{39}$$

The solution of these first order partial differential equations can be found by integrating the corresponding characteristic systems of ordinary differential equations which are here:

$$2\frac{du}{u} = \frac{ds}{s} \tag{40}$$

and

$$2\frac{du}{u} = \frac{ds}{s} = 2\frac{ds'}{s'} \tag{41}$$

so that, for (40)

$$\frac{s}{u^2} = C_0 \tag{42}$$

and for (41)

$$\frac{s}{u^2} = \mathcal{C}_1 \quad \text{and} \quad \frac{s'}{u} = \mathcal{C}_2 \tag{43}$$

where $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_2$ are constants.

The general solutions of (38) and (39) are arbitrary functions of $\mathcal{C}_0$ and of $\mathcal{C}_1$ and $\mathcal{C}_2$, that is $\phi(s/u^2)$ and $\psi(s/u^2, s'/u)$ respectively and the new variables p and q can be simply defined as the independent dimensionless functions

$$q = \frac{s}{u^2} \quad \text{and} \quad p = \frac{s'}{u} \quad . \tag{44}$$

Substituting in (28) s, s' and s'' by

$$s = qu^2, \quad s' = pu \quad \text{and} \quad s'' = (p-2q)\frac{dp}{dq} + p \tag{45}$$

one gets a first order equation which can be factorized into

$$[(m-1)p + 2nq]\{-2q(p-2q)[n(n-1)q^2 + m(m-1)]\frac{dp}{dq} + (mp+2nq)\cdot$$
$$\cdot[2m(m-1)p + 4(m-1)(n-1)q - (m-1)(n-1)qp^2 - 2n(n-1)q^2p]\} = 0 \, . \tag{46}$$

Thus, (46) splits into two equations, the simplest of which is

$$(m-1)p + 2nq = 0 \tag{47}$$

or

$$(m-1)us' + 2ns = 0 \tag{48}$$

and gives

$$s = \sigma u^{-\frac{2n}{m-1}} \quad (\sigma \text{constant}) \tag{49}$$

or

$$s = \sigma t^{-\frac{2}{3}\frac{n}{m-1}} \quad . \tag{50}$$

So, after (25),

$$\frac{S'(t)}{S(t)} = \frac{s'(t)}{s(t)} = -\frac{2}{3}\frac{n}{m-1}\frac{1}{t} \tag{51}$$

and, after (22),

$$kc^2 = \frac{4n}{9t^2}\frac{n+m-1}{m-1}\frac{1}{\frac{m(m-1)}{S^2} + \frac{n(n-1)}{R^2}} \quad . \tag{52}$$

This would imply $k = 1$, but a variety with a constant positive curvature is necessarily finite [11] and cannot have an Euclidean space as a subvariety. Thus this possibility has to be rejected here.

The second factor on the left member of (46) yields the equation

$$\frac{dp}{dq} = \frac{(mp+2nq)[2m(m-1)p+4(m-1)(n-1)q-(m-1)(n-1)qp^2-2n(n-1)q^2p]}{2q(p-2q)[n(n-1)q^2+m(m-1)]} \tag{53}$$

equivalent to the following autonomous system, written as a function of a dimensionless temporal variable θ (obviously not to be identified with u nor t):

$$\frac{dp}{d\theta} = (mp+2nq)[2m(m-1)p+4(m-1)(n-1)q-(m-1)(n-1)qp^2-2n(n-1)q^2p] \tag{54}$$

$$\frac{dq}{d\theta} = 2q(p-2q)[n(n-1)q^2+m(m-1)] \quad . \tag{55}$$

and which can be investigated using the methods of dynamical systems theory [14,15]. Here, as implied by the definitions of p and q (44), to each integral curve (or trajectory) of (54-55) in the (q,p) plane corresponds a differential equation for $s(u)$. This system has three singular (or equilibrium) points in the $q \geq 0$ half-plane – the only interesting region here (see fig. 1) – which are the zeroes of the vector field defined by the right member of (54 - 55):

$$1) \quad q_0 = 0, \quad p_0 = 0 \tag{56}$$

$$2) \quad q_1 = \sqrt{\frac{m-1}{n-1}}, \quad p_1 = 2\sqrt{\frac{m-1}{n-1}} \tag{57}$$

and the point at infinity. (For the second case, substituting $p = 2q$ into the brackets of (54-1) gives

$$4(m+n-1)q[m-1-(n-1)q^2] = 0$$

hence the result.)

It may also be noted that the straight lines $q = 0$ and $p = 2q$ are singular lines for the differential equation (53).

Let F and G stand for the right members of (54) and (55): these being polynomials can be expanded in Taylor polynomials around their zeroes (q_i, p_i) $(i = 0, 1)$:

$$F(q,p) = \left.\frac{\partial F}{\partial p}\right|_{(q_i,p_i)}(p-p_i) + \left.\frac{\partial F}{\partial q}\right|_{(q_i,p_i)}(q-q_i) + P(q-q_i, p-p_i) \tag{58}$$

$$G(q,p) = \left.\frac{\partial G}{\partial p}\right|_{(q_i,p_i)}(p-p_i) + \left.\frac{\partial G}{\partial q}\right|_{(q_i,p_i)}(q-q_i) + Q(q-q_i, p-p_i) \tag{59}$$

where P and Q start with terms which are at least quadratic in $(p-p_i)$ and $(q-q_i)$ and may be expected to be negligible compared with the linear terms when they exist.

Near the first zero – the origin – the field is purely nonlinear; it does not give rise to any interesting conclusions, and nor does the point at infinity as will be shown in the appendix. For the second zero,

$$\left.\frac{\partial F}{\partial p}\right|_{(q_1,p_1)} = -4(m-1)(m+n)(m+n-2)q_1 \tag{60}$$

$$\left.\frac{\partial F}{\partial q}\right|_{(q_1,p_1)} = -8(m+n)^2(m-1)q_1 \tag{61}$$

$$\left.\frac{\partial G}{\partial p}\right|_{(q_1,p_1)} = 2(m+n)(m-1)q_1 \tag{62}$$

$$\left.\frac{\partial G}{\partial q}\right|_{(q_1,p_1)} = -4(m+n)(m-1)q_1 \tag{63}$$

so the matrix of the linear system associated with (54-55) in the neighbourhood of (q_1, p_1) is

$$A = \begin{pmatrix} \left.\frac{\partial F}{\partial p}\right|_{(q_1,p_1)} & \left.\frac{\partial F}{\partial q}\right|_{(q_1,p_1)} \\ \left.\frac{\partial G}{\partial p}\right|_{(q_1,p_1)} & \left.\frac{\partial G}{\partial q}\right|_{(q_1,p_1)} \end{pmatrix} = 2(m+n)(m-1)q_1 A_1 \tag{64}$$

with

$$A_1 = \begin{pmatrix} -2(m+n-2) & -4(m+n) \\ 1 & -2 \end{pmatrix} \tag{65}$$

the characteristic equation of which is

$$\lambda^2 + 2(m+n-1)\lambda + 8(m+n-1) = 0 \tag{66}$$

hence the eigenvalues:

$$\lambda = -(m+n-1) \pm \sqrt{(m+n-1)(m+n-9)} \tag{67}$$

So, three cases have to be distinguished:

1) $m+n < 9$: the eigenvalues are complex, and the trajectories spiral indefinitely around the singular point – which is then a focus [14]; the real part of λ being negative, p and q are damped oscillatory functions of time;

2) $m+n = 9$: the eigenvalue $\lambda = -8$ is double, with an only eigenvector:

$$\vec{x} = \begin{pmatrix} -6 \\ 1 \end{pmatrix} \quad ; \tag{68}$$

A is not diagonalizable and all the trajectories are tangent to $\vec{x}$ at (q_1, p_1), which is an improper node of the second kind;

3) $m+n > 9$: the eigenvalues are real and have the same sign ($\lambda' < \lambda < 0$); all the trajectories tend to (q_1, p_1) tangentially to $\vec{x}$, the eigenvector which corresponds to λ, except the two trajectories which have the direction of $\vec{x}'$ (the eigenvector corresponding to λ'); (q_1, p_1) is an improper node of the first kind.

In the last two cases, p and q, when sufficiently close to p_1 and q_1, are monotonous functions of time.

The three preceding situations are among those for which the solutions of the initial nonlinear system in the vicinity of the singular point have the same behaviour as those of its linearization [15], and the above discussion allows one to conclude that for the initial equation (53):

– if $m+n<9$, dp/dq has no limit when $\theta \to \infty$, so that in this case no differential equation can be drawn from (53) for $s(u)$ when $u \to \infty$ (as shown later, $\theta \to \infty$ implies $u \to \infty$);

– if $m+n=9$, this limit exists and is unique, as well as a simple limit form for the differential equations which correspond to the trajectories of (54 - 55);

– if $m+n>9$, dp/dq has two possible limits: however one of them (the slope of $\vec{x}'$) is exceptional since every trajectory which (in the linear approximation) is not exactly directed along $\vec{x}'$ ends tangentially to $\vec{x}$.

Thus a qualitative difference exists between the time dependencies of p and q – and $S(t)$ – depending on the number of dimensions $D=m+n+1$ of space-time, and a simple approximate solution can be found if $D \geq 10$.

B) In the case of the radiative era ($\beta = 1/2$), equation (27) is already rational, and no change of variable has to be performed. Here also, the Lie group has an only generator – which can be obtained from the transformation $\bar{t} = \lambda^2 t, \bar{s} = \lambda s$:

$$U = 2t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} \tag{69}$$

In the same way as before, one finds for the first extension of U:

$$U^{(1)} = 2t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} - s'\frac{\partial}{\partial s'} \tag{70}$$

The new variables p and q, solutions of (37), have to satisfy the differential equations

$$2t\frac{\partial q}{\partial t} + s\frac{\partial q}{\partial s} = 0 \tag{71}$$

and

$$2t\frac{\partial p}{\partial t} + s\frac{\partial p}{\partial s} - s'\frac{\partial p}{\partial s'} = 0 \tag{72}$$

which are equivalent to

$$\frac{dt}{2t} = \frac{ds}{s} \tag{73}$$

and

$$\frac{dt}{2t} = \frac{ds}{s} = -\frac{ds'}{s'} \quad , \tag{74}$$

hence

$$\frac{s}{\sqrt{t}} = C_0 \quad \text{and} \quad s'\sqrt{t} = C_1 \tag{75}$$

with C_0 and C_1 constant, and the new variables

$$q = \frac{s}{\sqrt{t}} \quad \text{and} \quad p = s'\sqrt{t} \quad . \tag{76}$$

The substitutions

$$s = q\sqrt{t}, \quad s' = \frac{p}{\sqrt{t}} \quad \text{and} \quad s'' = \frac{q^3}{s^3}\Big[(p-\frac{q}{2})\frac{dp}{dq} - \frac{p}{2}\Big] \tag{77}$$

bring the left member of (27) with $\beta = 1/2$ into an expression which, once again, can be factorized as the product of a first degree polynomial in p and q and of a first order differential polynomial, hence:

$$[2(m-1)p + nq]\{-2q(2p-q)[n(n-1)q^2 + m(m-1)]\frac{dp}{dq} + (2mp+nq)\cdot$$
$$\cdot[2m(m-1)p + (m-1)(n-1)q - 4(m-1)(n-1)qp^2 - 2n(n-1)q^2p]\} = 0\,. \tag{78}$$

The polynomial factor yields a differential equation

$$2(m-1)t\frac{ds}{dt} + ns = 0 \tag{79}$$

the solution of which writes

$$s = \sigma t^{-\frac{1}{2}\frac{n}{m-1}} \quad (\sigma \text{ constant}) \tag{80}$$

hence, after (25):

$$\frac{S'(t)}{S(t)} = \frac{s'(t)}{s(t)} = -\frac{n}{2(m-1)}\frac{1}{t} \quad ; \tag{81}$$

as here

$$R = R_0\Big(\frac{t}{t_0}\Big)^{1/2} \quad , \tag{82}$$

so that

$$\frac{R'(t)}{R(t)} = \frac{1}{2t} \quad , \tag{83}$$

the numerator of kc^2 (equation (22)) writes

$$-\Big[m(m-1)\frac{S'^2}{S^2} + n(n-1)\frac{R'^2}{R^2} + 2mn\frac{R'}{R}\frac{S'}{S}\Big] = \frac{n}{4t^2}\frac{n+m-1}{m-1} \quad , \tag{84}$$

and for the same reasons as in the case of the dust universe, equation (79) must be rejected.

The differential polynomial yields a first order differential equation which, as in the previous case, has two singular points: the origin and the point $q_1', p_1')$ with

$$q_1' = \sqrt{\frac{m-1}{n-1}}, \quad p_1' = \frac{1}{2}\sqrt{\frac{m-1}{n-1}} \quad . \tag{85}$$

As before, it also has two singular lines: $q = 0$ and $2p = q$.

The matrix of the associated linear system is here:

$$A = (m+n)(m-1)q_1 A_1 \tag{86}$$

with

$$A_1 = \begin{pmatrix} -2(m+n-2) & -(m+n) \\ 4 & -2 \end{pmatrix} \tag{87}$$

This matrix has the same trace and the same determinant as the matrix A_1 of equation (65) and the same conclusions as in the case of the dust universe are obtained as concerns the solutions of the initial nonlinear equation and the influence of the number of dimensions of space-time.

From the preceding results can be deduced an approximate expression of $S(t)$ in the neighbourhood of the singular point (q_1, p_1) and hence for $t \to \infty$.

For $D = 10$ and the dust universe, the equation of the tangent to all the trajectories at (q_1, p_1) is

$$p + 6q = 8q_1 \tag{88}$$

hence the differential equation for s:

$$\frac{ds}{du} + \frac{6}{u}s = 8q_1 u \tag{89}$$

which integrates by the method of variation of parameters to

$$s(u) = q_1\left(u^2 + \frac{K}{u^6}\right) \tag{90}$$

and gives, after (26) with $\gamma = 2$ and $u = t^{1/3}$:

$$S(t) = q_1 R_0\left[\left(\frac{t}{t_0}\right)^{2/3} + \kappa\left(\frac{t_0}{t}\right)^2\right] \tag{91}$$

where

$$K = \kappa t_0^{8/3} \quad . \tag{92}$$

This yields

$$\frac{S}{R} = q_1\left[1 + \kappa\left(\frac{t_0}{t}\right)^{8/3}\right] \quad : \tag{93}$$

$\frac{S}{R} \to q_1 = \sqrt{\frac{8-n}{n-1}}$ when $t \to \infty$: there is no compactification of the extra dimensions, a possibility which is presently considered in superstring and M theories [16, 17].

As shown on figs. 1 or 2, any trajectory which has (q_1, p_1) as a limit point cuts the straight line $p = 2q$ at another point where, after (55), the tangent is parallel to the p-axis. In the vicinity of such a point with abscissa a, q remains approximately equal to a and

$$\frac{s}{u^2} \simeq a \tag{94}$$

hence

$$s \simeq at^{2/3} \tag{95}$$

and (26)

$$S = \Sigma_0 s \simeq aR_0\left(\frac{t}{t_0}\right)^{2/3} \quad : \tag{96}$$

then S varies in the same way as R and it seems natural to take t or $u \simeq 0$ at this point. (Taking the origin at $(a, 2a)$ and assuming, with $\xi = p - 2a$ and $\eta = q - a$, a parabolic form $\eta = A\xi^2$ for the trajectory near $q = a$, one can show that

$$\frac{s}{u^2} = a + \frac{1}{4A}\left(1 - \frac{u_0}{u}\right)^2 \quad ; \tag{97}$$

hence for (97) to make sense, u_0 can be taken small but not zero.)

For the radiation era, $S(t)$ also varies like $R(t)$ (as $t^{1/2}$) in the vicinity of points of the line $2p = q$ different from (q_1', p_1').

The evolution of the universe thus takes place on a portion of trajectory starting on the singular line $p = 2q$ (for the dust universe) at a point $q = a$ different from the node (q_1, p_1) and approaching (q_1, p_1) as $\theta -$ and $t - \to \infty$, hence above or under the line $p = 2q$ according as a is lower or greater than q_1: as shown in the appendix, the other parts of the integral curves of (54-55) can hardly be associated with physical solutions.

For $D \geq 11$ and the dust universe, the matrix A_1 has two distinct eigenvalues, noted λ and λ', given by (67), with eigenvectors

$$\vec{x} = \begin{pmatrix} \lambda + 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{x}' = \begin{pmatrix} \lambda' + 2 \\ 1 \end{pmatrix} \quad ; \tag{98}$$

the slope of $\vec{x}$ being $\lambda + 2$, the equation of the common tangent at (q_1, p_1) to the stable trajectories is:

$$p = (\lambda + 2)q - \lambda q_1 \tag{99}$$

hence the differential equation

$$\frac{ds}{du} - (\lambda + 2)\frac{s}{u} = -\lambda q_1 u \tag{100}$$

which integrates to

$$s(u) = q_1(u^2 + Ku^{\lambda+2}) \tag{101}$$

and gives:

$$S(t) = q_1 R_0\left[\left(\frac{t}{t_0}\right)^{2/3} + \kappa\left(\frac{t_0}{t}\right)^{-(\lambda+2)/3}\right] \tag{102}$$

where, as before

$$K = \kappa t_0^{-\lambda/3} \quad . \tag{103}$$

From

$$\frac{S}{R} = q_1\left[1 + \kappa\left(\frac{t_0}{t}\right)^{-\lambda/3}\right] \tag{104}$$

the same conclusions as previously can be drawn from (93) and the portions of the integral curves representing the evolution of the universe might also be described in the same way as for $D = 10$. As superstring or M theories yield a maximum value $D = 10$ or 11 for the number of non compact dimensions of space-time, from now on, in the frame of the simple approximation defined above ((89)and (100)), we shall focus our attention on these two cases.

5. Determining $c(t)$, the horizons and the luminosity distance

Starting from (22), with now $k = -1$ (as results from (108), taking $k = 1$ would imply $c^2 < 0$ for $t \to \infty$), one has:

$$\frac{c^2}{R^2} = \frac{m(m-1)\frac{S'^2}{S^2} + n(n-1)\frac{R'^2}{R^2} + 2mn\frac{R'}{R}\frac{S'}{S}}{n(n-1) + m(m-1)\frac{R^2}{S^2}} \quad , \tag{105}$$

hence in the vicinity of $t = 0$, where S and R vary as t^α ($\alpha = 2/3$ or $1/2$),

$$\frac{c}{R} \propto \frac{1}{t} \tag{106}$$

from which follows the divergence of the integral

$$\varrho_p = \int_0^{t_1} \frac{c(t)dt}{R(t)} \tag{107}$$

and thus the absence of particle horizon.

For $t \to \infty$, we have for the dust universe, with $\lambda = -8$ if $D = 10$ and the largest eigenvalue if $D \geq 11$

$$\frac{c^2}{R^2} = \frac{4}{9}\frac{1}{t^2}\frac{1}{n-1}\frac{1}{m + n[1+\kappa(\frac{t_0}{t})^{-\lambda/3}]^2}\Big\{m(m-1)\Big[1 + (1+\frac{\lambda}{2})\kappa\Big(\frac{t_0}{t}\Big)^{-\lambda/3}\Big]^2$$
$$+ 2mn\Big[1 + (1+\frac{\lambda}{2})\kappa\Big(\frac{t_0}{t}\Big)^{-\lambda/3}\Big]\Big[1+\kappa\Big(\frac{t_0}{t}\Big)^{-\lambda/3}\Big] + n(n-1)\Big[1+\kappa\Big(\frac{t_0}{t}\Big)^{-\lambda/3}\Big]^2\Big\} \tag{108}$$

hence, for $|\kappa(\frac{t_0}{t})^{-\lambda/3}| = |\kappa(\frac{R_0}{R})^{-\lambda/2}| < 1$ where t_0 is an instant of reference and R_0 the corresponding value of R,

$$\frac{c}{R} = \frac{2}{3}\frac{1}{t}\sqrt{\frac{m+n-1}{n-1}}\Big\{1 + \frac{1}{2}\frac{m(\lambda+2)}{m+n}\kappa\Big(\frac{t_0}{t}\Big)^{-\lambda/3}$$
$$- \frac{1}{8}\frac{mn[4(\lambda+3)(m+n-1)+\lambda^2]}{(m+n)^2(m+n-1)}\kappa^2\Big(\frac{t_0}{t}\Big)^{-2\lambda/3} + O\Big(\kappa^3\Big(\frac{t_0}{t}\Big)^{-\lambda}\Big)\Big\} \tag{109}$$

so that

$$\varrho_e = \int_{t_1}^{\infty} \frac{c(t)dt}{R(t)} \tag{110}$$

is divergent, implying the absence of event horizon.

The last expression can be integrated to give

$$\int \frac{cdt}{R} = \sqrt{\frac{m+n-1}{n-1}}\Big\{\frac{2}{3}\ln\frac{t}{t_0} + (1+\frac{2}{\lambda})\frac{m\kappa}{m+n}\Big(\frac{t_0}{t}\Big)^{-\lambda/3}$$
$$-\frac{1}{8\lambda}\frac{mn[4(\lambda+3)(m+n-1)+\lambda^2]}{(m+n)^2(m+n-1)}\kappa^2\Big(\frac{t_0}{t}\Big)^{-2\lambda/3} + O\Big(\kappa^3\Big(\frac{t_0}{t}\Big)^{-\lambda}\Big)\Big\} \qquad (111)$$

and, R_e and R_r being the values of the scale parameter R at the instants of emission t_e and of reception t_r of a radiation,

$$\int_{t_e}^{t_r} \frac{cdt}{R} = \sqrt{\frac{m+n-1}{n-1}}\Big\{\ln\frac{R_r}{R_e} + (1+\frac{2}{\lambda})\frac{m\kappa}{m+n}\Big[\Big(\frac{R_0}{R_r}\Big)^{-\lambda/2} - \Big(\frac{R_0}{R_e}\Big)^{-\lambda/2}\Big]$$
$$-\frac{1}{8\lambda}\frac{mn[4(\lambda+3)(m+n-1)+\lambda^2]}{(m+n)^2(m+n-1)}\kappa^2\Big[\Big(\frac{R_0}{R_r}\Big)^{-\lambda} - \Big(\frac{R_0}{R_e}\Big)^{-\lambda}\Big] + O(\kappa^3\ldots)\Big\} \quad . \qquad (112)$$

Noting that

$$\Big(\frac{R_0}{R_r}\Big)^{\nu} - \Big(\frac{R_0}{R_e}\Big)^{\nu} = \Big(\frac{R_0}{R_r}\Big)^{\nu}(1-(1+z)^{\nu})$$

this gives, with

$$\alpha = \kappa\Big(\frac{R_0}{R_r}\Big)^{-\lambda/2} \quad , \qquad (113)$$

$$\int_{t_e}^{t_r} \frac{cdt}{R} = \sqrt{\frac{m+n-1}{n-1}}\Big\{\ln(1+z) - (1+\frac{2}{\lambda})\frac{m}{m+n}\alpha[(1+z)^{-\lambda/2}-1]$$
$$+\frac{1}{8\lambda}\frac{mn[4(\lambda+3)(m+n-1)+\lambda^2]}{(m+n)^2(m+n-1)}\alpha^2[(1+z)^{-\lambda}-1] + O(\alpha^3\ldots)\Big\} \quad . \qquad (114)$$

With R_r expressed as

$$R_r = \frac{R'_r}{H_r} = \frac{c}{H_r}\frac{R'_r}{R_r}\frac{1}{c/R_r} \qquad (115)$$

(where c equals c_r and H_r is the present value of the Hubble constant H_0) and, after (20) and (109),

$$R_r = \frac{c}{H_r}\sqrt{\frac{n-1}{m+n-1}}\Big\{1 + \frac{1}{2}(\lambda+2)\frac{m}{m+n}\alpha$$
$$-\frac{1}{8}\frac{mn[4(\lambda+3)(m+n-1)+\lambda^2]}{(m+n)^2(m+n-1)}\alpha^2 + O(\alpha^3)\Big\}^{-1} \quad , \qquad (116)$$

the luminosity distance in the case of zero curvature

$$d_L = R_r(1+z)\int_{t_e}^{t_r}\frac{cdt}{R} \qquad (117)$$

writes here at first order

$$d_L \simeq \frac{c}{H_r\left[1+\frac{1}{2}(\lambda+2)\frac{m\alpha}{m+n}\right]}(1+z)\left\{\ln(1+z)-(1+\frac{2}{\lambda})\frac{m\alpha}{m+n}[(1+z)^{-\lambda/2}-1]\right\} \quad . \tag{118}$$

The general expression of d_L is

$$d_L = \frac{c}{H_0}(1+z)r_e(z) \tag{119}$$

where $r_e(z)$ writes for the Einstein - de Sitter model

$$r_e(z) = 2\left(1-\frac{1}{\sqrt{1+z}}\right) \tag{120}$$

which is a special case of

$$r_e(z) = \frac{1}{q_0^2}\frac{q_0 z+(q_0-1)(-1+\sqrt{2q_0 z+1})}{1+z} \quad , \tag{121}$$

valid for any Friedmann model whatever the curvature [11], and, for a zero curvature model with density parameters Ω_M and Ω_Λ ($\Omega_M+\Omega_\Lambda=1$) [8, 18],

$$r_e(z) = \int_0^z dz[(1+z)^2(1+\Omega_M z)-z(z+2)\Omega_\Lambda]^{-1/2} \quad . \tag{122}$$

Lines 1, 4 and 6 of table 1 show for various models the results obtained from the previous expressions of $r_e(z)$ for the parameter values – if any – which fit the recent observations of high-redshift supernovae [8].

Lines 2 and 3 give for $(1+\frac{2}{\lambda})\frac{m\alpha}{m+n} = -10^{-2}$ and $-2\ 10^{-2}$ respectively the values of the expression in braces in (118) for $m+n=9$ and 10 ($\lambda=-8$ and -6): they differ by at most 2.5% from the results of line 1 ($\Omega_M=0.2, \Omega_\Lambda=0.8$, zero curvature) and the second order terms in (114) are less than $7\ 10^{-3}$ in absolute value for $z=1$; the factor of H_r at the denominator of (118) is then 1.04 or 1.06 but H_0 is far from being experimentally known within 5% uncertainty (ref [8] ascribes it a value of 65 ± 7 kms^{-1}Mpc^{-1} and the indications for Ω_M and Ω_Λ reported there are independent of this determination).

Line 5 gives the values of the same expression (i. e. $\ln(1+z)$) for $m=0$ hence for the first version of the scale invariant model [6], which has no free parameter: they are quite close to those of line 4 ($\Omega_M=0.2$, $\Omega_\Lambda=0$, negative curvature).

The values of line 6 are those of the Einstein - de Sitter model ($\Omega_M=1, \Omega_\Lambda=0$, zero curvature).

With α negative as found here, the constant K in (90) is negative also and one may compute

$$\frac{dp}{du} = \frac{dp}{dq}\frac{d}{du}\left(\frac{s}{u^2}\right) = \frac{1}{u}\left(\frac{s'}{u}-\frac{2s}{u^2}\right)\frac{dp}{dq} \tag{123}$$

so, after (100) and since, in the neighbourhood of (q_1, p_1), $dp/dq \simeq \lambda + 2$ (with $\lambda = -8$ and -6 for $D = 10$ and 11 respectively), we have in this neighbourhood

$$\frac{dp}{du} \simeq \lambda(\lambda+2)Kq_1u^{\lambda-1} \tag{124}$$

so that dp/du has the same sign as K: here p is a decreasing function of u (if one goes towards the singular point (q_1, p_1) when t or $u \to \infty$) and the portion of curve which represents the evolution of the universe is situated above the line $p = 2q$ (see fig. 2).

Qualitatively, the variation with time of $S(t)$ is sketched on fig. 3, where

– the curve No 1 represents a function

$$S_1(t) = aR_0\left(\frac{t}{t_0}\right)^{2/3} \tag{125}$$

as given by (96); a is here the starting value of q ($< q_1$ for $K < 0$), corresponding to $t \simeq 0$

– the curve No 2 represents the limit form

$$S_2(t) = q_1R_0\left(\frac{t}{t_0}\right)^{2/3} \tag{126}$$

of $S(t)$ when $t \to \infty$ (equations (91 - 92) and (102 - 103))

– and $S(t)$ (curve No 3) starts as $S_1(t)$ for $t \simeq 0$ and approaches $S_2(t)$ asymptotically when $t \to \infty$, hence an acceleration of expansion when compared to $S_1(t)$: here this acceleration concerns the scale parameter of the extra dimensions; obviously, it originates in the coupling term which appears in (2') and (3') (if this term – the third one in these equations – is dropped, the same law in $t^{2/3}$ (as in [6]) can be taken for both S and R, and d_L then remains the same as in the model with one spatial scale parameter), and it can affect the values of d_L only through the ability for c to vary with time.

The preceding expression of d_L has been derived supposing the evolution of the universe follows one of the tangents described by (88) or (99) – which are particular trajectories of the linear approximation to (54 - 55). In order to check the accuracy of our results, we shall now study the other trajectories and turn to better approximations.

6. Exact solutions of the linear approximation for the dust universe

A. *Dimension* $D = 10$ $(m + n = 9)$

The matrix A_1 (equation (65)) has an only eigenvector $\vec{x}$ given by (68) and can be transformed to the triangular form D_1 introducing a vector $\vec{x}_1$ orthogonal to $\vec{x}$,

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 6 \end{pmatrix} , \tag{127}$$

to build the matrix

$$P = \begin{pmatrix} -6 & 1 \\ 1 & 6 \end{pmatrix} \quad , \tag{128}$$

the inverse of which is

$$P^{-1} = \frac{1}{37}\begin{pmatrix} -6 & 1 \\ 1 & 6 \end{pmatrix} = \frac{1}{37}P \quad , \tag{129}$$

so that

$$D_1 = P^{-1}A_1P = \begin{pmatrix} -8 & 37 \\ 0 & -8 \end{pmatrix} \tag{130}$$

and, for $A = \mu A_1$ with

$$\mu = 2(m+n)(m-1)q_1 = 18(m-1)q_1 \quad \text{here,} \tag{131}$$

the triangular form

$$D = \mu D_1 \quad . \tag{132}$$

From this one gets

$$e^{A\theta} = Pe^{D\theta}P^{-1} \tag{133}$$

where

$$e^{D\theta} = e^{-8\mu\theta}\begin{pmatrix} 1 & 37\mu\theta \\ 0 & 1 \end{pmatrix} \tag{134}$$

hence

$$e^{A\theta} = e^{-8\mu\theta}\begin{pmatrix} 1-6\mu\theta & -36\mu\theta \\ \mu\theta & 1+6\mu\theta \end{pmatrix} \tag{135}$$

and, defining

$$\xi = p - p_1, \quad \eta = q - q_1 \tag{136}$$

and

$$\xi_0 = p_0 - p_1, \quad \eta_0 = q_0 - q_1 \tag{137}$$

where the index 0 points to initial or reference values,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = e^{A\theta}\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} \quad . \tag{138}$$

Thus ξ and η are obtained as functions of the time variable θ and, in order to express them as functions of u or t, one has to find a relation between θ and u: this can be done most easily deriving with respect to u the expression (136) of η with $q = s/u^2$, which yields

$$u\frac{d\eta}{du} = \frac{1}{u}\frac{ds}{du} - 2\frac{s}{u^2} = p - 2q = \xi - 2\eta \tag{139}$$

or

$$\frac{d\eta/d\theta}{\xi - 2\eta}d\theta = \frac{du}{u} \quad ; \tag{140}$$

now, starting from (135) and (138), one gets

$$\frac{d\eta}{d\theta} = \mu(\xi - 2\eta) \quad , \tag{141}$$

so that

$$\mu\frac{d\theta}{du} = \frac{1}{u} \tag{142}$$

and, taking $\theta = 0$ when $u = u_0$,

$$\mu\theta = \ln\frac{u}{u_0} = \frac{1}{3}\ln\frac{t}{t_0} \quad : \tag{143}$$

the mathematical time θ, which appeared in (54-55), identifies, up to a constant factor, to the conformal time ϑ [6] which thus appears to be the natural time of this dynamical system.

This yields, substituting into (135) and (138),

$$\xi = \left(\frac{u_0}{u}\right)^8\left(-6\eta_0 + (\xi_0 + 6\eta_0)(1 - 6\ln\frac{u}{u_0})\right) \tag{144}$$

and

$$\eta = \left(\frac{u_0}{u}\right)^8\left(\eta_0 + (\xi_0 + 6\eta_0)\ln\frac{u}{u_0}\right) \quad . \tag{145}$$

which gives after (136), with $q = s/u^2$,

$$s = u^2\left\{q_1 + \left(\frac{u_0}{u}\right)^8\left[\eta_0 + (\xi + 6\eta_0)\ln\frac{u}{u_0}\right]\right\} \tag{146}$$

and

$$S(t) = R_0\left(\frac{t}{t_0}\right)^{2/3}\left\{q_1 + \left(\frac{t_0}{t}\right)^{8/3}\left[\eta_0 + (\xi_0 + 6\eta_0)\ln\left(\frac{t}{t_0}\right)^{1/3}\right]\right\} \quad , \tag{147}$$

where $u_0 = t_0^{1/3}$, and at last

$$S(t) = q_1 R_0\left\{\left(\frac{t}{t_0}\right)^{2/3} + \left(\frac{t_0}{t}\right)^2\frac{1}{q_1}\left[\eta_0 + \frac{1}{3}(\xi_0 + 6\eta_0)\ln\left(\frac{t}{t_0}\right)\right]\right\} \quad . \tag{148}$$

Writing

$$\kappa = \frac{\eta_0}{q_1} = \frac{q_0}{q_1} - 1, \quad \kappa' = \frac{1}{3\eta_0}(\xi_0 + 6\eta_0) \tag{149}$$

this can be given the form

$$S(t) = q_1 R_0\left\{\left(\frac{t}{t_0}\right)^{2/3} + \kappa\left(\frac{t_0}{t}\right)^2\left[1 + \kappa'\ln\left(\frac{t}{t_0}\right)\right]\right\} \tag{150}$$

to be compared with (91): the complete solution involves a logarithmic additional term depending on a constant κ' proportional to the distance, measured along a parallel to the p-axis, from the point (ξ_0, η_0) to the eigendirection $\xi + 6\eta = 0$.

Then $c(t)$ and the luminosity distance can be determined as in §4, using

$$S'(t) = \frac{q_1 R_0}{t_0}\left\{\frac{2}{3}\left(\frac{t}{t_0}\right)^{-1/3} - \kappa\left(\frac{t_0}{t}\right)^3\left[2-\kappa'\left(1-2\ln\frac{t}{t_0}\right)\right]\right\} \tag{151}$$

hence

$$\frac{S'}{S} = \frac{2}{3}\frac{1}{t}\frac{1-3\kappa\left(\frac{t_0}{t}\right)^{8/3}\left[1-\frac{1}{2}\kappa'\left(1-2\ln\frac{t}{t_0}\right)\right]}{1+\kappa\left(\frac{t_0}{t}\right)^{8/3}\left[1+\kappa'\ln\frac{t}{t_0}\right]} \tag{152}$$

which gives

$$\frac{c}{R} \simeq \frac{2}{3}\frac{1}{t}\sqrt{\frac{m+n-1}{n-1}}\left\{1-\frac{3m}{m+n}\left(1-\frac{1}{2}\kappa'\left(1-2\ln\frac{t}{t_0}\right)\right)\kappa\left(\frac{t_0}{t}\right)^{8/3} + O\left(\kappa^2\left(\frac{t}{t_0}\right)^{16/3}\right)\right\} \quad , \tag{153}$$

and

$$\int\frac{cdt}{R} = \sqrt{\frac{m+n-1}{n-1}}\left\{\frac{2}{3}\ln\frac{t}{t_0}+\frac{3}{4}\frac{m\kappa}{m+n}\left(\frac{t_0}{t}\right)^{8/3}\left[1+\kappa'\left(-\frac{1}{8}+\ln\frac{t}{t_0}\right)\right]+O\left(\kappa^2\left(\frac{t}{t_0}\right)^{16/3}\right)\right\} \tag{154}$$

hence, integrating between t_e and t_r,

$$\int_{t_e}^{t_r}\frac{cdt}{R} = \sqrt{\frac{m+n-1}{n-1}}\left\{\ln\frac{R_r}{R_e}+\frac{3}{4}\frac{m\kappa}{m+n}\left(1-\frac{\kappa'}{8}\right)\left[\left(\frac{R_0}{R_r}\right)^4-\left(\frac{R_0}{R_e}\right)^4\right]\right.$$
$$\left.+\frac{9}{8}\frac{m\kappa\kappa'}{m+n}\left[\left(\frac{R_0}{R_r}\right)^4\ln\frac{R_r}{R_0}-\left(\frac{R_0}{R_e}\right)^4\ln\frac{R_e}{R_0}\right]+O(\kappa^2\ldots)\right\} \tag{155}$$

or

$$\int_{t_e}^{t_r}\frac{cdt}{R} = \sqrt{\frac{m+n-1}{n-1}}\left\{\ln(1+z)+\frac{3}{4}\frac{m\kappa}{m+n}\left(1-\frac{\kappa'}{8}\right)\left(\frac{R_0}{R_r}\right)^4\left(1-(1+z)^4\right)\right.$$
$$\left.+\frac{9}{8}\frac{m\kappa\kappa'}{m+n}\left(\frac{R_0}{R_r}\right)^4\left[(1+z)^4\ln(1+z)+\left(1-(1+z)^4\right)\ln\left(\frac{R_r}{R_0}\right)\right]+O\left(\kappa^2\left(\frac{t}{t_0}\right)^{16/3}\right)\right\} \tag{156}$$

and, with α defined by (113) with $\lambda = -8$ and R_r as in (115), an expression of d_L similar to (118) in which the brackets in the denominator are to be replaced by the braces in (153) and the braces in the numerator by

$$r_e(z) \simeq \ln(1+z)-\frac{3}{4}\frac{m\alpha}{m+n}\left(1-\frac{\kappa'}{8}\right)\left((1+z)^4-1\right)$$
$$+\frac{9}{8}\frac{m\alpha\kappa'}{m+n}\left[(1+z)^4\ln(1+z)-\left((1+z)^4-1\right)\ln\left(\frac{R_r}{R_0}\right)\right] \quad . \tag{157}$$

So, all the new terms which appear in d_L are proportional to a constant κ' which, after (149), is close to zero if at the reference point (p_0, q_0) the trajectory $p = f(q)$ lies close enough to the common tangent Δ at (p_1, q_1); moreover, it may be noted that in the present approximation, according to (149), since the portions of trajectories starting from a point $(a, 2a)$ (94) are below Δ for $\eta < 0$ and above it for $\eta > 0$, κ' is necessarily positive.

B. Dimension $D \geq 11$

Now the matrix A_1 has two negative distinct eigenvalues λ and λ' ($\lambda > \lambda'$) which are given by (67) and their eigenvectors by (98). The exact solution of the linear approximation to (54-55) can be derived as for $D = 10$ from (133) and (138) where this time

$$P = \begin{pmatrix} \lambda+2 & \lambda'+2 \\ 1 & 1 \end{pmatrix} , \tag{158}$$

$$P^{-1} = \frac{1}{\lambda - \lambda'} \begin{pmatrix} 1 & -(\lambda'+2) \\ -1 & \lambda+2 \end{pmatrix} \tag{159}$$

and

$$e^{A\theta} = P \exp\left[\mu\theta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}\right] P^{-1} \tag{160}$$

where μ is given by (131). Thus,

$$e^{A\theta} = \frac{1}{\lambda - \lambda'} \begin{pmatrix} (\lambda+2)e^{\lambda\mu\theta} - (\lambda'+2)e^{\lambda'\mu\theta} & (\lambda+2)(\lambda'+2)(e^{\lambda'\mu\theta} - e^{\lambda\mu\theta}) \\ e^{\lambda\mu\theta} - e^{\lambda'\mu\theta} & (\lambda+2)e^{\lambda'\mu\theta} - (\lambda'+2)e^{\lambda\mu\theta} \end{pmatrix} \tag{161}$$

and

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{e^{\lambda\mu\theta}}{\lambda - \lambda'} \begin{pmatrix} (\lambda+2)[\xi_0 - (\lambda'+2)\eta_0] + (\lambda'+2)[-\xi_0 + (\lambda+2)\eta_0]e^{(\lambda'-\lambda)\mu\theta} \\ \xi_0 - (\lambda'+2)\eta_0 + [-\xi_0 + (\lambda+2)\eta_0]e^{(\lambda'-\lambda)\mu\theta} \end{pmatrix} , \tag{162}$$

which implies as formerly (141) and (143), so that, here also, the mathematical time θ identifies to the conformal time and η writes

$$\eta = \frac{1}{\lambda - \lambda'} \left[X_0 \left(\frac{u}{u_0}\right)^{\lambda} - Y_0 \left(\frac{u}{u_0}\right)^{\lambda'}\right] , \tag{163}$$

where

$$X_0 = \xi_0 - (\lambda'+2)\eta_0 \tag{164}$$

and

$$Y_0 = \xi_0 - (\lambda+2)\eta_0 . \tag{165}$$

This yields

$$s = u^2(q_1 + \eta) = q_1 u^2 + \frac{1}{\lambda - \lambda'} \left[X_0 \frac{u_0^{-\lambda}}{u^{-(\lambda+2)}} - Y_0 \frac{u_0^{-\lambda'}}{u^{-(\lambda'+2)}}\right] , \tag{166}$$

$$s(t) \;=\; q_1 t^{2/3} + \frac{1}{\lambda-\lambda'}\Big[X_0 \frac{t_0^{-\lambda/3}}{t^{-(\lambda+2)/3}} - Y_0 \frac{t_0^{-\lambda'/3}}{t^{-(\lambda'+2)/3}}\Big] \tag{167}$$

and

$$S(t) \;=\; \Sigma_0 s(t) \;=\; q_1 R_0 \Big\{\Big(\frac{t}{t_0}\Big)^{2/3} + \frac{1}{\lambda-\lambda'}\Big[\frac{X_0}{q_1}\Big(\frac{t_0}{t}\Big)^{-(\lambda+2)/3} - \frac{Y_0}{q_1}\Big(\frac{t_0}{t}\Big)^{-(\lambda'+2)/3}\Big]\Big\} \tag{168}$$

or

$$S(t) \;=\; q_1 R_0 \Big\{\Big(\frac{t}{t_0}\Big)^{2/3} + \kappa\Big(\frac{t_0}{t}\Big)^{-(\lambda+2)/3} + \kappa'\Big(\frac{t_0}{t}\Big)^{-(\lambda'+2)/3}\Big\} \tag{169}$$

where

$$\kappa \;=\; \frac{1}{\lambda-\lambda'}\,\frac{\xi_0 - (\lambda'+2)\eta_0}{q_1} \quad \text{and} \quad \kappa' \;=\; -\frac{1}{\lambda-\lambda'}\,\frac{\xi_0 - (\lambda+2)\eta_0}{q_1} \tag{170}$$

to be compared with the previous expressions of $S(t)$: here again, κ' is proportional to the distance measured along a parallel to the p-axis from (ξ_0, η_0) to the eigendirection corresponding to λ and, if $\kappa' = 0$, $\kappa = \eta_0/q_1$; in any case, $\kappa + \kappa' = \eta_0/q_1$. It can be seen that the exact solution (169) of the linear approximation differs from the approximate one (102) through a term which depends on κ' and λ' in the same way as (102) depends on κ and λ. The same calculations as formerly for $c(t)$ and d_L lead to

$$\begin{aligned} r_e(z) \;=\; \ln(1+z) - \frac{m}{m+n}\Big[&(1+\frac{2}{\lambda})\,\kappa\,\Big(\frac{R_0}{R_r}\Big)^{-\lambda/2}[(1+z)^{-\lambda/2}-1] \\ &+ (1+\frac{2}{\lambda'})\,\kappa'\,\Big(\frac{R_0}{R_r}\Big)^{-\lambda'/2}[(1+z)^{-\lambda'/2}-1]\Big] \quad , \end{aligned} \tag{171}$$

an expression which also involves similar terms for each eigenvalue.

As a last remark, it can easily be shown that $D = 10$ or 11 are the only cases which lead to integral values for λ and λ' – otherwise they are irrational numbers, together with the exponents derived from them: writing $N = m + n - 1$, the square root in (67) can be an integer if $N(N-1) = M^2$, with M integer only, hence

$$N \;=\; 4 \pm \sqrt{16 + M^2} \quad ;$$

thus $16 + M^2$ itself has to be a perfect square Q^2, so that

$$(Q - M)(Q + M) \;=\; 16 \quad ,$$

and $(Q - M)$ and $(Q + M)$ are to be chosen among the divisors of 16, which leads to $m + n = 0$, 1, 9 or 10.

C. Dimension $D \leq 9$

The previous results can be extended to this case writing equations (158) to (171) with complex values for λ and λ': defining here

$$\alpha \;=\; -(m+n-1) \quad \text{and} \quad \beta \;=\; \sqrt{(m+n-1)(9-m-n)} \tag{172}$$

so that

$$\lambda = \alpha + i\beta \quad \text{and} \quad \lambda' = \alpha - i\beta \tag{173}$$

yields

$$S(t) = q_1 R_0 \Big\{ \Big(\frac{t}{t_0}\Big)^{2/3} + \Big(\frac{t_0}{t}\Big)^{-(\alpha+2)/3} \frac{1}{q_1} \Big[\eta_0 \cos(\frac{\beta}{3}\ln\frac{t}{t_0}) + (\xi_0 - (\alpha+2)\eta_0)\frac{1}{\beta}\sin(\frac{\beta}{3}\ln\frac{t}{t_0})\Big]\Big\} \tag{174}$$

and

$$\begin{aligned} r_e(z) &= \ln(1+z) + \frac{2m}{m+n}\Big(\frac{R_0}{R_r}\Big)^{-\alpha/2}\Big[a_1\Big\{\cos(\frac{\beta}{2}\ln\frac{R_0}{R_r}) - (1+z)^{-\alpha/2}\cos(\frac{\beta}{2}\ln\frac{R_0}{R_r}(1+z))\Big\} \\ &\quad - a_2\Big\{\sin(\frac{\beta}{2}\ln\frac{R_0}{R_r}) - (1+z)^{-\alpha/2}\sin(\frac{\beta}{2}\ln\frac{R_0}{R_r}(1+z))\Big\}\Big] \end{aligned} \tag{175}$$

where

$$a_1 = (1 + \frac{2\alpha}{\alpha^2+\beta^2})\frac{\eta_0}{2q_1} - \frac{1}{\alpha^2+\beta^2}\frac{\xi_0 - (\alpha+2)\eta_0}{q_1}$$

and

$$a_2 = (1 + \frac{2\alpha}{\alpha^2+\beta^2})\frac{1}{2\beta}\frac{\xi_0 - (\alpha+2)\eta_0}{q_1} + \frac{\beta}{\alpha^2+\beta^2}\frac{\eta_0}{q_1}$$

from which it can be seen that, owing to the presence of the trigonometric functions, dp/dq has no definite limit when $t \to \infty$.

D. Comparison with observation

In order to determine values of the parameters κ and κ' which allow the theoretical expressions (157), (171) and (175) of $r_e(z)$ to fit the observational results at best, these expressions have been included in an optimization program which minimizes the sum of squares $\mathcal{S}$ of the differences between the experimental and the theoretical values of r_e. This has been performed with the help of a simple routine [19] using Nelder and Mead's nonlinear simplex method [20], which has the advantage of requiring no derivations nor matrix inversions. When both κ and κ' are available, the ratio R_0/R_r has been given a succession of values (between 0.6 and 1.3) which allow, solving (149) for η_0 and ξ_0, to get the points (q_0, p_0) associated with the corresponding reference times t_0 and hence to sketch the trajectory representing the evolution of the universe in the (q,p) plane which yields the best agreement with observation (see fig. 4). Table 2 shows the values of $r_e(z)$ and $\mathcal{S}$ obtained for z varying as in table 1 and for different hypotheses and values of D. As the sequence of values of r_e is very nearly independent of R_0/R_r, one sequence only is given for each case. For comparison, the $\mathcal{S}$ values corresponding to lines 2 and 3 of table 1 are $6.9\,10^{-4}$ and $1.7\,10^{-4}$, so they are almost optimal: on table 2, with $\kappa' = 0$, one gets $6.3\,10^{-4}$ and $1.6\,10^{-4}$ respectively. Of course $\mathcal{S}$ decreases when κ' is set free. It may be noted that the result obtained for $D = 7$ (line 7 of table 2) is of the same order as the other ones for the complete linear approximation: the observed values of d_L give no indication about the number of extra dimensions.

7. Higher order approximations

A differential equation such as (53) may always be cast [21] into the form of Briot and Bouquet, the general solution of which however requires the introduction of Ψ series (Malmquist [22]). Here, we shall use Poincaré's theory of normal forms [23] to obtain the first terms of the series which represent the solutions in the neighbourhood of an equilibrium point.

With from now on

$$x = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{176}$$

and ξ and η defined as in (136), a nonlinear differential system can be written

$$x' = Ax + \sum_{k=r}^{\infty} v_k(x) \tag{177}$$

where the $v_k(x)$ are vector-valued polynomials of degree $r \geq 2$ and where, here, due to the polynomial form of the second member of (54 - 55), the number of terms of the sum is finite.

Starting from a system with one only term v_r,

$$x' = Ax + v_r(x) \tag{178}$$

one performs the change of variable

$$x = y + h_r(y) \tag{179}$$

where h_r is also a vector-valued polynomial of degree r, which gives

$$y' + \frac{\partial h_r}{\partial y} y' = Ay + Ah_r(y) + v_r\big(y + h_r(y)\big) \quad , \tag{180}$$

or

$$y' = Ay + Ah_r(y) + v_r\big(y + h_r(y)\big) - \frac{\partial h_r}{\partial y}\Big[Ay + Ah_r(y) + v_r\big(y + h_r(y)\big) - \frac{\partial h_r}{\partial y} Ay\Big] + \cdots \tag{181}$$

which can be given the form

$$y' = Ay - \big(L_A h_r - v_r(y)\big) + u_{2r-1}(y) \tag{182}$$

where the linear operator

$$L_A h_r = \frac{\partial h_r}{\partial y} Ay - Ah_r(y) \tag{183}$$

is the homological operator, and where

$$u_{2r-1}(y) = v_r\big(y + h_r(y)\big) - v_r(y) - \frac{\partial h_r}{\partial y}\Big[Ah_r(y) + v_r\big((y + h_r(y)\big) - \frac{\partial h_r}{\partial y} Ay\Big] + \cdots \tag{184}$$

is indeed of order $2r-1$. Thus the resolution of the homological equation

$$L_A h_r = v_r \tag{185}$$

allows one to annihilate the terms of degree r in the initial nonlinear equation, which becomes

$$y' = Ay + \sum_{k=r+1}^{\infty} w_k(y) \tag{186}$$

where the w_k include terms coming from the above u and v. Successively eliminating the terms of degree 2, 3, ..., the original equation (177) is transformed into

$$y' = Ay \tag{187}$$

and one gets for x from (179) a series the first term of which is y, the solution of (187). After the theorems of Poincaré and Poincaré - Dulac [23], this series is convergent if the convex hull of the points which represent the eigenvalues of A in the complex plane does not contain zero – which is obviously true here, the eigenvalues being negative.

All of this is possible only if the homological equation (185) can be solved, that is if there are no resonance relations between the eigenvalues of A; here this occurs for $D = 11$, with $\lambda = -6$ and $\lambda' = -12$: then one has to start not from (187), but from a simple nonlinear differential system and the series thus obtained is still convergent.

There being nothing special if A cannot be diagonalized (here, for $D = 10$), the homological equation can be written for any m and n. For the quadratic terms one gets as coefficients of the vector valued polynomial h_2,

$$h_2(\xi,\,\eta) = \begin{pmatrix} h_2^{11}\xi^2 + h_2^{12}\eta^2 + h_2^{13}\xi\eta \\ h_2^{21}\xi^2 + h_2^{22}\eta^2 + h_2^{2}3\xi\eta \end{pmatrix} \tag{188}$$

rational fractions the denominators of which always include a factor $m+n-10$, thus indeed preventing the resolution of the homological equation in the case $D = 11$. So, in the general case ($D \neq 11$),

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}_2 = \begin{pmatrix} \xi \\ \eta \end{pmatrix} + h_2(\xi,\,\eta) \tag{189}$$

where ξ and η are given by the linear approximation. For $D = 10$, one gets:

$$h_2^{11} = \frac{1}{32q_1}(-6n+49), \quad h_2^{12} = \frac{3}{8q_1}(-34n+171)$$

$$h_2^{13} = \frac{1}{72q_1}(-106n+747), \quad h_2^{21} = \frac{1}{576q_1}(10n-63)$$

$$h_2^{22} = \frac{7}{144q_1}(38n-153), \quad h_2^{23} = \frac{1}{144q_1}(38n-189) \tag{190}$$

For $D = 11$, however, one has to use the eigenbasis to compute the available coefficients of h_2:

$$
\begin{aligned}
h_2^{11} &= \frac{1}{5q_1}(-n+15), \quad h_2^{12} = \frac{1}{3q_1}(-4n+35) \\
h_2^{13} &= \frac{3}{10q_1}(-4n+35), \\
h_2^{22} &= \frac{11}{10q_1}(2n-15), \quad h_2^{23} = \frac{3}{q_1}(n-6)
\end{aligned}
\tag{191}
$$

and, since h_2^{21} is lacking, to add a nonlinear term to (187).

In the case $D = 10$, the second order approximations (189) to ξ and η are expressed, through the results of the linear ones (138), as functions of the time variable θ which, as formerly, can be related to u and thus to t using (140). Starting from (189) and (190), one gets on the left hand side of (140):

$$
\frac{d\eta/d\theta}{\xi - 2\eta} = \mu \frac{36[8\mu(\xi_0 + 6\eta_0)\theta - \xi_0 + 2\eta_0] + (1/q_1)\exp(-8\mu\theta)P(\theta,\xi_0,\eta_0,\mu,n)}{36[8\mu(\xi_0 + 6\eta_0)\theta - \xi_0 + 2\eta_0] + (1/q_1)\exp(-8\mu\theta)Q(\theta,\xi_0,\eta_0,\mu,n)} \tag{192}
$$

where P and Q are polynomials of the second degree in θ, ξ_0, η_0 and μ and of the first one in n.

If $\xi_0 + 6\eta_0 \neq 0$, hence (149) if $\kappa' \neq 0$, the change of variable

$$
\theta_1 = 8\mu\theta + \frac{-\xi_0 + 2\eta_0}{\xi_0 + 6\eta_0} \tag{193}
$$

transforms this expression into

$$
\frac{d\eta/d\theta}{\xi - 2\eta} = \mu \frac{36(\xi_0 + 6\eta_0)\theta_1 + (1/2q_1)\exp((-\xi_0 + 2\eta_0)/(\xi_0 + 6\eta_0) - \theta_1)(\xi_0 + 6\eta_0)^2 P_1}{36(\xi_0 + 6\eta_0)\theta_1 + (1/q_1)\exp((-\xi_0 + 2\eta_0)/(\xi_0 + 6\eta_0) - \theta_1)(\xi_0 + 6\eta_0)^2 Q_1}, \tag{194}
$$

the polynomials P and Q factorizing into $(\xi_0 + 6\eta_0)^2 P_1/2$ and $(\xi_0 + 6\eta_0)^2 Q_1$ respectively, or, using the explicit expressions of P_1 and Q_1:

$$
\frac{d\eta/d\theta}{\xi - 2\eta} = \mu \frac{1 + (\kappa\kappa'/24)e^{\frac{8}{3\kappa'}-1}(e^{-\theta_1}/\theta_1)[(16n-63)\theta_1^2 + (24n-63)\theta_1 + 24n - 126]}{1 + (\kappa\kappa'/12)e^{\frac{8}{3\kappa'}-1}(e^{-\theta_1}/\theta_1)[(7n-36)\theta_1^2 + (11n-36)\theta_1 + 12n - 63]}. \tag{195}
$$

The linear approximation of η and ξ amounts to replace by 1 the fraction on the right-hand side of (195), or the exponentials $\exp(-\theta_1)$ by zero. θ or θ_1 being large and κ small, a second order approximation will be obtained developing this fraction to first order in $\exp(-\theta_1)$, hence

$$
\frac{d\eta/d\theta}{\xi - 2\eta} \simeq \mu[1 + \frac{\kappa\kappa'}{12}e^{\frac{8}{3\kappa'}-1}e^{-\theta_1}(n + \frac{9}{2})(\theta_1 + 1)] \quad . \tag{196}
$$

(140) thus integrates between 0 and θ into

$$\mu \int_{\frac{8}{3\kappa'}-1}^{\frac{8}{3\kappa'}-1+8\mu\theta} [1+\frac{\kappa\kappa'}{12} e^{\frac{8}{3\kappa'}-1} e^{-\theta_1}(n+\frac{9}{2})(\theta_1+1)] \frac{d\theta_1}{8\mu} \simeq \int_{u_0}^{u} \frac{du}{u} \tag{197}$$

yielding for θ the equation

$$8\mu\theta - \frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa'+8\mu\theta\kappa')\exp(-8\mu\theta) \simeq 8\ln\frac{u}{u_0} - \frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa') \tag{198}$$

which, with

$$v = (\frac{u_0}{u})^8, \quad x = 8\mu\theta, \quad A = \frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa') \quad \text{and} \quad C = \frac{\kappa\kappa'}{12}(n+\frac{9}{2}) \quad , \tag{199}$$

can be rewritten

$$x - (A+Cx)\exp(-x) \simeq -\ln v - A \quad . \tag{200}$$

Starting from

$$x = -\ln v - A \tag{201}$$

and iterating along

$$x_{n+1} = -\ln v - A + (A+Cx_n)\exp(-x_n) \quad , \tag{202}$$

an asymtotic expansion of the solution of (200) can be obtained, yielding at first order [14]

$$x \simeq -\ln v - A + [A - C(A+\ln v)]\exp(A)v + O(\kappa^2\exp(2A)v^2) \tag{203}$$

and

$$\exp(-x) \simeq \exp(A)v\{1 - [A - C(A+\ln v)]\exp(A)v + O(\kappa^2\exp(2A)v^2)\} \quad . \tag{204}$$

This is valid for x large enough, hence for $u_0/u << 1$, but more accurate conditions can be obtained observing that (200) is equivalent to

$$z = \Phi(z) \quad \text{where} \quad z = \exp(-x) \quad \text{and} \quad \Phi(z) = \exp(A)v\exp[(C\ln z - A)z] \tag{205}$$

from which (204) and (203) can be found again iterating once

$$z_{n+1} = \Phi(z_n) \quad \text{with} \quad z_0 = \exp(A)v \tag{206}$$

and expanding the z-depending exponential at first order, so that the conditions to be verified at the end of calculations are

$$|(C\ln z - A)z| << 1 \quad \text{and} \quad |\Phi'(z)| < 1 \quad , \tag{207}$$

the last one being the condition $\Phi(z)$ has to fulfil to be a contractive mapping.

Thus, in terms of u, θ, κ, ...,

$$\begin{aligned} 8\mu\theta \ &\simeq\ -8\ln\frac{u_0}{u} - \frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa') + \{\frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa') \\ &- \frac{\kappa\kappa'}{12}(n+\frac{9}{2})[\frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa') + 8\ln\frac{u_0}{u}]\}e^{\frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa')}(\frac{u_0}{u})^8 \\ &+ O(\kappa^2\exp(2A)(\frac{u_0}{u})^{16}) \end{aligned} \tag{208}$$

instead of (143) and

$$\begin{aligned} \exp(-8\mu\theta) \ &\simeq\ e^{\frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa')}(\frac{u_0}{u})^8 \\ &-\{\frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa') - \frac{\kappa\kappa'}{12}(n+\frac{9}{2})[\frac{\kappa}{12}(n+\frac{9}{2})(\frac{8}{3}+\kappa')+8\ln\frac{u_0}{u}]\}e^{\frac{\kappa}{6}(n+\frac{9}{2})(\frac{8}{3}+\kappa')}(\frac{u_0}{u})^{16} \\ &+ O(\kappa^2\exp(3A)(\frac{u_0}{u})^{24}) \end{aligned} \tag{209}$$

which implies that since, at first order (see (143 - 145) and (149))

$$\eta \ =\ \kappa q_1\exp(-8\mu\theta)(1+3\kappa'\mu\theta) \tag{210}$$

and

$$\xi = \ 3\kappa q_1\exp(-8\mu\theta)(\kappa'-2-6\kappa'\mu\theta) \quad , \tag{211}$$

the above expressions for $8\mu\theta$ and $\exp(-8\mu\theta)$ allow one to compute η and ξ up to the second order in $\kappa(u_0/u)^8$ and hence the second order approximations of η and ξ given by (189) and (190).

If $\kappa' = 0$, (192) reduces to

$$\frac{d\eta/d\theta}{\xi-2\eta} \ =\ \mu\frac{\exp(-8\mu\theta)(16n-63)\eta_0/q_1+9}{\exp(-8\mu\theta)(14n-72)\eta_0/q_1+9} \tag{212}$$

or, owing to (149),

$$\frac{d\eta/d\theta}{\xi-2\eta} \ =\ \mu\frac{1+(16n-63)\kappa\exp(-8\mu\theta)/9}{1+(14n-72)\kappa\exp(-8\mu\theta)/9} \quad . \tag{213}$$

Bringing this expression into (140) and integrating, one gets

$$\int_0^\theta \frac{d\eta}{\xi-2\eta} \ =\ \mu\theta + \frac{2n+9}{8(14n-72)}\ln\frac{1+(14n-72)\kappa/9}{1+(14n-72)(\kappa/9)\exp(-8\mu\theta)} \ =\ \ln\frac{u}{u_0} \tag{214}$$

as a relation between the mathematical time θ and the time variable u. This yields at first order in κ

$$8\mu\theta + (1+2n/9)\kappa - (1+2n/9)\kappa\exp(-8\mu\theta) \ =\ 8\ln\frac{u}{u_0} \tag{215}$$

which is nothing but (198) with $\kappa' = 0$, so that from this point on, the case $\kappa' = 0$ can be dealt with in the same way as the general case.

From (209) can be deduced an asymptotic expansion of $S(t)$ and, using (211), an expression of S'/S which can be obtained without deriving S:

$$\frac{ds}{dt} = \frac{ds}{du}\frac{du}{dt} = up\frac{du}{dt} = \frac{1}{3}t^{-1/3}(2q_1 + \xi) \tag{216}$$

hence

$$\frac{S'}{S} = \frac{1}{s}\frac{ds}{dt} = \frac{1}{3t}\frac{2 + \xi/q_1}{1 + \eta/q_1} \quad . \tag{217}$$

Thus c/R can be expressed as

$$\frac{c}{R} \simeq \frac{2}{3}\frac{1}{t}\sqrt{\frac{m+n-1}{n-1}}[1 + (c_{01} + c_{11}\ln v)v + (c_{02} + c_{12}\ln v + c_{22}\ln^2 v)v^2] \tag{218}$$

where the coefficients c_{ij} depend on κ, κ' and n and where v is given by (199). Integrating the last expression between t_e and t_r leads to

$$r_e(z) \simeq \ln(1+z) + \frac{2}{3}(\alpha\tau_1 + \alpha^2\tau_2) \tag{219}$$

where

$$\alpha = \exp[(2n+9)(3\kappa'+8)\kappa/72] \quad , \tag{220}$$

$$\begin{aligned} \tau_1 = {} & \{[(2n+9)(3\kappa'+8)\kappa\kappa' + 96(\kappa'-2) + 72\kappa'(4\ln\frac{R_0}{R_r} - 1)][(1+z)^4 - 1] \\ & + 288\kappa'(1+z)^4\ln(1+z)\}\frac{9-n}{3\,.\,2^9}\kappa(\frac{R_0}{R_r})^4 \quad , \end{aligned} \tag{221}$$

and

$$\begin{aligned} \tau_2 = {} & -\Big\{576\{\{16[12(n-7)-(n-18)\kappa'] - (n-7)[(2n+9)(3\kappa'+8)\kappa\kappa' + 36\kappa'(4\ln\frac{R_0}{R_r} - 1)]\}. \\ & .\{\ln\frac{R_0}{R_r} - (1+z)^8\ln[(1+z)\frac{R_0}{R_r}]\} + 144(n-7)\kappa'(1+z)^8\ln(1+z)\ln[(1+z)\frac{R_0}{R_r}]\}\kappa' \\ & - \{\{32[12(n-7) - (n-18)\kappa'] - (2n+9)(n-7)(3\kappa'+8)\kappa\kappa'\}(2n+9)(3\kappa'+8)\kappa\kappa' \\ & -24[4(107\kappa'^2 n - 477\kappa'^2 + 80\kappa' n + 144\kappa' + 384n - 2688) - 3(2n+9)(n-7)(3\kappa'+8)\kappa\kappa'^2]\}. \\ & .\{(1+z)^8 - 1\}\Big\}\frac{9-n}{3\,.\,2^{17}}\kappa^2(\frac{R_0}{R_r})^8 \quad , \end{aligned} \tag{222}$$

as given by Reduce after some reordering and where the τ_1 term can easily be seen to give the same result as (156) at first order in κ.

Comparison with observation

Line 4 of table 2 shows the results given by expressions (219 - 222), with the last two ones directly converted by Reduce into Fortran instructions. As for the preceding approximations, R_0/R_r has been given values between 0.6 and 1.3 (in the last case, u_0/u is not small, but the conditions (207) are still satisfied). The value of $\mathcal{S}$ obtained is still better than on lines 2 and 3, thus confirming the validity of the present computation. This can be seen also on fig. 4, where the trajectories representing the first and the second order approximations get closer and closer when approaching the singular point (q_1, p_1). The second order trajectory even shows a maximum in the same way as the theoretical integral curves. A few theoretical trajectories have been computed starting from points of the fitted second order one and using the Runge - Kutta integration method of Maple, and a number of them cross the points of the fitted trajectory which are closer to the singular point (q_1, p_1), showing the quality as well as the limits of this second order approximation.

In view of these results, and since in this case the calculations have to be reformulated from the beginning, we did not go further than the first order for $D = 11$.

8. Conclusion

Starting from most general assumptions about the metric, we have obtained a model of the universe which does not require inflation nor the cosmological constant and in which the present acceleration of expansion is attributed to the extra dimensions. Moreover, the qualitative behaviour of the solutions obtained for the scale factor of these extra dimensions when $t \to \infty$ suggests the possibility of a meeting point with superstring or M theories as concerns the number of dimensions of spacetime. Of course the big-bang model is strongly relativized: in conformal (i. e. physical) time ϑ, the origin of the universe is rejected to $-\infty$ and no expansion is actually observable [6].

Lie groups play an important part in the theory, especially through the generator of the inhomogeneous scale transformation associated with Kepler's third law: it belongs to the Lie algebra of equation (17) and appears in the resolution of equation (28) and also of the equation which can be deduced from (A - 5') for s.

Computer algebra has been used more than once in the derivation of these results: besides SPDE [24], the packages CHANGEVR [25] and TAYLOR [26] of Reduce [27] were especially useful; several factorizations, derivations of functions, ... had to be worked out, as well as, with Maple [28], asymptotic developments. The graphic representations of the dynamical systems trajectories have been performed using Maple.

Appendix – The singular points at the origin and at infinity

A1. The origin

Taking $|p|$ and $|q| << 1$ in (53) gives

$$\frac{dp}{dq} \simeq \frac{(mp+2nq)(mp+2(n-1)q)}{mq(p-2q)} \quad , \qquad (A-1)$$

an homogeneous equation which writes, with $z = p/q$,

$$q\frac{dz}{dq} \simeq \frac{m(m-1)z^2+4mnz+4n(n-1)}{m(z-2)} \quad . \qquad (A-2)$$

The last equation has as singular integrals the roots z_1 and z_2 of the numerator of the fraction on its right-hand side:

$$z_1 = -\frac{2}{m(m-1)}(mn - \sqrt{mn(m+n-1)}) \qquad (A-3)$$

$$z_2 = -\frac{2}{m(m-1)}(mn + \sqrt{mn(m+n-1)}) \qquad (A-3')$$

and its general solution can be obtained through integration:

$$\int \frac{dq}{q} \simeq m\int \frac{z-2}{m(m-1)z^2+4mnz+4n(n-1)}dz \qquad (A-4)$$

$$\simeq \frac{1}{m-1}\int \frac{z-2}{(z-z_1)(z-z_2)}dz \quad ,$$

hence

$$q \simeq \mathcal{C}\frac{|z-z_1|^{\alpha_1}}{|z-z_2|^{\alpha_2}} \quad \text{with} \quad \alpha_i = \frac{z_i-2}{(m-1)(z_1-z_2)}, \quad i = 1, 2 \quad \text{and } \mathcal{C} \text{ constant} \qquad (A-5)$$

yielding, together with

$$p \simeq \mathcal{C}z\frac{|z-z_1|^{\alpha_1}}{|z-z_2|^{\alpha_2}} \qquad (A-5')$$

a parametric representation of the integral curves. Fig 5 shows a number of these curves, which can also be interpolated from fig 1. As results from (A - 5) with $z = z_2 + \epsilon$, the integral curves are tangent to the singular solution $z = z_2$ in the neighbourhood of the origin, so that

$$p \simeq z_2 q \quad \text{or} \quad \frac{ds}{du} \simeq z_2\frac{s}{u}$$

and, noting s' the derivative of s with respect to t,

$$s \simeq \sigma t^{z_2/3}(\sigma \text{ constant}) \quad \text{and} \quad \frac{s'}{s} \simeq \frac{z_2}{3}\frac{1}{t}$$

This gives for the numerator of c^2 in (22)

$$m(m-1)\frac{S'^2}{S^2}+n(n-1)\frac{R'^2}{R^2}+2mn\frac{R'}{R}\frac{S'}{S} \simeq (m(m-1)z_2^2+4mnz_2+4n(n-1))\frac{1}{9t^2}\,, \quad (A-6)$$

which thus tends to zero in the vicinity of the origin for all the trajectories (and gets a zero value anywhere on the singular straight lines $z = z_1$ and $z = z_2$), so that these solutions have to be excluded.

A2. The point at infinity

For $|p|$ and $|q| >> 1$, (53) becomes

$$\frac{dp}{dq} \simeq -\frac{p(mp+2nq)((m-1)p+2nq)}{2nq^2(p-2q)} \quad (A-7)$$

which, as previously, can be transformed into

$$q\frac{dz}{dq} \simeq -\frac{z}{2n(z-2)}(m(m-1)z^2+4mnz+4n(n-1)) \quad (A-8)$$

where the second degree polynomial is the same as in (A - 2), so that we have here three singular solutions: $z = 0$ and, as formerly, $z = z_1$ and $z = z_2$.

Integrating (A - 8) gives for the general solution

$$-\int\frac{dq}{q} \simeq 2n\int\frac{z-2}{z(m(m-1)z^2+4mnz+4n(n-1))}dz \quad (A-9)$$
$$\simeq \frac{2n}{m(m-1)}\int\frac{z-2}{z(z-z_1)(z-z_2)}dz$$

hence, with C still standing for a constant,

$$q \simeq C|z|^{\frac{1}{n-1}}\frac{|z-z_2|^{\beta_2}}{|z-z_1|^{\beta_1}} \quad \text{where} \quad \beta_i = \frac{2n}{m(m-1)}\frac{z_i-2}{z_i(z_1-z_2)}, \quad i = 1, 2 \quad (A-10)$$

so that, the β_i being positive, $q \to \infty$ when $z \to z_1$: the integral curves are asymptotic to the singular solution $z = z_1$ and, as formerly (A - 6), $c^2 \simeq 0$ in this case also.

From (A - 10) we get for $|z| \to \infty$

$$q \to C|z|^{\frac{1}{n-1}}|z|^{\beta_2-\beta_1}$$

where, as can easily be verified,

$$\beta_2 - \beta_1 = -\frac{1}{n-1}$$

so that q tends to remain constant, as can be seen on the parts of the trajectories parallel to the p-axis on figs. 1 and 6. Thus here

$$s \simeq Cu^2 \simeq Ct^{2/3}$$

and S follows the same law as R, which brings nothing interesting.

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z	0.16	0.33	0.50	0.66	0.83	1.0
(1)	0.156	0.312	0.457	0.584	0.709	0.823
(2)	0.156	0.306	0.446	0.573	0.706	0.843
(3)	0.159	0.312	0.453	0.578	0.707	0.833
(4)	0.148	0.283	0.405	0.508	0.610	0.705
(5)	0.148	0.285	0.405	0.507	0.604	0.693
(6)	0.143	0.266	0.367	0.448	0.522	0.586

Table 1 – Comparison of the r_e values given by various models

(1) $\Omega_M = 0.2,\ \Omega_\Lambda = 0.8,\ k = 0$
(2) present theory with $D = 10$
(3) present theory with $D = 11$
(4) $\Omega_M = 0.2,\ \Omega_\Lambda = 0,\ k = -1$
(5) scale invariant cosmology [1]
(6) Einstein - de Sitter model ($\Omega_M = 1,\ \Omega_\Lambda = 0,\ k = 0$)

z	0.16	0.33	0.50	0.66	0.83	1.0	$\mathcal{S}$
(1)	0.156	0.312	0.457	0.584	0.709	0.823	
(2)	0.156	0.306	0.445	0.570	0.703	0.838	$6.3\,10^{-4}$
(3)	0.161	0.315	0.457	0.582	0.707	0.824	$4.9\,10^{-5}$
(4)	0.160	0.315	0.457	0.583	0.709	0.823	$2.8\,10^{-5}$
(5)	0.159	0.312	0.452	0.577	0.705	0.831	$1.6\,10^{-4}$
(6)	0.161	0.315	0.457	0.582	0.707	0.824	$5.3\,10^{-5}$
(7)	0.160	0.314	0.457	0.582	0.707	0.825	$3.6\,10^{-5}$

Table 2 – Comparison of the r_e values given by different forms and orders of approximation of the present theory

(1) $\Omega_M = 0.2,\ \Omega_\Lambda = 0.8,\ k = 0$
(2) $D = 10$ – linear approximation with $\kappa' = 0$
(3) $D = 10$ – complete linear approximation
(4) $D = 10$ – second order approximation
(5) $D = 11$ – linear approximation with $\kappa' = 0$
(6) $D = 11$ – complete linear approximation
(7) $D = 7$ $(n = 4,\ m = 2)$ – complete linear approximation

Figure captions

Figure 1: Trajectories of the field defined by equations (54 - 55) in the half-plane $q > 0$

Figure 2: The two kinds of trajectories of the field defined by equations (54 - 55) and ending at (q_1, p_1)

Figure 3: Variation of $S(t)$ between the two limit forms $S_1(t)$ and $S_2(t)$

Figure 4: Best fit trajectories in the vicinity of the singular point (q_1, p_1). The points of the first order approximation are represented by small circles, the points of the second order one by crosses; the two straight lines are the singular line $p = 2q$ and the tangent common to all the trajectories; the dotted line is the locus of the maxima of the curved trajectories; the arrow indicates the point for which $R_0/R_r = 1$ (present time).

Figure 5: The trajectories in the vicinity of the singular point at the origin

Figure 6: The trajectories in the vicinity of the singular point at infinity

Figures 1, 2 and 4, 5, 6 correspond to the case $D = 10$, $n = 4$, $m = 5$ ($n = 4$ has been chosen to allow for the embedding of the (ordinary) Euclidean space in the n-space)

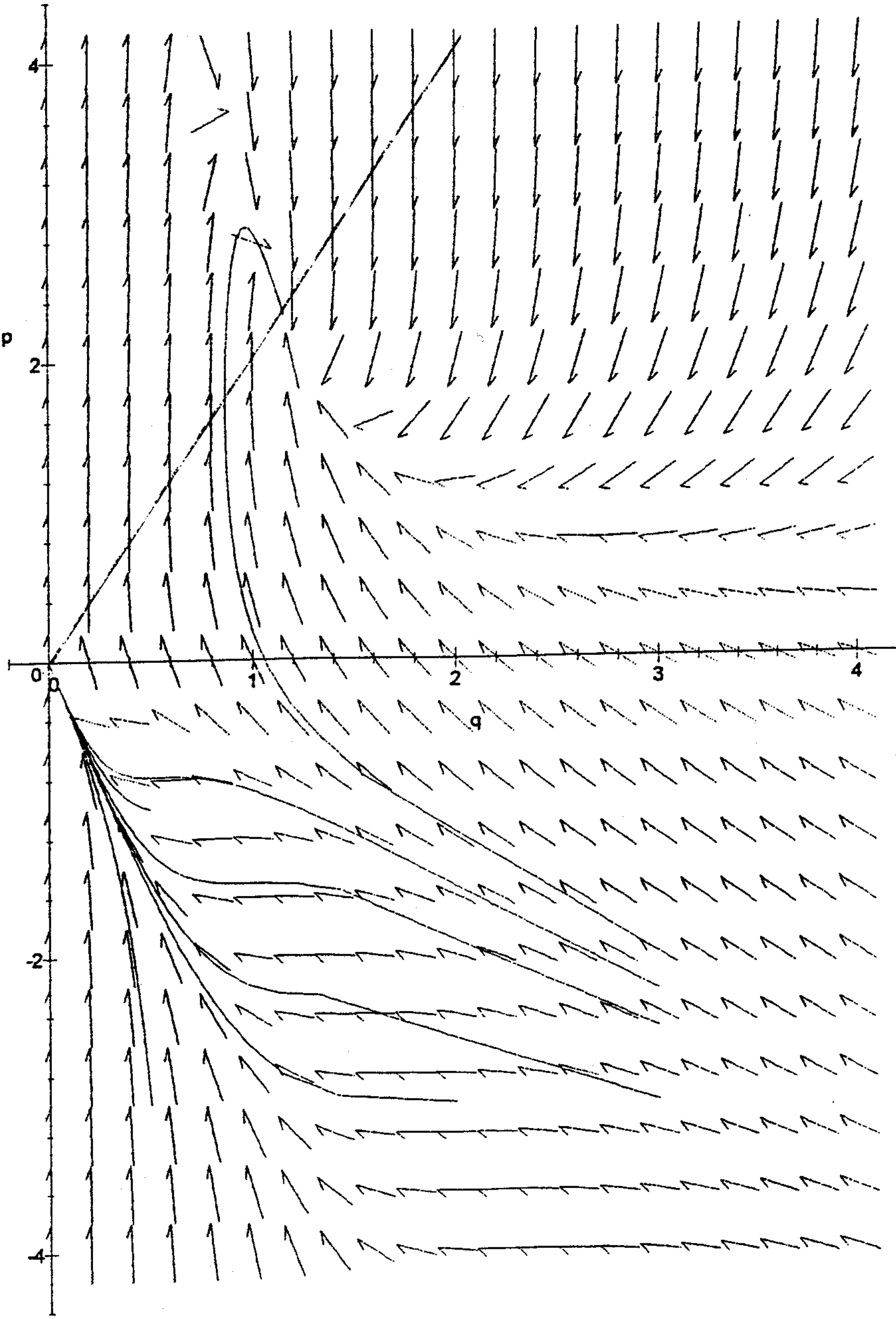

Fig. 1

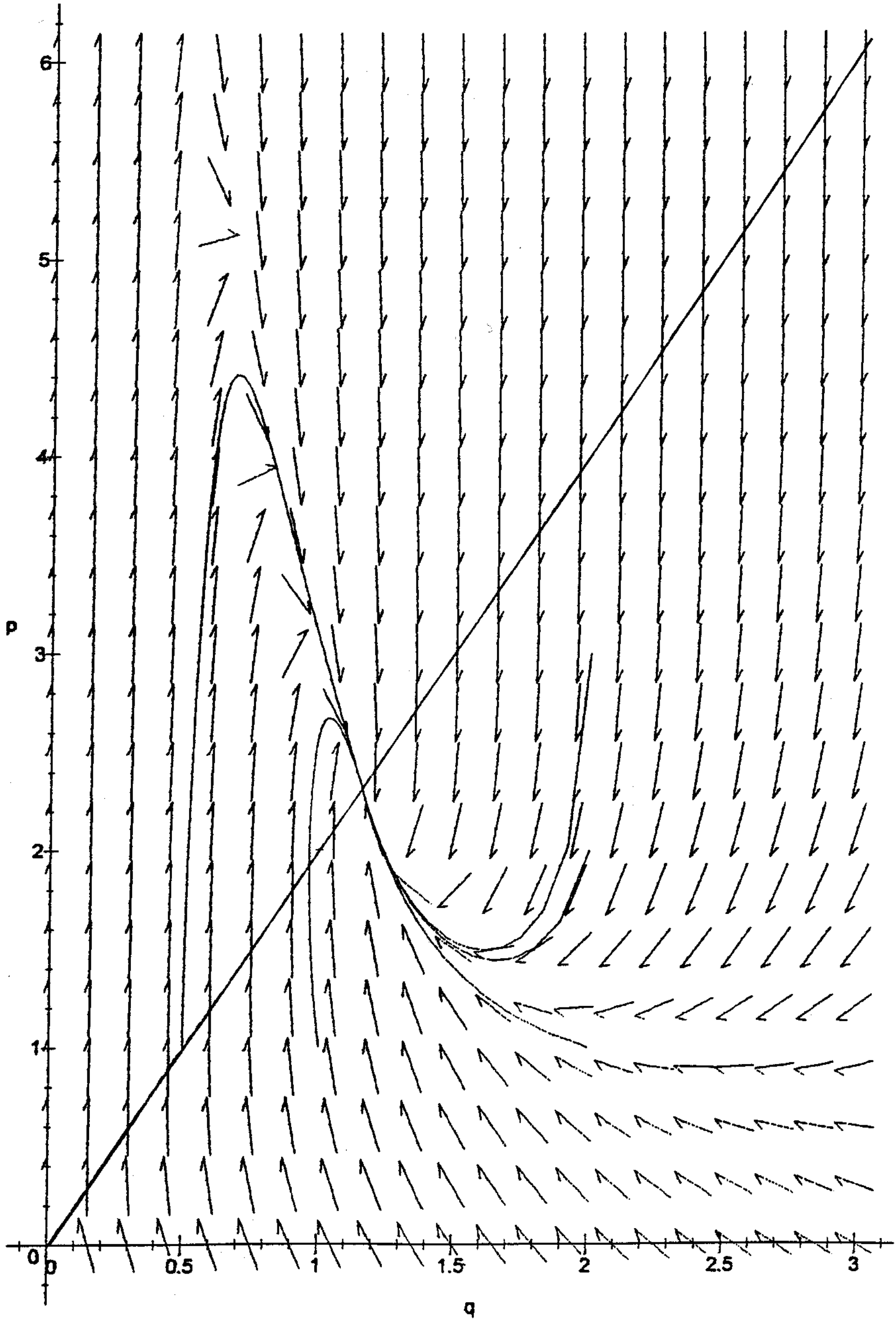

Fig. 2.

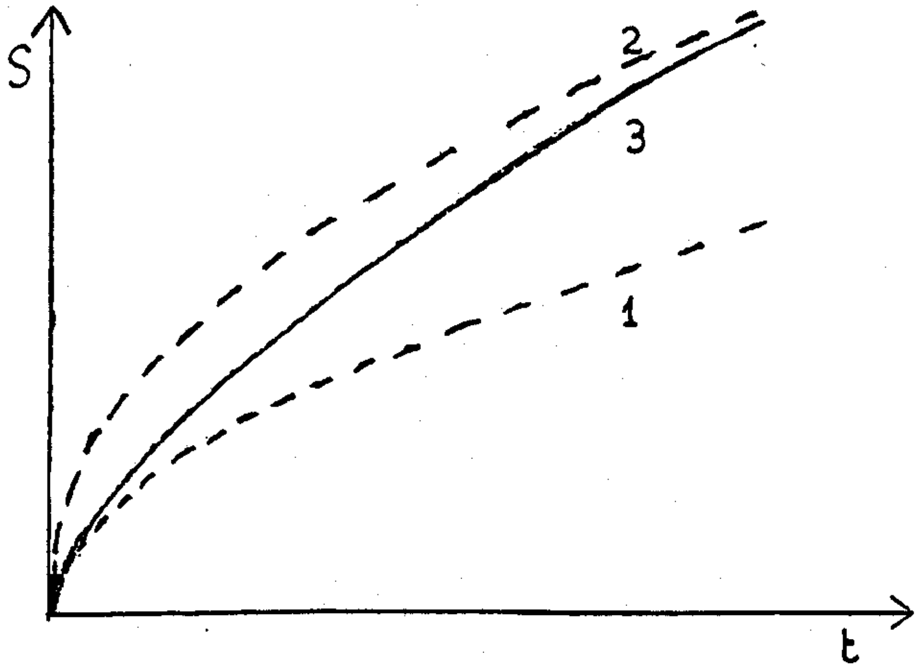

Fig. 3

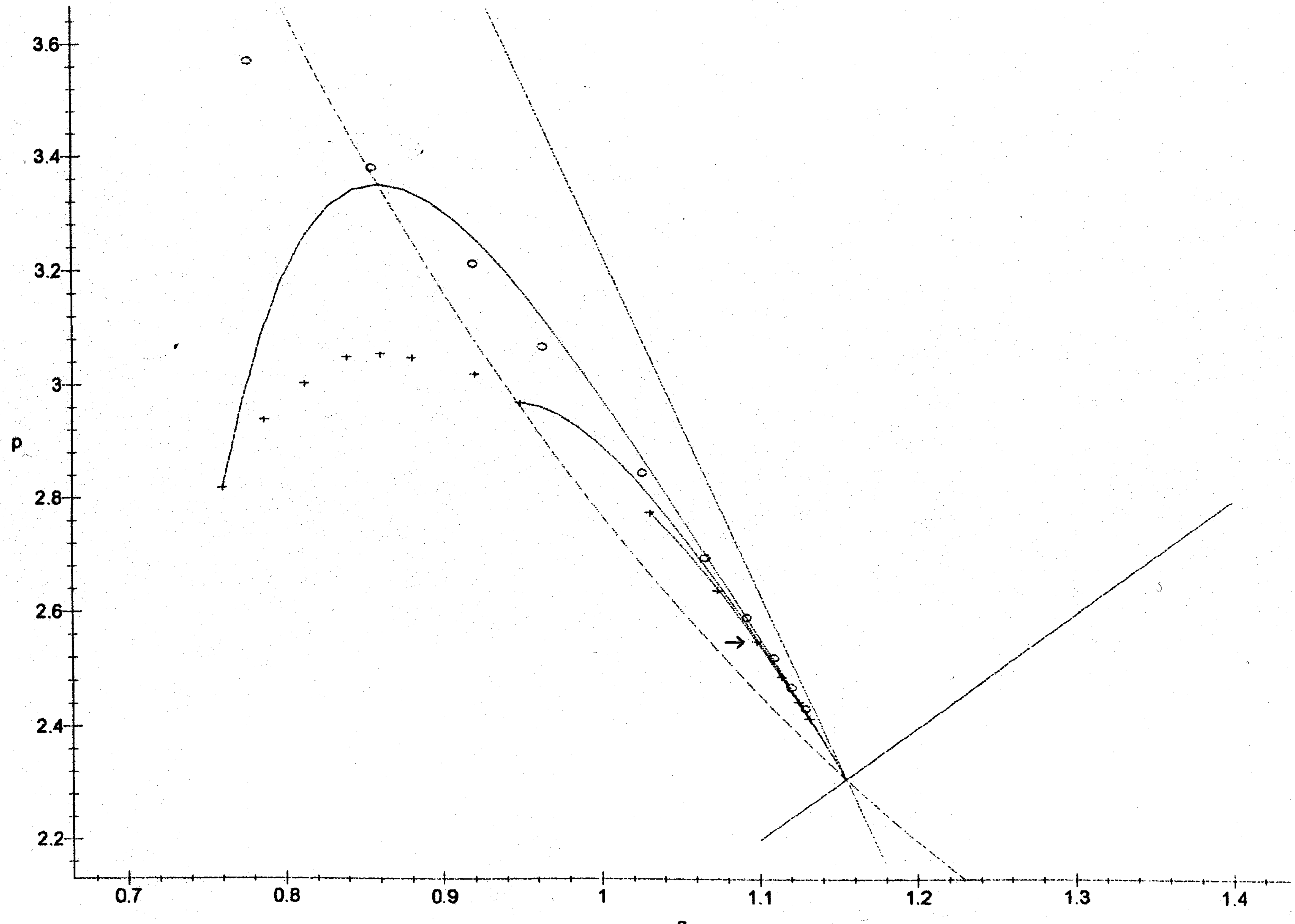

Fig. 4

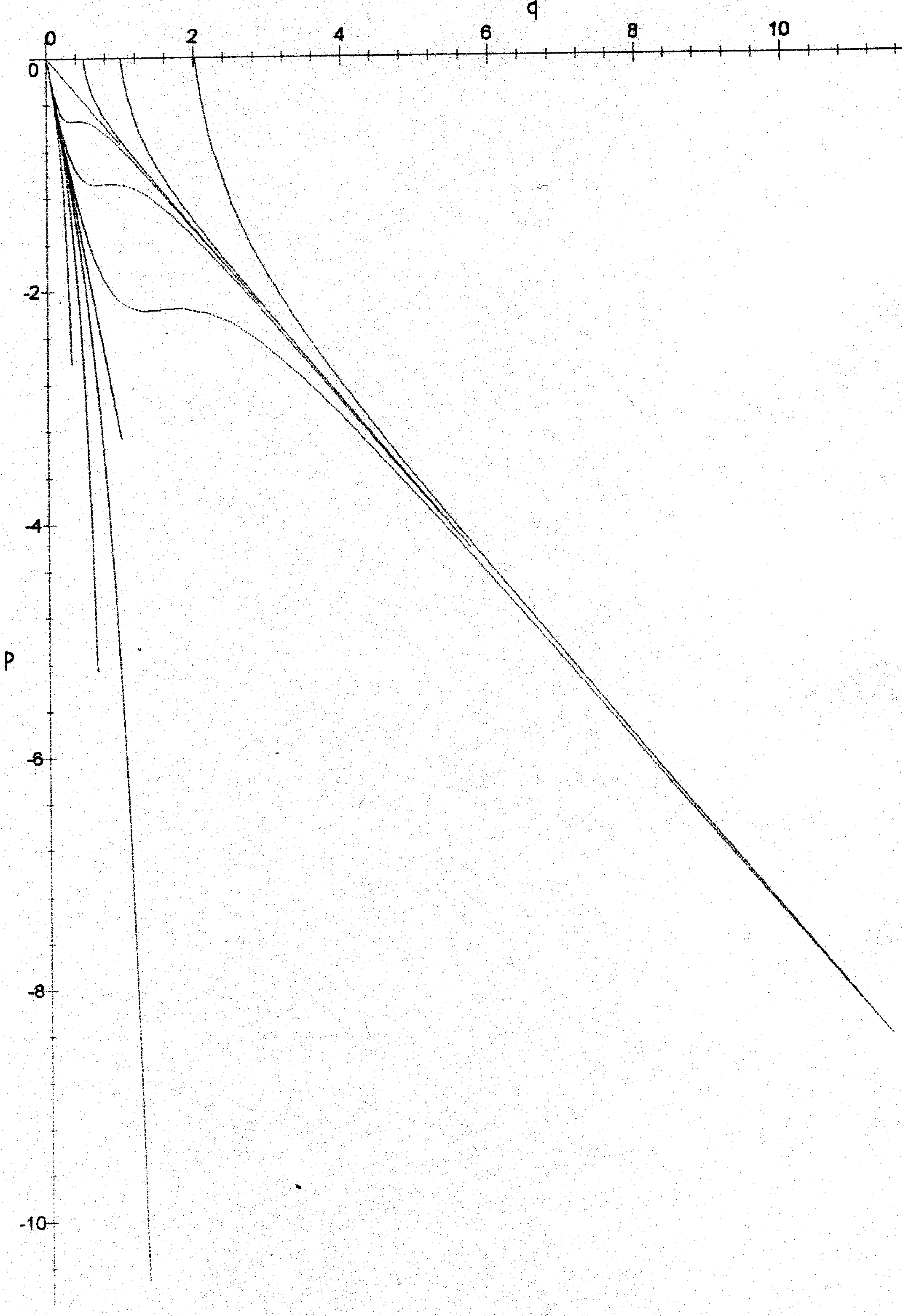

Fig. 5

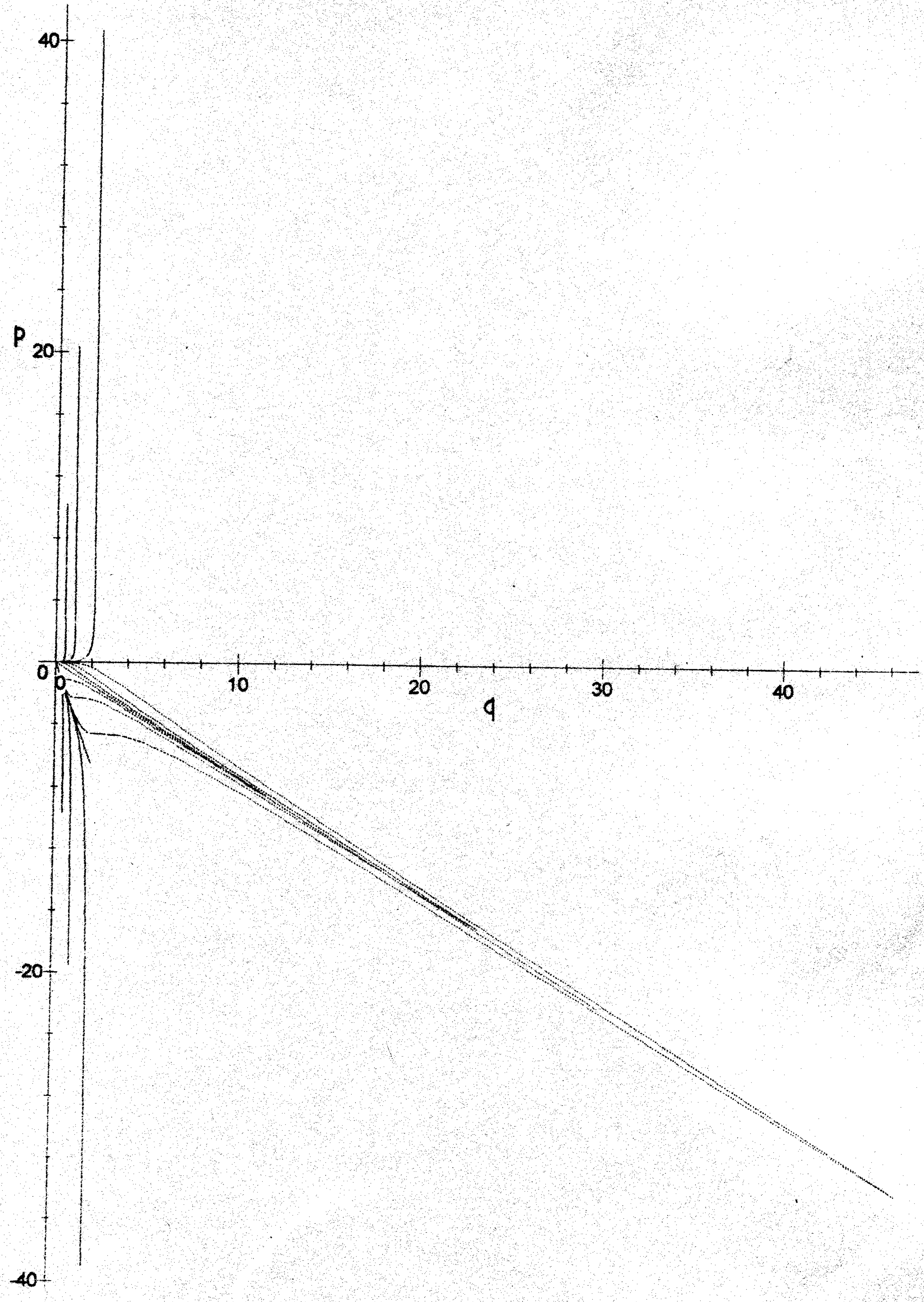

Fig. 6