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Reasoned Challenge to T. Damour's Rejection of the Janus Model, as a Pathway Out of the Deepening Crisis in Cosmology.

Key words : Janus Cosmological bimetric model

Abstract : We discuss the deep crisis in which cosmology and astrophysics are sinking, and how the Janus model offers a fruitful solution to these problems. We detail the criticisms made by academician T.Damour, refuting them one by one.

Foreword :

This will be a very long article. It is commensurate with the stakes involved. In the first part, we present the various elements that make up the profound crisis that has been affecting cosmology, astrophysics and fundamental physics in general for decades. The second part looks at the history of the Janus cosmological model, and how it offers a way out of this crisis. The third part will present T.Damour's own attempt, showing that the Janus model derives from it, modulo a simple change of sign in the second member of the second equation. Finally, we'll take up his criticisms, one by one, and refute them. This is a necessary step. Indeed, posted on his page on the website of the Institut des Hautes Etudes Scientifiques (french Institute for Advanced Studies) since 2919, these mathematically and physically incoherent articles, although unread, have had a devastating impact in France and abroad.

I - **The general crisis in cosmology, astrophysics and theoretical physics**.

In 1915, Einstein published his famous field equation, the basis of his theory of general relativity. Initially devoid of any cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \chi \, T_{\mu\nu}$$

The successes piled up. Just a few months later, the mathematician Karl Schwarzschild published two articles describing the geometry outside and inside a sphere filled with incompressible matter of density \rho. The trajectories of the planets were then assimilated to the geodesics located in an empty space surrounding the masses. The solution, first sketched in its linearized form by Einstein and restored in the form of an exact solution by Schwarzschild, accounts for the advance of Mercury's perihelion. The unsteady solutions proposed by Friedmann account for cosmic dynamics. Advances in quantum mechanics led scientists to consider the state of the cosmos in its primitive era, when this medium was brought to a very high temperature. When the age of the universe was less than one hundredth of a second, the cosmic fluid was believed to be a mixture, in equal parts, of matter and antimatter. Matter particles constantly merge with their antimatter counterparts to produce high-energy photons.

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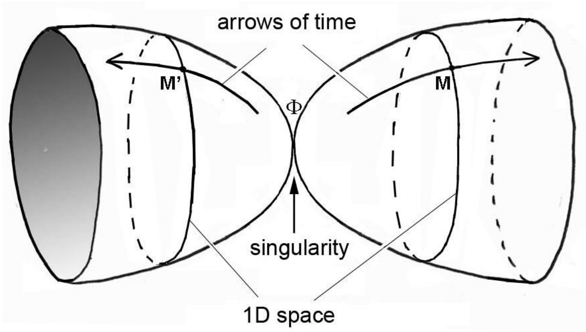
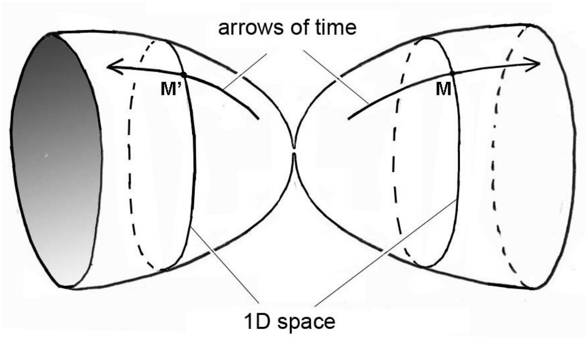
Symmetrically, these same photons transform into matter-antimatter pairs. But the expansion cools the cosmic fluid. The index lengths of the photons, by expanding along with the scale factor of the universe, lose energy. They are therefore no longer able to compensate for the annihilation of pairs of matter particles and antimatter particles. These begin to disappear, and logically, this process of self-destruction should have left only a gas of photons in their place. But, for some unexplained reason, one matter particle in a million survives. Astrophysicists then seek to identify and locate this antimatter counterpart. We imagine that what observation places before our eyes is a collection of galaxies and antigalaxies. Indeed, this antimatter, with a positive mass, is a priori capable of forming conglomerates through gravitational instability. It can theoretically give birth to stars, planets, galaxies, and this sort of antiworld would then send us photons, indistinguishable from those sent to us by the stars made of matter in our galaxies. Maison notes that encounters between galaxies are not rare events. Images of interacting galaxies are multiplying. Under these conditions, collisions between galaxies into antigalaxies should occur, a phenomenon that would give rise to a very powerful emission of gamma rays. However, no trace of this has been found. Thus, this idea of a partition of the universe into galaxies and antigalaxies is abandoned.

I.1 – **The Absence of Cosmological Antimatter: Andrei Sakharov's Conjecture**

So, from the start, we lose no less than half the universe along the way. The only one who proposed something, in 1967[3], was the Russian Andrei Sakharov. He reasoned as follows. Theoretical physicists have constructed a model according to which baryons are sets of quarks; three in number. Antibaryons, for their part, are made up of antiquarks. Sakharov, starting from the asymmetry between these two sectors, that of matter and that of antimatter, differs that the rate of production of matter from quarks could have been faster than that of production of antibaryons from antiquarks, and this by a ratio of one in a million. In what he then considers as one of the universes of a pair, baryons and antiquarks should remain in the free state, in addition to the very numerous photons resulting from annihilations, in a ratio of 3 to 1. The opposite situation occurs in a second universe, which Sakharov calls a twin, made up of antimatter and an equivalent remainder this time of quarks, still in a ratio of 3 to 1. These two universes would be linked by an initial singularity: the Big Bang. He also suggests that the timelines of the two universes would unfold in opposite directions. Below is a 2D didactic image where we imagine that space, limited to a single dimension, would also be of finite extension. A simple remark in passing; topologically, it is then possible to replace the singularity by a throat, through which the inverse of the time coordinate operates

[3]] A.D.Sakharov , (1980). Cosmological Model of the Universe with a Time Vector Inversion. ZhETF (Tr. JETP 52, 349-351) (79): 689–693





2D representation of A. Sakharov's model, with and without singularity

In his article, the fact that the second universe is populated by antimatter suggests C-symmetry. Since the equations of physics are all CPT-symmetric, he goes so far as to consider that this cosmic structure is composed of two space-times linked by a CPT-symmetry. This idea is now enjoying renewed interest among people like Boyle and Turok. It is surprising that in their articles they do not cite A. Sakharov.

I - 2 : **A missing mass.**

1933, the mass of galaxy clusters, such as Virgo, was estimated. From this, the escape velocity was calculated:

$$V_e = \sqrt{\frac{2\,G\,M}{R}}$$

The Doppler effect allowed Fritz Zwicky to evaluate the residual velocities of the galaxies in this cluster. He observed that these largely exceeded this escape velocity, meaning that this cluster should have dispersed long ago. He concluded that a gravitational force must be holding these clusters together. Thus, the idea of a missing mass appeared for the first time. At the very beginning of the 1970s, measurements of the rotational velocities of gas elements in spiral galaxies and the orbital velocities of stars were still marred by large error bars. But when these measurements became more precise, two observations became clear. Based on the mass deduced from optical observations, affecting matter mainly concentrated at the centers of galaxies, we deduced the maximum value of the orbital velocity of the galaxy's components, a speed large compared to their residual velocities, which gave them quasi-circular orbits. It turns out that the resulting centrifugal force greatly exceeds the gravitational field emanating from the visible mass of the galaxies. In addition, these rotation curves show a large plateau at the periphery.

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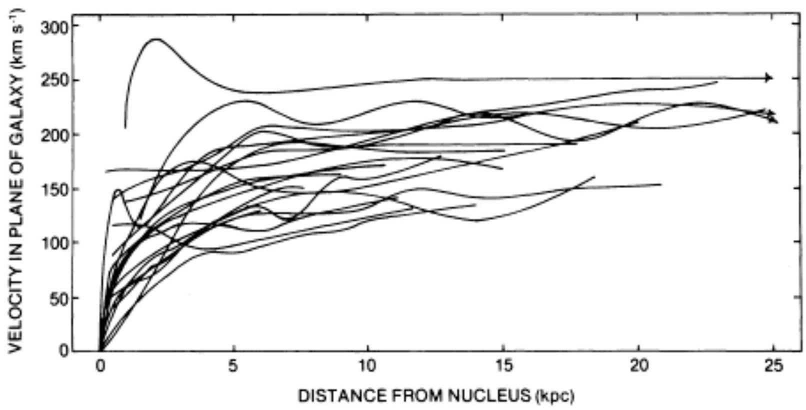


FIG. 6.—Superposition of all 21 Sc rotation curves. General form of rotation curves for small galaxies is similar to initial part of rotation curve for large galaxies, except that small galaxies often have shallower nuclear velocity gradient and tend to cover the low velocity range within the scatter at any *R*.

Rotation curves in different galaxies

It therefore becomes essential to recognize that the universe contains a hidden fraction of its mass, which we call dark matter, roughly five times greater than the visible mass. The constancy of orbital velocity implies that the centrifugal force must be 1/r. The same must be true for the gravitational field as a gradient of a potential.

$$g = -\frac{\partial \Psi}{\partial r} \propto \frac{1}{r}$$

This indicates a potential $\Psi \propto r^2$, therefore, according to Poisson's equation, a quasi-constant density over a range of distances. We deduce that all galaxies are associated with a large halo of dark matter whose diameter largely exceeds that of the elements accessible to observation. Researchers then use observational, photometric data to calculate the density profile of the different halos. Various hypotheses are then put forward concerning the possible nature of these elements escaping observation. One of them concerns MACHO (Massive Compact Objects of the Halo). But none proves satisfactory, given the significant funds devoted to tracking this dark matter. There remains the hypothesis that it is in the form of new particles and more precisely superparticles. The one that focuses all the attention is the neutralino, a neutral, supersymmetric particle associated with the neutron. This neutralino is then tracked in underground laboratories installed in mines or tunnels, so that the experiment is not disturbed by cosmic rays. The most promising setup is that of the Italian laboratory installed under the Gran Sasso, where it is hoped to trap the particles in a mass of liquid xenon. But test campaigns involving increasing masses of xenon are all unsuccessful. The same failure was observed for an experiment mounted on board the space station. The conclusion seems to be clear, to which astrophysicists are reluctant to accept: this dark matter simply does not exist. It should be noted that, in parallel, experiments conducted in particle accelerators to highlight these superparticles are all failures, even when ensuring an increasing increase in the energy involved. In these colliders, protons of mass m are brought to meet head-on, at relativistic speeds. The energies implemented are then:

$$\frac{2\,m\,c^2}{\sqrt{1-\frac{v^2}{c^2}}} \;>>\; 2\,m\,c^2$$

Under these conditions, it is no longer masses that collide, but concentrations of energy, then measured in Tev. In these experiments, the hope is that if the energy involved exceeds that associated with the hypothetical particles, they will be synthesized in the process. The mass of the neutralino, measured in equivalent energy, is one Tev (100 GeV). However, the CERN collider manages to create events where the energy involved is 13 TeV. The neutralino should have appeared. When experiments were set up to reveal phenomena involving new objects in particle physics, it was necessary to use the corresponding equivalent energies. 1 MeV for antielectrons, 2 GeV for antiprotons. It worked each time. But what happens in the CERN accelerator? When a maximum energy of 13 TeV is used, a host of particles are created, each more unstable than the other, the sum of these energies and the emitted radiation corresponding to these 13 TeV. But none of these particles individually has a mass greater than 1 GeV. High-energy physicists are therefore faced with what seems to be a limit of the model. There is an FCC Future Circular Collider project leading to an energy of 100 TeV, 7.6 times greater than the energy of the LHC. What will happen then, if not the production of even larger showers, but without the creation of new objects. We see that physics is facing a major crisis, both in the "infinitely large" and in the "infinitely small"? Still, the first idea was to add to the model of general relativity a "cold" dark matter in the sense that the speed of its components would be low compared to c. The standard model then became the CDM model (with Cold Dark Matter).

I.3 – **The Mond Theory Alternative**.

Mond stands for "Modified Newton Dynamics." If the centrifugal force at a distance varies as 1/r, the Israeli Mordechai Milgrom suggests that the gravitational field, as 1/r2) at small or medium distances, would become 1/r at the periphery of galaxies. Milgrom considered an empirical formulation, but he cannot find anything common to explain the anomaly in galaxies and clusters.

1.4 – 2011 : **The discovery of the acceleration of cosmic expansion. Dark energy**.

Here again, the modeling resulting from Friedmann's three solutions could only manage a deceleration. Three scenarios then presented themselves, depending on whether the density $\rho_{cr} = 10^{-29}\,gm/cm^3$. f this density was lower, the force of gravity became negligible and the expansion continued at constant speed. If it was higher, after a phase of maximum extension, the movement reversed, towards a cosmic implosion, a Big Crunch. If the density was equal to this critical value; we had an intermediate, parabolic expansion, with an expansion law in

$$a \propto t^{2/3}$$

The discovery of this acceleration led to the abandonment of Friedmann's models. Consideration was then given to reintroducing the cosmological constant into the field equation. This then gives rise to a phenomenon of acceleration of the expansion corresponding to an exponential law. The new model was then called Standard ΛCDM.

I.5 – **The problem of the extreme homogeneity of the primitive universe.**

In 1989, the COBE satellite was launched, revealing the extreme homogeneity of the early universe, with fluctuations not exceeding one hundred millimeter. A Russian, Andrei Linde, then proposed an inflationary model in which the universe would have, at a time between 10^{-36} and 10^{-32} seconds, an expansion by a factor of 1026. This fantastic expansion would then be attributable to an inflaton field. Today, there are as many inflaton models as there are researchers working on this subject. At this stage, we can say that researchers only ask questions they feel they can control. The question of dark matter is one of them. We could even say that it is the central question on which researchers have focused for decades, in vain. Regarding the acceleration of expansion, there are dozens of models aiming to account for the nature of this dark energy, none of which has the slightest credibility. They are just words placed at the end of each other. For dark matter, one will cite a 2022 article published in Monthly Notices which suggests that it is simply composed of… darkinos!

Some say that the identification of this dark energy is not necessary, given that the presence of the cosmological constant in the field equation perfectly accounts for it. But we must not lose sight of the fact that, if the density of energy in the form of matter is decreasing, due to its dilution, the same is not true for the density of dark energy which, if the presence of the cosmological constant accounts for it, is then constant. This acceleration must then faithfully follow the exponential curve corresponding to the value of this constant. However, recent measurements point to a conclusion according to which "the density of dark energy would have been greater in the past." However, it is impossible to provide the field equation with a cosmological constant... variable. Then it no longer results from an action and the entire geometric-mathematical functioning of the model collapses.

Regarding this great homogeneity of the early universe, attempts have been made to consider that it could be explained by a secular variation in the speed of light. This was considered by Moffat and Magueijo. But Lorentz invariance is immediately broken. All the physics of the cosmos must be consistent with group theory and, in cosmology, with the Poincaré group and its subgroup the Lorentz group. Similarly, our local representation in a three-dimensional Euclidean space must fit with the Euclid group. We must be able to perform translations and rotations on objects without changing their nature. In the world of special relativity, the key subgroup is the Lorentz group, which is nothing other than the equivalent of complex rotations in four dimensions. In the Euclidean world, rotations conserve a length, for example, the ratio of a sphere. In space-time these complex rotations conserve a scalar quantity $E^2 - p^2c^2$, where E is the energy, p the momentum and c the speed of light. The Lorentz metric is the metric of the space-time of special relativity, Minkowski space-time. The Poincaré group is its isometry group (conserving this length). Lengths are conserved by space-time translations. They are also conserved by the action of the Lorentz group. By varying c we destroy this geometric property. Special relativity ceases to work.

There is then a way to preserve this Lorentz invariance, which is to consider that this gauge phenomenon affects all components of the metric, whether it is the proper time s, which has the dimension of a length, lengths and values of the time coordinate. This was considered as early as 1988 by the author[4]. What is more, he even considered a generalized gauge phenomenon affecting lengths, times and all constants of physics. The common thread; that this generalized gauge phenomenon preserves the invariance of all equations of physics. We then find that all characteristic lengths vary as the spatial scale factor a, and that all characteristic times vary as the temporal scale factor. We also note that this gauge phenomenon conserves all energies. Since the redshift is based on a measurement of the energy of the photons, in such a phase there is no redshift. The variations of the constants, related to the spatial scale factor (where T is the temporal scale factor) are:

$$c \propto \frac{1}{\sqrt{a}}, h \propto a^{3/2}, G \propto \frac{1}{a}, e \propto \sqrt{a}, m \propto a\ , \varepsilon_o = cst\ , \mu_o \propto \frac{1}{a}, T \propto a^{3/2}$$

But the model places this gauge phenomenon before decoupling, 380,000 years ago. What is the observable? There is only one: the homogeneity of the universe. Why? Because the cosmological horizon then increases as the spatial scale factor a? This experiment, from 1988 and then 1995[5], i.e., dating back more than thirty years, has generated no interest within the community, despite its great geometric and mathematical coherence.

I.6 – **The so-called ad hoc coherence theory**.

From the CMB fluctuation map, it is possible to extract a curve that represents the power spectrum of angular anisotropies. Modeling this curve involves choosing the values of 6 parameters: the value of the cosmological constant, the parameters of the inflation models, the modeling of dark matter, and the Hubble constant.

I.7 – **The catastrophy of the first observations of the** JWST

The first images from a new space telescope reveal the existence of massive, fully formed galaxies, 500, then 400, then 350 million years old. A phenomenon that the Standard Model is unable to account for.

II : **The Janus model, its history and development.**

II.1. **The initial heuristic hypothesis.**

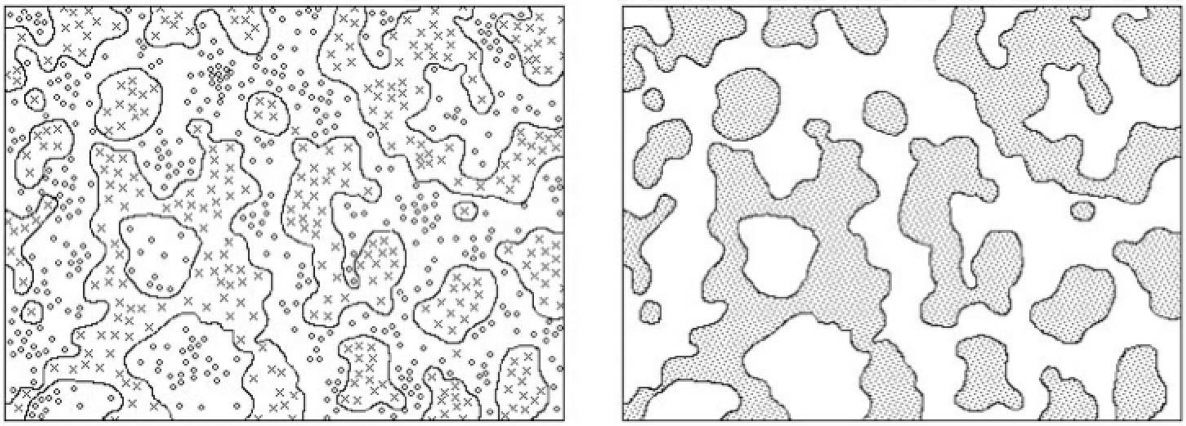
In the late 1980s, the following heuristic hypotheses were made: - Positive masses attract each other according to Newton's law - Negative masses attract each other according to Newton's law - Masses of opposite signs repel each other according to "anti-Newton" law.

[4] J.P.PETIT : Cosmological model with variable velocity of light. The interpretation of redshifts. Col.3 n° 88 (1988) 1733-1744.

[5] J. P. Petit, Astrophys. Space Sci. Twin universe cosmology **226**, 273 (1995).

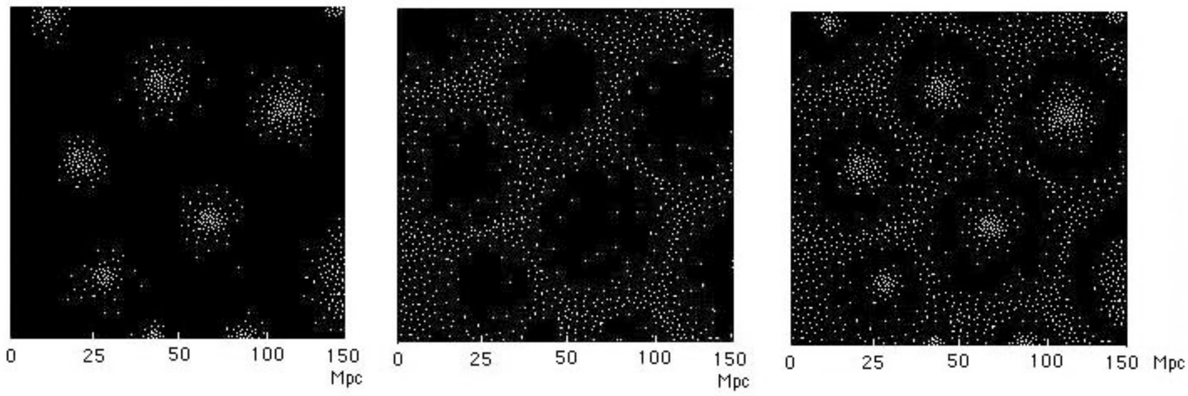
II.2 **: The Very Large Structure**.

In 1992, when numerical simulations were carried out on computers to explore the behavior of a universe made up of these two types of masses, with equal absolute densities and identical thermal agitation speeds.



Evolution of the system from fully symmetrical conditions $\left|\rho^{(-)}\right| = \rho^{(+)}$

The result does not seem to lead to shapes that can be compared with any observations. It is then that we introduce, still heuristically, a strong asymmetry, by giving the other mass, of negative volume density, a greater absolute value. A phenomenon immediately manifests itself. The negative masses form a regular set of spheroidal conglomerates, which forces the positive mass to fit into the remaining space, thus adopting a lacunar structure, evoking "joining bubbles".

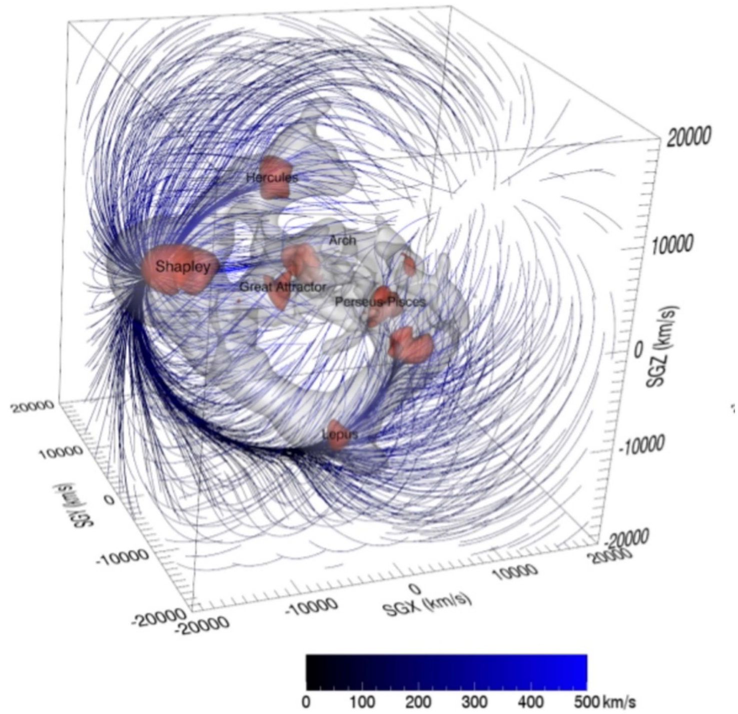


First modeling of the structure on a very large scale[6]

The idea that this second matter "of negative mass" would emit negative energy photons, which would therefore escape our means of observation. In 2017, a very large-scale mapping of the

[6] J.P.PETIT, Twin Universe Cosmology, Astrophys. and Sp. Science, **226**, 273-307, 1995

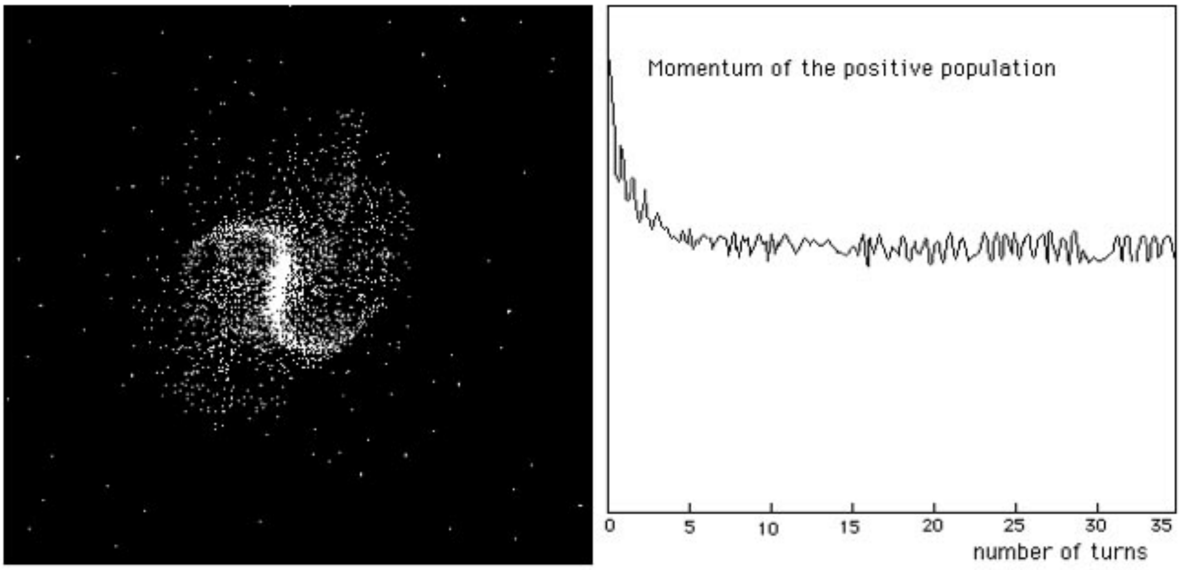
universe revealed the existence of a "repeller dipole," an immense void 100 million light years in diameter, located 600 million light years from our galaxy.



It should be noted that since then, half a dozen other formations of this type have been identified. According to the "standard" explanation, these voids result from the simple expansion of primitive fluctuations detected in the CMB. Moreover, the Janus model is the only one that can claim to explain the formation of the Big Ring. The interpretation of this "Janus model" leads to a falsifiable model, in the sense that these objects at the center of this large void should reveal its outline when a map of the brightness of objects located in the background is drawn, the latter being attenuated by the negative lensing effect.

II.3 – **Second agreement: speed curves and the galactic spiral structure**.

Also in 1992, a modeling attempt was made in two ways. The first was to describe galaxies as confined in negative, repulsive mass gaps. A fairly advanced modeling was carried out using solutions to two Vlasov equations cast by Poisson's equation. This model then accounts for the flatness of the rotation curve. Numerical simulations then produce a magnificent barred spiral, whose structure is maintained for more than 30 turns.



Spiral structure. Braking curve on the right.[7]

This work explains the purpose of these density waves: they allow the galaxy, a collision-free entity, to exchange momentum and energy with its surroundings through "dynamic friction." Since this transport cannot be achieved through collisions, density waves take care of it, producing a very small loss of angular momentum. Several conclusions:

- Barred spiral structures form at the same time as the galaxy forms.

- These structures will persist and remain visible, through fluorescence, as long as the galaxy has gas.

II.4 – **The explanation of the early birth of galaxies and stars[8]**.

When the very large-scale structure forms, the negative mass conglomerates sandwich the positive mass, structured into flat plates, membranes.

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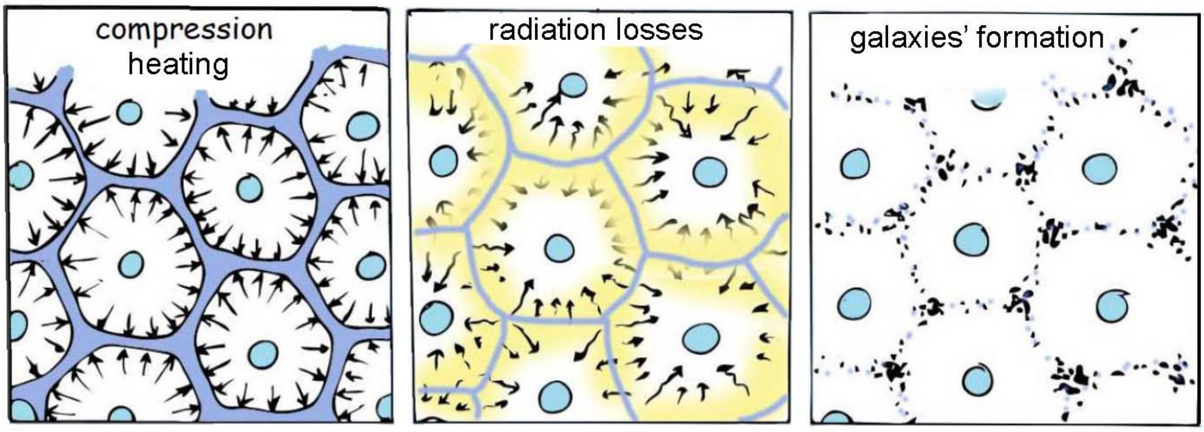
[7] J.P.PETIT, P.MIDY & F.LANDSHEAT : Twin matter against dark matter. Intern. Meet. on Atrophys. and Cosm. "Where is the matter ? ", Marseille 2001 june 25-29

[8] J.P.Petit, F.Margnat, H.Zejli : A bimetric cosmological model on Andreï’s twin universe approach. Th European Physical Journal. Vol. 84 :N°1126 (2024)



Scheme of rapid galaxy formation

These are immediately highly compressed and heated. But the plate structure allows for very rapid and efficient energy loss through radiation. This positive mass is thus destabilized, and the prediction is that stars and galaxies, in their fully mature forms, all appear within the first hundred million years. This has the advantage of being consistent with the JWST observations. However, the basis of this model cannot be satisfied with a simple heuristic approach. A problem immediately arises: the heuristically chosen force laws are incompatible with what emerges from Einstein's equation, from the model of general relativity. We must therefore consider breaking out of this straitjacket by extending it to a broader context. The idea is to consider that positive masses follow the geodesics from a metric $g_{\mu\nu}$ and that negative masses follow the geodesics from their own metric $\bar{g}$. Metrics which must be the solutions of a system of equations translating a geometrico-mathematical, "bimetric" context.

III – **Damour's attempt**.

II.1 : **His** 2002 **article in Physical Review** D[9]

For more than five decades, theoretical physics and cosmology have been lost in chimerical essays, and the most prominent journals have echoed these attempts. For example, let us cite objects called gravastars. These are balls of dark energy, surrounded by a thin layer of dark matter. All this has been published and developed for twenty years. At the beginning of the 2000s, another theme became fashionable: branes. Academician Thibault Damour, his arms laden with scientific prizes, now holds the position of undisputed French specialist in cosmology. In the early 2000s, he published an article of nearly forty pages in the sanctuary of cosmology and theoretical physics, Physical Review D. The idea is to imagine that in the universe, a large number of these "branes" interact. Very quickly, the authors fall back on two branes, called "left" and "right." The geometric context is not specified. Each one is then equipped with its own metric. We therefore have the first bimetric model with the metrics

$$g^L_{\mu\nu} \text{ et } \; g^R_{\mu\nu}$$

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[9] Damour T. , Kogan I I. Effective Lagrangians and universality classes of nonlinear bigravity Phys. Rev. D **66** (2002) 104024. hep-th/0206042. 40 pages.

The model uses gravitons with a mass spectrum. Considering that the second co-author has "demonstrated" that within this mass spectrum there is a gap between light gravitons and heavy gravitons, both choose to neglect the action of the latter.

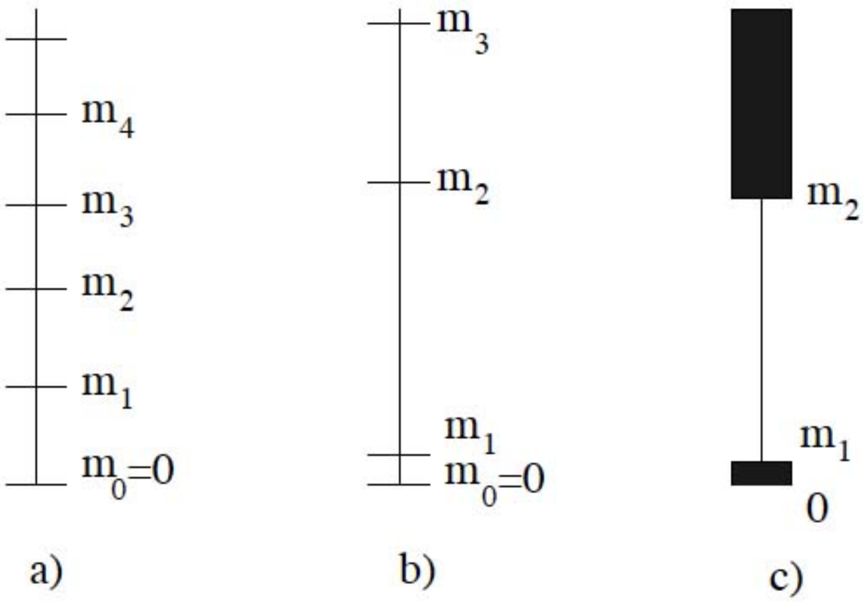

Figure 1: Regular spectrum on (Fig.1 a) versus bigravity (Fig.1 b) or quazi-localized gravity (Fig. 1 c). The last spectrum is continuous but the first band is very narrow in comparison with the gap between bands.

The mass spectrum of gravitons (Damour & Kogan)

The model must then be derived from an action. We know that in the Hilbert-Einstein couple approach, the action is constructed using Lagrangian densities, based on the elementary hypervolume:

$$\sqrt{-g}\ d^4x$$

Damour heuristically constructs his own by using a hypothetical equivalent hypervolume:

$$(\, g^L g^R\,)^{1/4}$$

Where g_L and g_R are the determinants of the two metrics

His action :

$$S = \int d^4x \sqrt{-g_L} \left(M_L^2 R(g_L) - \Lambda_L \right) + \int d^4x \sqrt{-g_L}\, L(\Phi_L, g_L) +$$

$$\int d^4x \sqrt{-g_R} \left(M_R^2 R(g_R) - \Lambda_R \right) + \int d^4x \sqrt{-g_R}\, L(\Phi_R, g_R)$$

$$-\mu^4 \int d^4x \left(g_R\, g_L\right)^{1/4} V(g_L, g_R)\,.$$

$R_{(g_L)}$ is the Ricci scalar derived from the "left" metric

$R_{(g_R)}$ s the Ricci scalar derived from the "right" metric

Λ_L abd Λ_R are two cosmological constants.

$L(\Phi_L, g_L)$ is the Lagrangian of the "left" matter

$L(\Phi_R, g_R)$ is the Lagrangian of the matter "right"

A variational calculation, not explained, leads the authors to the following system of two field equations:

$$2\,M_L^2\left(R_{\mu\nu}(g^L) - \frac{1}{2}\,g^L_{\mu\nu}\,R(g^L)\right) + \Lambda_L\,g^L_{\mu\nu} = t^L_{\mu\nu} + T^L_{\mu\nu}$$

$$2\,M_R^2\left(R_{\mu\nu}(g^R) - \frac{1}{2}\,g^R_{\mu\nu}\,R(g^R)\right) + \Lambda_R\,g^R_{\mu\nu} = t^R_{\mu\nu} + T^R_{\mu\nu}$$

In order to unify the notations and to be able to make a comparison with the Janus system, where the metrics are noted:

$$g_{\mu\nu} \text{ et } \bar{g}_{\mu\nu}$$

Neglecting cosmological constants, this system can be written:

$$R_{\mu\nu} - \frac{1}{2}\,\mathrm{R}\,g_{\mu\nu} = T_{\mu\nu} + K_{\mu\nu}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}\,R\,\bar{g}_{\mu\nu} = \bar{T}_{\mu\nu} + \bar{K}_{\mu\nu}$$

Both materials will be associated with masses m and $\overline{m}$

M_L and M_R are two constants.

$T_{\mu\nu}$ *and* $\bar{T}_{\mu\nu}$ are the source tensors of the two materials.

$K_{\mu\nu}$ is a interaction tensor translating the way in which the masses $\overline{m}$ contribute to the field to which the masses m are sensitive.

$\bar{K}_{\mu\nu}$ is an interaction tensor reflecting how the masses m contribute to the field to which the masses are sensitive $\overline{m}$.

It can be noted that the way of defining action, by Damour and Kogan, is not the most general.

II.2 : **How the Janus model derives from the approach of Damour and Kogan**.

To do this, it is necessary to show two Einstein constants, as follows:

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$$S = \int d^4x \sqrt{-g_L} \left(M_L^2 R(g_L) - \Lambda_L \right) - \chi^L \int d^4x \sqrt{-g_L} L(\Phi_L, g_L) +$$
$$\int d^4x \sqrt{-g_R} \left(M_R^2 R(g_R) - \Lambda_R \right) - \chi^R \int d^4x \sqrt{-g_R} L(\Phi_R, g_R)$$
$$-\mu^4 \int d^4x \, (g_R \, g_L)^{1/4} \, V(g_L, g_R) \, .$$

Hence the Damour system of field equations with, this time, the presence of these two "Einstein constants":

$$2 M_L^2 \left(R_{\mu\nu}(g^L) - \frac{1}{2} g^L_{\mu\nu} R(g^L) \right) + \Lambda_L g^L_{\mu\nu} = \chi^L \left(t^L_{\mu\nu} + T^L_{\mu\nu} \right)$$

$$2 M_R^2 \left(R_{\mu\nu}(g^R) - \frac{1}{2} g^R_{\mu\nu} R(g^R) \right) + \Lambda_R g^R_{\mu\nu} = \chi^R \left(t^R_{\mu\nu} + T^R_{\mu\nu} \right)$$

It is clear that this approach is equivalent to doing $\chi^L = \chi^R = 1$. Now let's consider thi option:

$$\chi^L = - \chi^R = \chi$$

With our notations, we obtain the following system:

$$R_{\mu\nu} - \frac{1}{2} \mathrm{R} \, g_{\mu\nu} = \chi \left(T_{\mu\nu} + K_{\mu\nu} \right)$$

$$\bar{R}_{\mu\nu} - \frac{1}{2} R \, \bar{g}_{\mu\nu} = - \chi \left(\bar{T}_{\mu\nu} + \bar{K}_{\mu\nu} \right)$$

Considering that the masses m of the first population are the masses of general relativity and that the masses$\bar{m} < 0$ are negative masses, we obtain the system of equations of the Janus model.

From metrics $g_{\mu\nu}$ et $\bar{g}_{\mu\nu}$ we construct the covariant derivation operators:

$$\nabla^\nu \text{ et } \bar{\nabla}^\nu$$

By posing:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \mathrm{R} \, g_{\mu\nu}$$

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} R \, \bar{g}_{\mu\nu}$$

We know that we have:

$$\nabla^\nu G_{\mu\nu} = 0$$

$$\bar{\nabla}^\nu \bar{G}_{\mu\nu} = 0$$

Which leads to the conditions of mathematical consistency:

$$\nabla^{\nu} T_{\mu\nu} = 0$$

$$\bar{\nabla}^{\nu} \bar{T}_{\mu\nu} = 0$$

$$\nabla^{\nu} K_{\mu\nu} = 0$$

$$\bar{\nabla}^{\nu} \bar{K}_{\mu\nu} = 0$$

In the Janus model it is assumed that the two populations are comparable to perfect fluids, which allows the following source tensors to be expressed as:

$$T_{\mu}^{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -\frac{p}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{p}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{p}{c^2} \end{pmatrix} \qquad \bar{T}_{\mu}^{\nu} = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 \\ 0 & -\frac{\bar{p}}{\bar{c}^2} & 0 & 0 \\ 0 & 0 & -\frac{\bar{p}}{\bar{c}^2} & 0 \\ 0 & 0 & 0 & -\frac{\bar{p}}{\bar{c}^2} \end{pmatrix}$$

In the Newtonian approximation:

$$T_{\mu}^{\nu} \cong \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \bar{T}_{\mu}^{\nu} \cong \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can now determine the interaction laws for different cases. When only masses m are present, the system then reduces to:

$$R_{\mu\nu} - \frac{1}{2} \mathrm{R}\, g_{\mu\nu} = \chi\, T_{\mu\nu}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2} R\, \bar{g}_{\mu\nu} = -\chi\, \bar{K}_{\mu\nu}$$

The first equation is then identified with Einstein's equation. We conclude that positive masses attract each other. Let's move on to a region where only negative masses are present. The system is written:

$$R_{\mu\nu} - \frac{1}{2} \mathrm{R}\, g_{\mu\nu} = \chi\, K_{\mu\nu}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2} R\, \bar{g}_{\mu\nu} = -\chi\, \bar{T}_{\mu\nu}$$

With $\bar{\rho} < 0$ the second equation, translating the interaction between negative masses $\bar{m} < 0$, will translate this as a mutual attraction.

III : **Damour's rejection of the Janus model**.

In 2019, Academician Thibault Damour published an article on the website of the french Institute of Advanced Studies[10] where he intends to denounce what he considers to be the fatal flaws weighing down the Janus cosmological model. To do this, he starts from two articles published in 2014 in the journals Astrophysics and Space Science[11] and Modern Physics letter A[12]. These two articles focus on a particular presentation of the second member of a system of two coupled field equations. In the first article, these are equations (3a) and (3b):

...relativity is based on a M_4 manifold, associated to a single Riemannian metric, ... tion of Einstein's field equation. On such grounds, negative matter cannot cohabit with positive ... What about a different geometrical framewok? A first bimetric description was proposed in Petit [18, 19] and Petit et al. [20], then in Henry-Couannier [11] and Hossenfelder [12]. Let us consider the following coupled field equations system:

$$R^{(+)}_{\mu\nu} - \frac{1}{2}R^{(+)}g^{(+)}_{\mu\nu} = \chi\left(T^{(+)}_{\mu\nu} + \varphi T^{(-)}_{\mu\nu}\right) \tag{3a}$$

$$R^{(-)}_{\mu\nu} - \frac{1}{2}R^{(-)}g^{(-)}_{\mu\nu} = -\chi\left(\phi T^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu}\right) \tag{3b}$$

$g^{(+)}_{\mu\nu}$ metric refers to positive energy (and mass if the have one) particles. $g^{(-)}_{\mu\nu}$ refers to *negative* energy (and mass if the have one) particles. $R^{(+)}_{\mu\nu}$ and $R^{(-)}_{\mu\nu}$ are Ricci tensors, built from metrics $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$. For ... of brevity we will sometimes use the notation: $f \in \{+,-\}$. In mixed form, we will write the tensors

Copy of equations (3a) and (3g) from the article published
in Astrophysics and Space Science in 2014[13].

The article is strictly limited to the construction of an unsteady solution where the two populations are homogeneous and uniform. This is a special case of the general system:

$$R^{(+)}_{\mu\nu} - \frac{1}{2}R^{(+)} = \chi\left(T^{(+)}_{\mu\nu} + \hat{T}^{(-)}_{\mu\nu}\right)$$

$$R^{(-)}_{\mu\nu} - \frac{1}{2}R^{(-)} = -\chi\left(\hat{T}^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu}\right)$$

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[10] https://www.ihes.fr/~damour/publications/JanusJanvier2019-1.pdf

[11] J-P Petit, G. D'Agostini : Negative mass hypothesis in cosmology and the nature of dark energy Astroph. And Space SciVolume 354, pages 611–615, (2014) , http://www.jp-petit.org/papers/cosmo/2014-ModPhysLett1.pdf

[12] J.P.Petit et G.D'Agostini : Cosmological bimetric model with interacting positive and negative masses and two different speeds of light in agreement with the observed acceleration of the Universe.. Modern Physics Letters A, Vol 29, No.34 (2014) 1450182. http://www.jp-petit.org/papers/cosmo/2014-AstrophysSpaceSci.pdf

[13] https://www.ihes.fr/~damour/publications/JanusJanvier2019-1.pdf

Where $T_{\mu\nu}^{(+)}$ *and* $T_{\mu\nu}^{(-)}$ are the tensors related to the two materials present, of positive mass and negative mass and where $\hat{T}_{\mu\nu}^{(-)}$ andt $\hat{T}_{\mu\nu}^{(+)}$ are "interaction tensors" that are not defined in their generality. But, in this first article, this system is not mentioned at any point. In this 2014 article, we assume that these interaction tensors take the form:

$$\hat{T}_{\mu\nu}^{(-)} = \varphi \ T_{\mu\nu}^{(-)}$$

$$\hat{T}_{\mu\nu}^{(+)} = \phi \ T_{\mu\nu}^{(+)}$$

Where φ and ϕ would be functions of time. The symmetry assumptions lead to metric solutions of the form FRLW (Friedman Lemaitre Robertson Walker):

$$\left(ds^{(+)}\right)^2 = c^2\, dt^2 - \left(a^{(+)}\right)^2 \frac{du^2 + u^2\,(d\theta^2 + sin^2\theta\, d\varphi^2)}{(1 + k^{(+)}u^2/4)}$$

$$\left(ds^{(-)}\right)^2 = c^2\, dt^2 - \left(a^{(-)}\right)^2 \frac{du^2 + u^2\,(d\theta^2 + sin^2\theta\, d\varphi^2)}{(1 + k^{(-)}u^2/4)}$$

Each field equation produces its own pairs of equations. The compatibility of these four equations leads to a relation (10):

$$\varphi = \left(\frac{a^{(-)}}{a^{(3)}}\right)^3 \qquad \phi = \left(\frac{a^{(+)}}{a^{(-)}}\right)^3 \qquad \phi = \varphi^{-1}$$

Which leads to a law (equation 9) expressing the generalized conservation of energy and energy:

$$E = \rho^{(+)}\, c^2 \left(a^{(+)}\right)^3 + \rho^{(-)}\, c^2 \left(a^{(-)}\right)^3 = Cst$$

Differential equations (13a) and (13b) then give the evolution of the scale factors.

$$\left(a^{(+)}\right)^2 \frac{d^2a^{(+)}}{dt^2} = -\frac{4\,\pi\, G}{c^4}\, E$$

$$\left(a^{(-)}\right)^2 \frac{d^2a^{(-)}}{dt^2} = +\frac{4\,\pi\, G}{c^4}\, E$$

This immediately suggests interpreting the acceleration of the expansion, in the positive sector, by assuming that the system's energy E is negative, that is, that negative masses are dominant. We then obtain an expansion profile corresponding to a negative curvature index ("hyperbolic" model) or, at infinity, this expansion is reduced to a linear law. The expansion law also corresponds to a solution previously found by William Bonnor[14]. See equation (14) of the mentioned paper. In everything that follows, we will discuss extensively the "Bianchi conditions," the fact that "the covariant derivatives of the right-hand sides of the equations must

[14] H.Bonnor Negative mass and general relativity. Gen. Relat. And Gravit. Nov 1989 vol 21 pp 1143-1157

be zero. Let us denote by $\nabla^{(+)\nu}$ et $\nabla^{(-)\nu}$ The covariant differentiation operators, each constructed with their own metric. This translates into the following conditions:

$$\nabla^{(+)\nu}\left(T^{(+)}_{\mu\nu} + \hat{T}^{(-)}_{\mu\nu} \right) = 0$$

$$\nabla^{(-)\nu}\left(\hat{T}^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu} \right) = 0$$

In general relativity, considering Einstein's equation, this reduces to:

$$\nabla^{\nu} T_{\mu\nu} = 0$$

In cosmology, we only consider two types of situations.

- Either a set or the medium is homogeneous and isotropic, where the quantities depend only on time. Then these relationships will reflect a conservation of energy (or of mass, assuming that c is constant).

- Or we limit ourselves to a stationary medium, with spherical symmetry (Schwarzschild metric solutions, external and internal). Then this relationship reflects how the pressure varies within the mass. A relationship which, when considered in the case of the Newtonian approximation, becomes the classic Euler equation, reflecting the balance between pressure and gravity.

Reading this paper, at no point does Mr. Damour realize that he is faced with a mathematically and physically coherent solution, which also accounts for a major physical phenomenon, and that his famous "Bianchi conditions" are being satisfied before his very eyes. Let's move on to the second paper. This one repeats the hypothesis of the previous article but, instead of imposing the same value of the speed of light, c, in both media, shows that the model can be extended with each having its own speed of light:

- $c^{(+)}$ for positive-energy photons (which can be observed)

- $c^{(-)}$ for negative-energy photons (which cannot be observed). The conclusions are similar, but the compatibility relationship is enriched according to

$$E = \rho^{(+)} \left(c^{(+)} \right)^2 \left(a^{(+)} \right)^3 + \rho^{(-)} \left(c^{(+)} \right)^2 \left(a^{(-)} \right)^3 = Cst$$

And here lies a typo without which all this commotion could have been avoided. In the article, the system of field equations presented corresponds to the image:

the approach by offering a bimetric description of the Universe. Though it has nothing to do with the bimetric models of Refs. 21 and 22 where the second metric refers to gravitons with nonzero mass. Strictly speaking, these models have not produced anything.

In our model, the Universe is an M_4 manifold associated not to one single metric, but to two: $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$, the former linked to species of positive mass and energy, the latter to species of negative mass and energy. From these metrics, one can build the associated Ricci tensors, $R^{(+)}_{\mu\nu}$ and $R^{(-)}_{\mu\nu}$. A system of two coupled field equations was then proposed:[20]

$$R^{(+)}_{\mu\nu} - \frac{1}{2}R^{(+)}g^{(+)}_{\mu\nu} = \chi\left(T^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu}\right), \qquad (2a)$$

$$R^{(-)}_{\mu\nu} - \frac{1}{2}R^{(-)}g^{(+)}_{\mu\nu} = -\chi\left(T^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu}\right). \qquad (2b)$$

where the tensors $T^{(+)}_{\mu\nu}$ and $T^{(+)}_{\mu\nu}$ represent positive and negative energy contents (and positive and negative mass contents as well). Previously, in 1957, Bondi[23] did study the possibility of introducing negative masses into the Einsteinian model with a single metric. As a result, positive masses attracted everything and negative masses repelled everything. It then led to a phenomenon that was called "runaway": when a mass $+m$ met a mass $-m$, the positive mass

The red arrows indicate an error. These are not matter tensors of the positive and negative species, but interaction tensors.

We should have written:

the approach by offering a bimetric description of the Universe. Though it has nothing to do with the bimetric models of Refs. 21 and 22 where the second metric refers to gravitons with nonzero mass. Strictly speaking, these models have not produced anything.

In our model, the Universe is an M_4 manifold associated not to one single metric, but to two: $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$, the former linked to species of positive mass and energy, the latter to species of negative mass and energy. From these metrics, one can build the associated Ricci tensors, $R^{(+)}_{\mu\nu}$ and $R^{(-)}_{\mu\nu}$. A system of two coupled field equations was then proposed:[20]

$$R^{(+)}_{\mu\nu} - \frac{1}{2}R^{(+)}g^{(+)}_{\mu\nu} = \chi\left(T^{(+)}_{\mu\nu} + \widehat{T}^{(-)}_{\mu\nu}\right), \qquad (2a)$$

$$R^{(-)}_{\mu\nu} - \frac{1}{2}R^{(-)}g^{(+)}_{\mu\nu} = -\chi\left(\widehat{T}^{(+)}_{\mu\nu} + T^{(-)}_{\mu\nu}\right). \qquad (2b)$$

where the tensors $T^{(+)}_{\mu\nu}$ and $T^{(+)}_{\mu\nu}$ represent positive and negative energy contents (and positive and negative mass contents as well). Previously, in 1957, Bondi[23] did study the possibility of introducing negative masses into the Einsteinian model with a single metric. As a result, positive masses attracted everything and negative masses repelled everything. It then led to a phenomenon that was called "runaway": when a mass $+m$ met a mass $-m$, the positive mass

The two equations with, in red, what should have been added.

The "hats" are missing on the interaction tensors, which, a priori, differ from the matter tensors of the two populations. But Mr. Damour is starting with this erroneous system and nothing can stop him. Indeed, we knew perfectly well that by identifying these interaction tensors with the source tensors of the two matters, this would immediately violate the conditions of mathematical compatibility. For example, in an unsteady situation, this imposed the trivial solution. $g^{(+)}_{\mu\nu} = g^{(-)}_{\mu\nu}$ giving $\rho^{(+)} + \rho^{(-)} = 0$. That is, with a universe equipped with scale factors $a^{(+)} = a^{(-)}$, equal, increasing linearly over time.

At the same time as he informed us of his conclusions through a registered letter addressed to my home, he posted on his IHES page, in January 2019, an article that he imagined to be a definitive condemnation, without appeal, and also announced that he would give it maximum coverage:

Thibault Damour
IHES, Bures-sur-Yvette,
7 Janvier 2019

à Jean-Pierre Petit, BP 55, 84122 Pertuis

Copies: Je me réserve le droit d'envoyer des copies de cette lettre à toutes les personnes que vous citez sur votre site, dans vos vidéos, et dans vos lettres, ainsi qu'à toute personne s'intéressant au "modèle Janus".

Objet: "modèle Janus".

Monsieur,

Translation :

I reserve the right to send a copy of this letter to anyone you mention in your website, letters, videos, and to anyone interested in the "Janus model."

We tried to point out to him that he had based his work on a system of equations containing a typo. In fact, we had already worked in the previous months on this thorny issue of the "Bianchi conditions," focusing on the second type of problem to be addressed, the one where the field is created by a positive or negative mass. We had been able to show that these conditions could be satisfied when working within the framework of the Newtonian approximation:

- Moderate curvature

- Small velocities compared to c.

A solution where the interaction tensors can be assimilated to the matter tensors, with the sign of the pressure terms reversed. We sent him a copy of the article, published in the peer-reviewed journal Progress in Physics, and asked for an interview to provide the requested clarifications. No response. We then created a very comprehensive document, including all the calculation details, which we forwarded to him. No Response Two years passed without us being able to obtain a response from anyone, whether it was the mathematician Emmanuel Ullmo, director of the IHES (Institut des Hautes Etudes de l'École Polytechnique), on whose website the article was posted, or the geometer Etienne Ghys, Permanent Secretary of the Paris Academy of Sciences, responsible for these theoretical questions. Not knowing what to do, I posted these calculation details on my website. Colleagues, physicists, astrophysicists, mathematicians, and physics professors, examined these calculations carefully and, equally perplexed, contacted me. I suggested they send Mr. Damour a collective letter stating that they had found no errors in the document, enclosing it and addressing one to Mr. Ullmo and to Mr. Ghys. Mr. Damour reacted immediately, expressing his irritation at what he perceived as a kind of "scientific harassment." His response was swift, in the form of a new article posted on his page on the IHES website in

December 2022, three years after the publication of his first article in 2019. We must therefore base our decision on the content of this new article.

III.1 : **His mistake regarding the meaning of forces.**

Dans cet article[15] que Mr Damour met dans sa page du site de l'IHES en décembre 2022, intitulé « Incohérence physique et mathématique du Modèle Janus de JP Petit et coll. », il écrit :

...on oublie cette incohérence, et si on étudie les conséquences ph... ...s deux équations (1), on va montrer que l'on obtient encore deux autres incohérences physico-mathématiques.

Une première nouvelle incohérence concerne l'idée de base du modèle Janus (tel qu'il a été défini dans un cadre newtonien), cad le fait que, dans ce modèle, *les masses positives attirent les masses positives; les masses négatives attirent les masses négatives, mais les masses positives et négatives se repoussent.*

Une conséquence particulière de ce principe fondamental du modèle Janus doit donc être qu'une étoile à masse négative, dont l'extérieur est décrit, d'après l'équ. (21) de PDD19, par une solution de Schwarzschild contenant une masse négative ($-m$ remplaçant $+m$) doit *attirer les masses d'épreuve négatives dans son voisinage.* Mais en fait les éqs (1) impliquent le contraire: *les masses d'épreuve négatives dans le voisinage d'une solution de Schwarzschild ayant une masse négative sont repoussées.*

En effet, si l'on applique la deuxième équation (1) au cas d'une distribution de matière négative, $T^-_{\mu\nu}$ (spatialement séparée de la distribution de matière ordinaire $T^+_{\mu\nu}$, ou, pour simplifier, en absence de matière ordinaire), l'identité de Bianchi, $\nabla^\nu_- E^-_{\mu\nu} \equiv 0$, satisfaite par le tenseur d'Einstein, $E^-_{\mu\nu}$, implique que $T^-_{\mu\nu}$ doit satisfaire la loi de conservation

$$\nabla^\nu_- T^-_{\mu\nu} = 0 . \tag{4}$$

Cette loi de conservation (par rapport à la connexion ∇_- de la métrique $g^-_{\mu\nu}$) implique, comme il est bien connu, qu'une particule d'épreuve à masse négative doit suivre une *géodésique* de la métrique $g^-_{\mu\nu}$. En particulier, une particule d'épreuve à masse négative autour d'une solution de Schwarzschild de masse négative, sera *repoussée*, et non attirée par la masse centrale négative. Nous avons donc ici une violation frappante d'une des idées de base du modèle Janus. Cela montre que les deux équations de champ (1) ne réussissent pas à donner une description relativiste de la situation physique qu'elles sont censées décrire.

Une autre incohérence (cette fois purement mathématique) des éqs. (1) apparait quand on considère les analogues de l'équation de Tolman-Oppenheimer-... ...nant la variation radiale de la pression d'une étoile de ma... or-

[15] https://www.ihes.fr/~damour/publications/JanusDecembre2022.pdf

Translation of excerpts, underlined, from the article
posted by T.Damour in December 2022 on his IHES page.

- A first mathematical inconsistency concerns the basic idea of the Janus model (as defined in a Newtonian framework), that is, the fact that, in this model, positive masses attract positive masses, negative masses attract negative masses, but positive and negative masses repel each other.

....

*Indeed, it is well known that a negative mass test particle will be **repelled**, not attracted, by a negative mass. Here we have a striking violation of the basic ideas of the Janus model. This shows that its two field equations fail to provide a relativistic description of the physical reality they are intended to describe.*

These underlined sentences show that Mr. Damour, three years after first criticizing our work, probably did not understand, or even simply did not read, what was published on the Janus model. By adopting the idea that negative masses repel each other, he is locked into the reasoning derived from the equation of general relativity. Or, by looking at things differently, he is basing himself on his own bimetric approach, or in both equations, he has implicitly chosen to give the same value to the equivalents of Einstein's constant in his two equations.

The quoted excerpt shows that it is Mr. T. Damour's criticisms that are inconsistent.

In the above, we have simply shown that the Janus model can be deduced from the Damour-Kogan bimetric model by simply adding equal and opposite Einstein constants, which then changes absolutely nothing to the alleged mathematical consistency of the two models. In their approach the authors assume that their interaction tensors, although they do not provide explicit expressions, satisfy the necessary condition

$$\nabla^{\nu} K_{\mu\nu} = 0$$

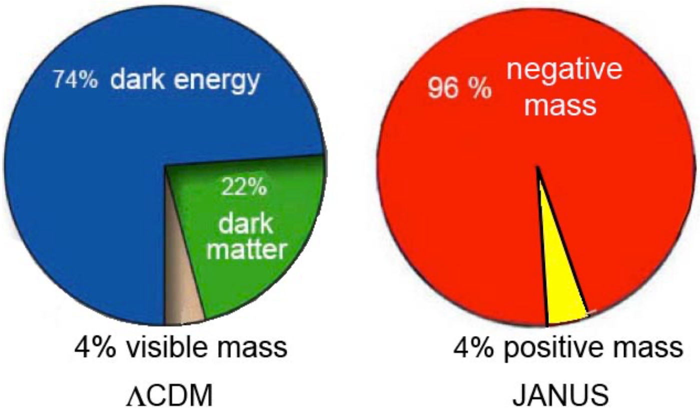
$$\overline{\nabla}^{\nu} \overline{K}_{\mu\nu} = 0$$

III.2 : **Refutation of this criticism. Fruitfulness of the Janus approach.**

The geometric context of the model has been clarified, as well as its bases with respect to the theory of dynamical groups. In order not to weigh down the article, which is focused on refuting T. Damour's errors, the reader should refer to the articles published in 2024 in European Physical Journal C and Reviews in Mathematical Physics. Incidentally, this provides support for Sakharov's conjecture. The Janus model is therefore the only one that confers a perfectly defined identity to the invisible components of the universe. These are antihydrogen and antihelium of negative mass. In this world of negative masses, there are no galaxies, stars, or planets. Life is absent. There are two antimatters, one with positive mass, located in our universe sheet ("falling downwards in the gravitational field of the Earth, prediction confirmed 2023[16]) and the other of negative mass, replacing both dark matter and dark energy. The Janus

[16] By the ALPHA-g experiment team at CERN, Switzerland.

model is thus the only one to show the profound asymmetry between the two entities, which will be justified in work to be published.



In the Janus model, negative mass replaces both dark matter and dark energy.

What are the additional assumptions made in the Janus model? They are both explicit and simple. It is assumed that under the conditions of the Newtonian approximation:

$$\text{For } \mu \neq 0 \quad K_\mu^\mu \cong 0 \quad \text{and} \quad \overline{K}_\mu^\mu \cong 0$$

$$K_0^0 < 0 \text{ and } \overline{K}_0^0 > 0$$

This means that masses of opposite signs repel each other. It should be remembered that these assumptions, concerning the force laws, were chosen heuristically to serve as a basis for the numerical simulations of 1992, which led to fruitful results. These results are recalled in the article published in 2024 in European Physical Journal C[17]. We have just seen what follows from the Newtonian form of the model (large-scale structure, dipole repeller, spiral structure).

III.3. : **The explanation of strong gravitational lensing effects.**

Regarding gravitational lensing effects, the diagram below compares these effects, depending on whether they are attributed to a dark matter halo or to a negative mass environment.

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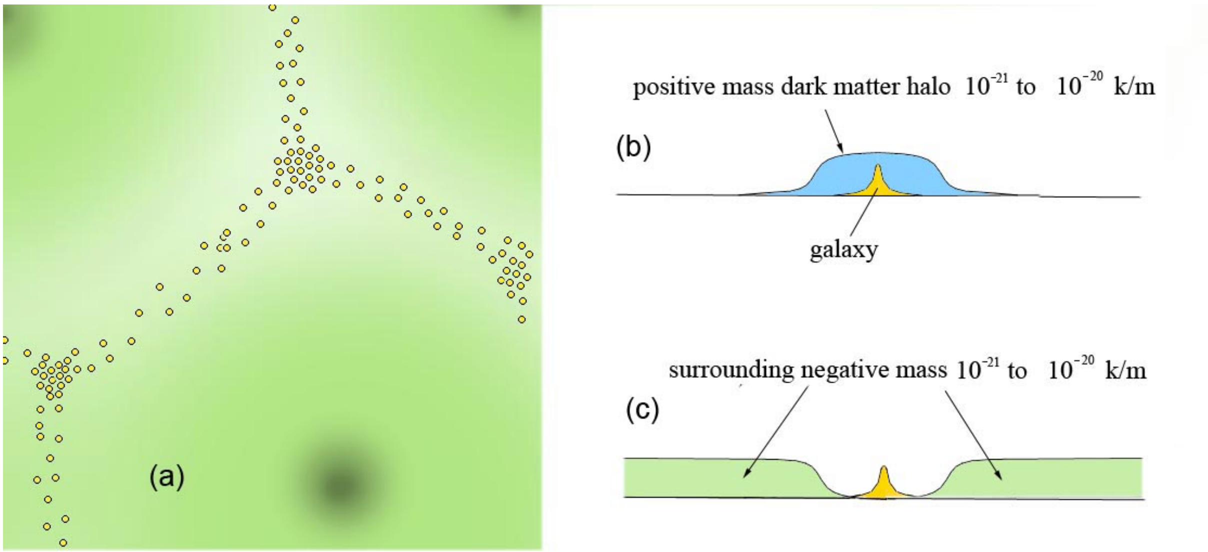
[17] J.P.Petit, F.Margnat, H.Zejli : A bimetric cosmological model on Andreï's twin universe approach. Th European Physical Journal. Vol. 84 :N°1126 (2024)



The hypothetical halo of dark matter, of positive mass, is only the image, in negative, of the gap in the distribution of negative mass

In figure (a) in yellow, the positive mass, in green the negative mass. Galaxies form in the planar structures of the very large-scale structure in contiguous bubbles. Gravitational instability encourages them to gather in the segments, common to three cells, and in the "nodes" of this structure to form galaxy clusters. The negative pass constitutes the spheroidal conglomerates located at the center of each cell. This negative mass infiltrates between the galaxies at the time of their formation. In figure (b) the galaxy model with, in blue, its dark matter halo caste. In (c) an equivalent structure where the confining field is then created by a gap in the negative mass, where the density is of the same order of magnitude, except for the sign, as that of the dark matter halo, of positive mass. The gravitational lensing effects will therefore be identical. The same conclusion applies to galaxy clusters, themselves located in a gap in the negative mass.

III.3.5: **Explanation of the early formation of galaxies and stars**.

The diagram of the formation of the very large-scale structure of the universe provides a different diagram of galaxy formation. At the moment this structure is formed, the positive mass is violently backcompressed by the two adjacent negative mass conglomerates. It thus experiences a sudden rise in temperature. However, its plate structure then proves optimal for ensuring no less rapid cooling through radiative loss. See the image in section II.4. This leads to a scenario in which all galaxies form at the same time, with their currently known masses. Simulations show that barred spiral structures also form before the first hundred million years, which is consistent with the JWST observational data.

III.3.6: **Explanation of the acceleration of cosmic expansion.**

In 2014, it was suggested that, in the matter phase, under the assumption of uniformity and isotropy, the interaction tensors should be given the following specific form:

$$K_{\mu\nu} = \phi \, \overline{T}_{\mu\nu} \qquad K\overline{T}_{\mu\nu} = \varphi \, T_{\mu\nu}$$

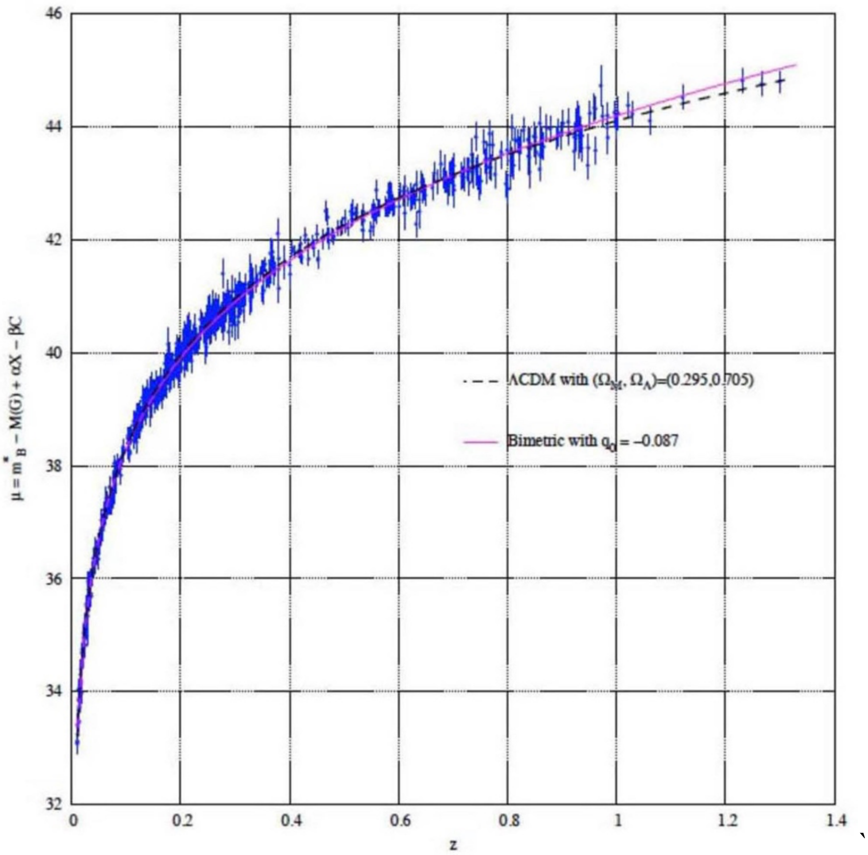
Where φ and ϕ are functions of time. The geometric hypothesis then leads to giving the metrics FRLW forms. An exact solution then emerges, where these functions, inverses of each other, correspond to:

$$\varphi = \frac{\bar{a}^3}{a^3} = \phi^{-1}$$

This solution also provides a mathematical compatibility condition:

$$E = \rho \, c^2 \, a^3 + \bar{\rho} \, \bar{c}^2 \, \bar{a}^3 = \text{Cst}$$

Which is none other than the generalized conservation of energy. Thus, the "Bianchi conditions" (zero divergence of the right-hand sides of the two equations) are satisfied. Damour at no point mentions this solution or this article, whose existence he decided to ignore. The exploitation of this solution then provides an alternative interpretation of the acceleration of cosmic expansion.

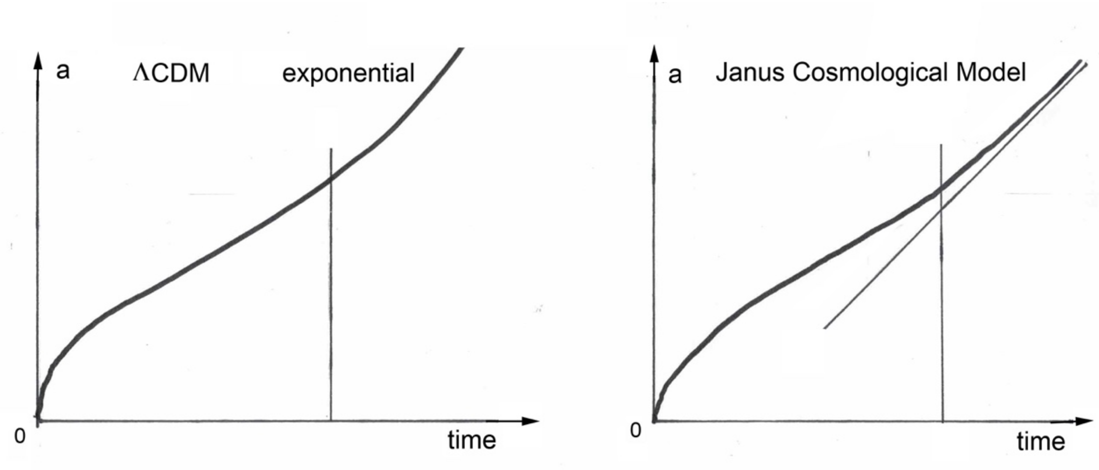


Magnitude des sources en fonction du redshift[18]

The law of expansion, in this matter phase, is then identified with the solution of W. Bonnor

[18] G. DAgostini and J.P.Petit : Constraints on Janus Cosmological model from recent observations of supernovae type Ia, Astrophysics and Space Science, (2018), 363:139.https://doi.org/10.1007/s10509-018-3365-3

[19]. It tends to infinity towards a linear law.



Comparative schematic expansions, ΛCDM and Janus

In the ΛCDM model, the expansion law is exponential, which corresponds to "a constant dark energy density." Recent measurements appear to invalidate this point. Under these conditions, the model can no longer be modified, as this would contradict the geometric and mathematical structure of the model. Finally, we come to the second aspect of the criticisms formulated by T. Damour over the past six years. This concerns the structure of the interaction tensors. An important point must be recalled. A model must produce elements that can be compared with observations. III.3.7: Justification of the heuristic hypothesis on the direction of forces. Leaving the domain of Newtonian approximation, let us consider the geometry associated with a neutron star. The system is then:

$$R_{\mu\nu} - \frac{1}{2} \mathrm{R}\, g_{\mu\nu} = \chi\, T_{\mu\nu}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2} R\, \bar{g}_{\mu\nu} = -\chi\, \bar{K}_{\mu\nu}$$

As early as 1916, the Austrian mathematician Karl Schwarzschild constructed an exact solution describing this geometry, later adopted by Tolman, Oppenheimer, and Volkoff. The presence of this mass creates an induced geometry in the second layer, the effects of which we can qualitatively consider. This positive mass must repel the control masses $\bar{m}$, according to the non-zero geodesics resulting from the corresponding metric solution $\bar{g}_{\mu\nu}$. This positive mass must also exert its effect on negative-energy photons according to the zero-length geodesics resulting from the metric $\bar{g}_{\mu\nu}$. But all this escapes observation. We are therefore under no obligation to specify the form of the tensor $\bar{K}_{\mu\nu}$.

Finally, let us consider the last configuration, corresponding to the Dipole Repeller, that is to say the case where we have an induced geometry effect, in the positive sector, created by a negative mass that we can represent as a sphere of radius R filled with a fluid of constant density $\bar{\rho} < 0$. The system is then

[19] W.B.Bonnor : Negative mass and general relativity. General Relativity and Gravitation Vol.21, N°11, 1989

$$R_{\mu\nu} - \frac{1}{2}\,\mathrm{R}\,g_{\mu\nu} = \chi\,K_{\mu\nu}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}\,R\,\bar{g}_{\mu\nu} = -\chi\,\bar{T}_{\mu\nu}$$

Simulations suggest that this mass remains of moderate density, and thus falls within the context of the Newtonian approximation. We could construct the geodesics followed by the witness masses and by the negative energy photons by assimilating this negative fluid to an ideal gas, that is, by giving the source tensor $\bar{T}_{\mu\nu}$ the form:

$$\bar{T}_{\mu}^{\nu} = \begin{pmatrix} \begin{matrix} \bar{\rho} & 0 \\ 0 & -\frac{\bar{p}}{\bar{c}^2} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -\frac{\bar{p}}{\bar{c}^2} & 0 \\ 0 & -\frac{\bar{p}}{\bar{c}^2} \end{matrix} \end{pmatrix}$$

IV: **Mathematical consistency of the Janus model.**

The calculation of the geometry within this sphere then leads to:

$$\bar{\mu}(r) = \frac{G}{\bar{c}^2}\int_0^r 4\,\pi\,r^2\,\bar{\rho}\;dr = \frac{G}{\bar{c}^2}\,\bar{M}(r)$$

i.e. to the classic relationship
:

$$\bar{p}' = -\frac{(\bar{\rho} + \bar{p}/\bar{c}^2)\,(\,\bar{\mu}(r) + 4\pi\,G\,\bar{p}\;r^3/\;\bar{c}^4)\bar{c}^2}{r\,(\,r - \bar{\mu}(r)\,)}$$

Since we are in the Newtonian approximation this relationship becomes the classic Euler equation, meaning that the pressure forces are balanced by the force of gravity.

$$\bar{p}' = -\frac{G\,\bar{\rho}\,\bar{M}(r)}{r^2}$$

Which simply means that the gravitational field to which a mass $\bar{m}$ is subjected, inside a sphere of constant density, is equivalent to the field created by a mass $\bar{M}(r)$ representing the amount of matter contained inside a sphere centered on the origin, of radius r, concentrated at this point. We must now consider the geodesics resulting from the metric g_{\mu\nu}. This is then an induced geometry (by the presence of this mass $\bar{M}(r)$negative). To construct this solution, we need to specify the form of the interaction tensor $K_{\mu\nu}$. Knowing that this tensor must satisfy the condition

$$\nabla^{\nu}\,K_{\mu\nu} = 0$$

As shown in 2019[20] then in the detailed calculations immediately sentto Mr. Damour this condition can be satisfied in the Newtonian approximation regime by giving the interaction tensor the form:

$$K_\mu^\nu = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 \\ 0 & \frac{\bar{p}}{\bar{c}^2} & 0 & 0 \\ 0 & 0 & \frac{\bar{p}}{\bar{c}^2} & 0 \\ 0 & 0 & 0 & \frac{\bar{p}}{\bar{c}^2} \end{pmatrix}$$

C'est-à-dire en inversant le sens des termes de pression du tenseur $\bar{T}_\mu^\nu$. Le calcul conduit alors à la relation :

$$\bar{p}' = -\frac{(\bar{\rho} - \bar{p}/\bar{c}^2)\,(\bar{\mu}(r) - 4\pi\, G\, \bar{p}\; r^3/\; \bar{c}^4)\bar{c}^2}{r\,(r + \bar{\mu}(r))}$$

Which is identified with the previously found Euler equation, precisely because it is within the framework of a Newtonian approximation. Thus the approximate form of the tensor K_μ^ν is perfectly suitable. Three years late (he surely hasn't read any of the documents I sent him giving all the details of these calculations) Mr. Damour finally realizes that this is suitable and he writes:

[20] J.P.Petit, G. D'Agostini, Nathalie Debergh : Physical and mathematical consistency of the Janus Cosmological Model (JCM). Progress in Physics **2019** Vol.15 issue 1

où la source $T^{+}_{\mu\nu}$ est stationnaire et à symétrie sphérique. Ces solutions ont été écrites[2] dans les éqs. (45), (46) de PDD19, c-a-d (avec $' = d/dr$)

$$
\begin{aligned}
p'_{+} &= -G\left(\rho_{+} + \frac{p_{+}}{c^2}\right)\frac{M_{+}(r) + 4\pi p_{+} r^3/c^2}{r(r - 2GM_{+}(r)/c^2)}\,,\\
p'_{+} &= -G\left(\rho_{+} - \frac{p_{+}}{c^2}\right)\frac{M_{+}(r) - 4\pi p_{+} r^3/c^2}{r(r + 2GM_{+}(r)/c^2)}\,,
\end{aligned} \tag{7}
$$

où $p_{+}(r)$ est la pression (de la matière ordinaire), $\rho_{+}(r)$ sa densité, et $M_{+}(r) = 4\pi \int_0^r drr^2 \rho_{+}(r)$ est la masse (positive) contenue dans le rayon r. Notons que l'on passe de la première équation (7) à la seconde par les changements: $p_{+} \to -p_{+}$ et $G \to -G$.

Il est vrai que si l'on prend formellement la limite newtonienne $\frac{1}{c^2} \to 0$ dans les équations (7), ces deux équations deviennent compatibles, car elles deviennent toutes deux identiques à l'unique équation de structure newtonienne (6).

Mais, il est physiquement inacceptable de négliger ainsi le fait que le modèle Janus-2019 prédit que *la variation radiale de la pression dans une étoile de matière ordinaire doit satisfaire deux équations incompatibles entre elles.* En effet, si l'on considère par exemple une étoile à neutrons, les termes relativistes supplémentaires dans (7), cad $\pm p_{+}/c^2$, $\pm 4\pi p_{+} r^3/c^2$, et $\pm 2GM_{+}(r)/c^2$, sont numériquement très significatifs (de l'ordre de 10%), et conceptuellement très importants car ils modifient grandement la valeur de la masse maximum d'une étoile à neutrons. Comme il est rappelé dans l'éq. (5), les analogues des équations (7) dans Janus-2014 avaient des membres droits qui différaient d'env

How Mr Damour finally realizes, three years late, that the coherence
of the Janus system can be assured in the Newtonian approximation

In this excerpt from the article posted by Mr. Damour in December 2022 on his page on the IHES website, he acknowledges, three years late, that the compatibility of the equations, that is to say the mathematical coherence of the system, is ensured in Newtonian. Below is the translation of the underlined passages.

- *Below are the equations giving the pressure gradient, from the two field equations, as they appear in the 2019 article in Progress in Physics*

$$
\begin{aligned}
p'_{+} &= -G\left(\rho_{+} + \frac{p_{+}}{c^2}\right)\frac{M_{+}(r) + 4\pi p_{+} r^3/c^2}{r(r - 2GM_{+}(r)/c^2)}\,,\\
p'_{+} &= -G\left(\rho_{+} - \frac{p_{+}}{c^2}\right)\frac{M_{+}(r) - 4\pi p_{+} r^3/c^2}{r(r + 2GM_{+}(r)/c^2)}\,,
\end{aligned} \tag{7}
$$

It is true that then, if we formally take the Newtonian limit, these equations become compatible, because they then identify with the Newtonian equation, the Euler equation[21].

[21] Reflecting the balance between pressure forces and the force of gravity.

These are the equations that I published in 2019 in the journal Progress in Physics.[22].

It took him four years to decide to read this article and finally understand that compatibility is assured under Newtonian conditions. But he immediately focuses on the case of the neutron star, where such an expression for an interaction tensor $\overline{T}_{\mu}^{\nu}$ obtained by simply inverting the pressure term, would not be suitable. But if the neutron star is in the positive world, we are not required to provide the explicit form of this tensor. This would be the case if hypermassive objects existed in the negative world. However, simulations indicate that this is not the case. So why persist in constructing a solution that could not be linked to any physical object? His criticism becomes irrelevant.

But if the neutron star is in the positive world, we are not required to provide the explicit form of this tensor. This would be the case if there were hypermassive objects in the negative world. However, simulations indicate that this is not the case. So why persist in constructing a solution that could not be linked to any physical object? His criticism becomes irrelevant.

V: **Advanced, and final form of the system of Janus field equations**.

We showed in Section II.2 that the model can be considered to derive from the initial approach of T. Damour and I. Kogan, with a simple change in the sign of the Einstein constant appearing in the right-hand side of the second field equation. To the extent that this system derives from an action, this can be considered to invalidate Damour's claim that the Janus model does not derive from an action. In unsteady conditions, if we consider that both entities satisfy the homogeneity and isotropy assumptions, the model then derives from the exact solution provided by the system:

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \chi \left[T_{\mu\nu} + \frac{\bar{a}^3}{a^3} \overline{T}_{\mu\nu} \right]$$

$$\overline{R}_{\mu\nu} - \frac{1}{2} \overline{R} \, \bar{g}_{\mu\nu} = -\chi \left[\overline{T}_{\mu\nu} + \frac{a^3}{\bar{a}^3} T_{\mu\nu} \right]$$

The condition of compatibility of the equations resulting in a generalized law of the conservation of energy: an element completely overlooked by Mr. T. Damour, for lack of having taken the trouble to read the articles in question. We then note that the scalar coefficients present in the interaction terms are identified with the ratios of the cubes of the determinants of the FRLW metric solutions, where x° is the common chronological parameter with:

$$x^0 = c\,t = \bar{c}\,\bar{t}$$

[22] J.P.Petit, G. D'Agostini, Nathalie Debergh : Physical and mathematical consistency of the Janus Cosmological Model (JCM). Progress in Physics **2019** Vol.15 issue 1

$$ds^2 = (dx^0)^2 - \frac{dr^2}{1 - k\,r} - r^2(\,d\theta^2 + sin^2\theta\, d\varphi^2\,)$$

$$d\bar{s}^2 = (dx^0)^2 - \frac{dr^2}{1 - \bar{k}\,r} - r^2(\,d\theta^2 + sin^2\theta\, d\varphi^2\,)$$

We are therefore tempted to conjecture that the system can be written:

$$R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} = \chi\left[T_{\mu\nu} + \sqrt{\frac{\bar{g}}{g}}\;\bar{T}_{\mu\nu}\right]$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}\,\bar{R}\,\bar{g}_{\mu\nu} = -\chi\left[\bar{T}_{\mu\nu} + \sqrt{\frac{g}{\bar{g}}}\;T_{\mu\nu}\right]$$

In Appendix I, it is shown that this system derives from an action. Regarding the stationary situation, the sub-radical terms can be considered constant over large distance ranges. In any case, in science, nothing should be considered definitive. A theoretical model progresses when it manages to account for more phenomena. It is in this sense that the Janus model can be considered as progress. But one day, it itself will have to give way to something that will perhaps be more efficient, although quite different, integrating for example a quantized structure of space-time, unifying gravitation and quantum mechanics.

Conclusion :

What this second article shows is that the so-called criticisms published by academic T. Damour, positioned on his page of the Institut des Hautes Études Scientifiques, are both incoherent (the error on the meaning of the forces) and irrelevant (referring to the unfounded hypothesis of the existence of negative mass neutron stars). They betray the fact that their author did not understand, and perhaps simply did not even read with sufficient attention the documents he intended to criticize. But one can understand the displeasure felt by a scientist who, after having failed in 2002 to bring out something coherent from his bimetric approach, notes that others, at the cost of a tiny adjustment that escaped him (the inversion of the sign of Einstein's constant in the right-hand side of the second equation) have on the contrary opened an extremely fruitful and promising path.

Strictly speaking, given the importance of the issue, it would have been desirable for Mr. Damour to publish these criticisms in a peer-reviewed journal. But it is doubtful that a serious journal would agree to publish such an incoherent article, both physically and mathematically.

The impact of this approach, both in France and abroad, has been devastating and continues to be so, among scientists, the public and the journalistic world, although it is very unlikely that anyone has actually read these documents. Academician T. Damour is considered in France and abroad as a reference in this field, personalities. Thus, personalities from the French scientific world such as Mr. Emmanuel Ullmo, mathematician and director of the IHES, Mrs. Françoise Combes, astrophysicist and President of the Paris Academy of Sciences and Mr.

Etienne Ghys, mathematician, geometer and Permanent Secretary of the Academy for these theoretical questions, declare that they agree with Mr. Damour's conclusions, while admitting without embarrassment that they have not read the articles in question.

I add that none of them agreed to meet with me, or to read the documents I had sent them, along with my legitimate protest. Let us clarify, with regard to Mr. E. Ghys, that the role of the permanent secretary of the Academy (for these theoretical questions) is precisely to follow up on this type of problem, which it is lawful to qualify as scientific defamation, and which would merit in this case the exercise of a right of scientific reply within the Academy.

As a result, the entire scientific world, both French and foreign, has been dissuaded from taking an interest in the articles we have published, presenting the ins and outs of our Janus cosmological model. This adds to the damage this has caused to the advancement of French research.

I will conclude with a question, formulated in a rather pessimistic tone: It's 2025. How many months, years, decades will pass without the citizens and taxpayers of different countries, stunned by the siren songs of the various figures in their respective scientific communities, who will leave no shadow of a memory in the history of science, reacting to the fact that they continue to take at face value an opinion emanating from a supposed authority in the relevant field, fraught with consequences, which stifles any interest in an innovative and fruitful endeavor, without taking the trouble to verify its content, out of laziness, herd mentality, compromise, or simple cowardice.

References

Annexe I **: Derivation of the system of Janus equations from an action.**

This is presented as a requirement neither ne qua non by Damour, in his critiques. We can indeed consider several ways of proceeding. The one implemented by Damour and Kogan in 2002 is one of them. But before looking into the question, let us clarify that a theory is in no way obliged to satisfy such a requirement, to the extent that the mathematical and geometric imperatives are satisfied. In the case of the Janus model, they are mentioned and managed in the appendix below, to have obeyed the imperatives of zero covariant derivations. Its geometric context also has the advantage of being clearly specified. See the article published in 2024 in European Physical Journal C, which is not the case for the article by Damour and Kogan from 2002, with a large number of "branes" floating in a higher-dimensional space, interacting with the help of "gravitons with a mass spectrum". Let's return to the title of this article: Effective Lagrangians and universality classes of nonlinear bigravity Translation: Effective Lagrangians and universality classes of nonlinear bigravity What ambition for a forty-page article that leads to no results of any kind!

Let us return to this question of construction from a Lagrangian by minimizing an action. In general relativity, the metric $g_{\mu\nu}$ is involved in the construction of the action. This belongs to the functional space of Lorentzian metrics. A variation $\delta g_{\mu\nu}$ is then carried out which leads to a variation δS of the action. And it is by canceling it that we obtain a system of differential equations which, modulo symmetries, determine the form of this metric.

In bimetrics, the metrics$g_{\mu\nu}$ and $\bar{g}$ are involved in the construction of the action S as well as the Lagrangian and what will give rise to the interaction tensors. When we operate a variation, it would be logical to consider it as a "bivariation"$(\,\delta g_{\mu\nu}\,,\delta\bar{g}_{\mu\nu})$.. Although this is not explicitly developed, it is the strategy used by Damour and Kogan. In general relativity, we use the Ricci scalar R, multiplied by the elementary volume, which gives R $\sqrt{|g|}\; d^4$x. Damour and Kogan therefore place in their action similar terms R $\sqrt{|g|}$ and $\bar{\text{R}}$ $\sqrt{|\bar{g}|}\; d^4$x , using the same notations. Beyond that; as "differential bigeometry" remains to be constructed, the theoretician can only introduce a heuristic contribution. Damour and Kogan introduce a sort of equivalent elementary space $(\,g\;\bar{g}\,)^{1/4} d^4$x . In the Janus model, for the moment, we prefer to construct the action on the basis of the hypervolumes $\sqrt{|g|}\; d^4$x and $\sqrt{|\bar{g}|}\; d^4$x, which does not mean that the approach is the best and is unique. Let's say it's a way to construct the system of equations from an action. We believe that these mathematical aspects require the contribution of mathematician-geometers, the only ones capable of defining this new "differential biogeometry." That being said, here is this derivation.

We've changed the font choices. Artificial intelligence, which is expected to play an increasing role in the scientific sphere, doesn't differentiate between different fonts. The action choices are therefore

x

$$\text{S} = \int_{D4}\left[\,\frac{1}{2\chi}R\,+\,L\,+\,\Sigma\,\right]\sqrt{|g|}\;d^4x \quad +\int_{D4}\left[\,\frac{1}{2\bar{\chi}}\bar{R}\,+\,\bar{L}\,+\,\bar{\Sigma}\,\right]\sqrt{|\bar{g}|}\;d^4x$$

$$\delta\text{S} \;=\; \int\limits_{D4}\left[\,\frac{1}{2\chi}\left(\frac{\delta R}{\delta g^{\mu\nu}}+\frac{R}{\sqrt{|g|}}\frac{\delta\sqrt{|g|}}{\delta g^{\mu\nu}}\right)+\frac{1}{\sqrt{|g|}}\frac{\delta\left(L\sqrt{|g|}\right)}{\delta g^{\mu\nu}}+\frac{1}{\sqrt{|g|}}\frac{\delta\left(\Sigma\sqrt{|g|}\right)}{\delta g^{\mu\nu}}\right]\sqrt{|g|}\;\delta g^{\mu\nu}d^4x$$

$$+\int\limits_{D4}\left[\,\frac{1}{2\bar{\chi}}\left(\frac{\delta\bar{R}}{\delta\bar{g}^{\mu\nu}}+\frac{\bar{R}}{\sqrt{|\bar{g}|}}\frac{\delta\sqrt{|\bar{g}|}}{\delta\bar{g}^{\mu\nu}}\right)+\frac{1}{\sqrt{|\bar{g}|}}\frac{\delta\left(\bar{L}\sqrt{|\bar{g}|}\right)}{\delta\bar{g}^{\mu\nu}}\;\;\frac{1}{\sqrt{|\bar{g}|}}\frac{\delta\left(\Sigma\sqrt{|\bar{g}|}\right)}{\delta\bar{g}^{\mu\nu}}\right]\sqrt{|\bar{g}|}\;\delta\bar{g}^{\mu\nu}d^4x$$
$$=\;0$$

Introduce the tensors:

$$T_{\mu\nu}\;=\;-\;\frac{1}{2\sqrt{|g|}}\frac{\delta\left(L\sqrt{|g|}\right)}{\delta g^{\mu\nu}}\;=\;-\,2\;\frac{\delta L}{\delta g^{\mu\nu}}\;+\;g_{\mu\nu}\,L$$

$$\bar{T}_{\mu\nu}\;=\;-\;\frac{1}{2\sqrt{|\bar{g}|}}\frac{\delta\left(\bar{L}\sqrt{|\bar{g}|}\right)}{\delta\bar{g}^{\mu\nu}}\;=\;-\,2\;\frac{\delta\bar{L}}{\delta\bar{g}^{\mu\nu}}\;+\;\bar{g}_{\mu\nu}\,\bar{L}$$

$$K_{\mu\nu}\;=\;-\;\frac{1}{2\sqrt{|\bar{g}|}}\frac{\delta\left(\,\Sigma\sqrt{|g|}\right)}{\delta g^{\mu\nu}}$$

$$\bar{K}_{\mu\nu} = -\frac{1}{2\sqrt{|g|}} \frac{\delta\left(\bar{\Sigma}\sqrt{|\bar{g}|}\right)}{\delta \bar{g}^{\mu\nu}}$$

Whence :

$$\sqrt{\frac{\bar{g}}{g}}\, K_{\mu\nu} = \sqrt{\frac{\bar{g}}{g}}\, \frac{-2}{\sqrt{|\bar{g}|}} \frac{\delta\left(\Sigma\sqrt{|g|}\right)}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{|g|}} \frac{\delta\left(\Sigma\sqrt{|g|}\right)}{\delta g^{\mu\nu}} = -2\frac{\delta\Sigma}{\delta g^{\mu\nu}} + g_{\mu\nu,}\Sigma$$

$$\sqrt{\frac{g}{\bar{g}}}\, \bar{K}_{\mu\nu} = \sqrt{\frac{g}{\bar{g}}}\, \frac{-2}{\sqrt{|g|}} \frac{\delta\left(\Sigma\sqrt{|\bar{g}|}\right)}{\delta \bar{g}^{\mu\nu}} = \frac{-2}{\sqrt{|\bar{g}|}} \frac{\delta\left(\bar{\Sigma}\sqrt{|\bar{g}|}\right)}{\delta \bar{g}^{\mu\nu}} = -2\frac{\delta\bar{\Sigma}}{\delta \bar{g}^{\mu\nu}} + \bar{g}_{\mu\nu,}\bar{\Sigma}$$

We pose a $\bar{\chi} = -\chi$. Whence the Janus field equations system :

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \chi \left[T_{\mu\nu} + \sqrt{\frac{\bar{g}}{g}}\, K_{\mu\nu} \right]$$

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{R}\, \bar{g}_{\mu\nu} = -\chi \left[\bar{T}_{\mu\nu} + \sqrt{\frac{g}{\bar{g}}}\, \bar{K}_{\mu\nu} \right]$$

Annexe II **: Newtonian coherence calculation**.

It would have been logical to present this calculation when the geometry created a negative mass, that of the conglomerate responsible for the effect of the "dipole repeller" comes from the second equation, according to:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\,\bar{g}_{\mu\nu} = -\chi\,\bar{T}_{\mu\nu}$$

While observables respond to the positive mass equation:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \chi\sqrt{\frac{\bar{g}}{g}}\;K_{\mu\nu}$$

To the extent that in the negative mass conglomerate the curvature remains moderate and the speed of light low compared to c, it is then possible to use the Newtonian approximation and give the tensor K_{\mu\nu}a form such that the exploitation of the solution, in both solutions, leads to the same Euler equation translating, within the negative mass, the balance between gravity forces and pressure forces:

$$\bar{p}' = -\frac{G\,\bar{\rho}\,\bar{M}(r)}{r^2}$$

But, given the context, it seemed preferable to us to reproduce, identically, the calculation, totally symmetrical, corresponding to the system,

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \chi\,T_{\mu\nu}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\,\bar{g}_{\mu\nu} = -\chi\sqrt{\frac{g}{\bar{g}}}\;\bar{K}_{\mu\nu}$$

As it appeared in the memorandum sent to Mr. Damour in 2019, representing the details of the calculation in our January 2019 article. If he had agreed to read it six years ago, this would have avoided finding ourselves in the situation we are in today.

Both metrics are given the form :

(A2-1) $$ds^2 = e^{\nu}dx^{\circ 2} - e^{\lambda}dr^2 - r^2d\theta^2 - r^2 sin^2\theta d\varphi^2$$

(A2-2) $$d\bar{s}^2 = e^{\bar{\nu}}dx^{\circ 2} - e^{\bar{\lambda}}dr^2 - r^2d\theta^2 - r^2 sin^2\theta d\varphi^2$$

As Tolman, Oppenheimer and Volkoff did in 1939, reproducing the result of Karl Schwarzschild's February 1916 paper. The introduction of strictly positive exponential functions reflects the concern to introduce signature invariance (+ - - -).

Assuming that the positive mass creating the field corresponds to a sphere filled with an incompressible fluid of constant density ρ , and that this fluid can be assimilated to a perfect fluid, we have :

(A2-3)

$$g^{\mu\nu} = \begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}\theta \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} e^{\nu} & 0 & 0 & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ 0 & 0 & -r^{2} & 0 \\ 0 & 0 & 0 & -r^{2}\sin^{2}\theta \end{pmatrix} \qquad g^{\nu}_{\mu} = \delta^{\nu}_{\mu}$$

The Ricci tensor components are:

(A2-4)

$$R_{oo} = e^{\nu-\lambda}\left[-\frac{\nu''}{2} + \frac{\nu'\lambda'}{4} - \frac{\nu'^2}{4} - \frac{\nu'}{r} \right] \qquad R^0_0 = -e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu'}{r} \right)$$

$$R_{11} = \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r} \qquad R^1_1 = -e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} - \frac{\lambda'}{r} \right)$$

$$R_{22} = e^{-\lambda}\left[1 + \frac{\nu' r}{2} - \frac{\lambda' r}{2} \right] - 1 \qquad R^2_2 = -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{2r} - \frac{\lambda'}{2r} \right) + \frac{1}{r^2}$$

$$R_{33} = R_{22}\sin^2\theta \qquad R^3_3 = R^2_2$$

And Ricci's scalar:

(A2-5)

$$R = R^{\mu}_{\mu} = e^{-\lambda}\left[2\left(-\frac{\nu''}{2} + \frac{\nu'\lambda'}{4} - \frac{\nu'^2}{4} \right) - \frac{\nu'}{r} + \frac{\lambda'}{r} - \frac{2}{r^2} - \frac{2\nu'}{2r} + \frac{2\lambda'}{2r} \right] + \frac{2}{r^2}$$

This gives the components of the Einstein tensor:

(A2-6) $$G^0_0 = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2}$$

(A2-7) $$G^1_1 = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2}$$

(A2-8) $$G^2_2 = e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} \right]$$

Let's write the equations corresponding to the first of the two field equations, in mixed notation.

(A2-9) $$G^{\nu}_{\mu} = \chi\, T^{\nu}_{\mu}$$

(A2-10)
$$e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}=\chi T_0^0$$

(A2-11)
$$e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)-\frac{1}{r^2}=\chi T_1^1$$

(A2-12)
$$e^{-\lambda}\left[\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right]=\chi T_2^2$$

Et aussi :

(A2-13)
$$\chi T_0^0-\chi T_1^1=-\frac{\nu'+\lambda'}{r}e^{-\lambda}$$

We'll now consider the outer metric, where the second members of the equations are zero. As is conventionally done :
(A2-14)
$$e^{\nu}=e^{-\lambda}=1-\frac{2m}{r}$$

(A2-15)
$$ds^2=\left(1-\frac{2m}{r}\right)dx^{\circ 2}-\frac{dr^2}{1-\frac{2m}{r}}-r^2\,(d\theta^2+\sin^2\theta\, d\varphi^2)$$

With :

(A2-16)
$$m=\frac{GM}{c^2}$$

M being the (positive) mass of the object creating the field.
Let's move on to the classic construction of the interior metric. We have :

(A2-17)
$$T_\mu^{\ \nu}=\begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & -\frac{p}{c^2} & 0 & 0\\ 0 & 0 & -\frac{p}{c^2} & 0\\ 0 & 0 & 0 & -\frac{p}{c^2}\end{pmatrix}$$

The field equations are as follows:

(A2-18)
$$e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}=\chi\rho$$

(A2-19)
$$e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)-\frac{1}{r^2}=-\chi\frac{p}{c^2}$$

(A2-20)
$$e^{-\lambda}\left[\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right]=-\chi\frac{p}{c^2}$$

(A2-21)
$$-\frac{\nu'+\lambda'}{r}e^{-\lambda}=\chi\left(\rho+\frac{p}{c^2}\right)$$

Whence :

(A2-22)
$$e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)-\frac{1}{r^2}=e^{-\lambda}\left[\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right]$$

(A2-23)
$$\frac{e^{\lambda}}{r^2}=\frac{1}{r^2}-\frac{\nu'^2}{4}+\frac{\nu'\lambda'}{4}+\frac{\nu'+\lambda'}{2r}-\frac{\nu''}{2}$$

For the resolution, we pose :

(A2-24)
$$e^{-\lambda}\equiv 1-\frac{2m_{(r)}}{r}\quad \text{soit}\quad 2m_{(r)}=r\left(1-e^{-\lambda}\right)$$

We derive this expression:

(A2-25)
$$2m'=\left(1-e^{-\lambda}\right)+r\lambda'e^{-\lambda}$$

(A2-27)
$$-\frac{2m'}{r^2}=\frac{-1+e^{-\lambda}-r\lambda'e^{-\lambda}}{r^2}=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)$$

(A2-28)
$$m'=-\frac{r^2\chi\rho}{2}=4\pi r^2\frac{G}{c^2}\rho$$

Whence :

(A2-29)
$$m_{(r)}=\int_0^r m'_{(r)}dr=\frac{4}{3}\pi r^3\rho\frac{G}{c^2}$$

(A2-30)
$$\nu'=\frac{r}{r\left(r-2m\right)}\left(-\chi\frac{p}{c^2}r^2+1\right)-\frac{\left(r-2m\right)}{r\left(r-2m\right)}$$

(A2-31)
$$\nu'=2\frac{m+4\pi G p r^3/c^4}{r\left(r-2m\right)}$$

By deriving the equation

(A1-23)

(A2-32)
$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-\lambda'e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)+e^{-\lambda}\left(\frac{-2}{r^3}+\frac{\nu''}{r}-\frac{\nu'}{r^2}\right)$$

$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-e^{-\lambda}\left(\frac{\lambda'}{r^2}+\frac{\lambda'\nu'}{r}+\frac{2}{r^3}-\frac{\nu''}{r}+\frac{\nu'}{r^2}\right)$$

$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-2\frac{e^{-\lambda}}{r}\left(\frac{\lambda'}{2r}+\frac{\lambda'\nu'}{2}+\frac{1}{r^2}-\frac{\nu''}{2}+\frac{\nu'}{2r}\right)$$

$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-2\frac{e^{-\lambda}}{r}\left(\frac{1}{r^2}-\frac{\nu'^2}{4}+\frac{\lambda'\nu'}{4}+\frac{\lambda'+\nu'}{2r}-\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\lambda'\nu'}{4}\right)$$

Combining to (A1-21) we get :

(A2-33)
$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-2\frac{e^{-\lambda}}{r}\frac{e^{\lambda}}{r^2}-2\frac{e^{-\lambda}}{r}\left(\frac{\nu'^2}{4}+\frac{\lambda'\nu'}{4}\right)$$

(A2-34)
$$-\chi\frac{p'}{c^2}=-e^{-\lambda}\frac{\nu'}{2r}(\nu'+\lambda')$$

Using (A2-21) we get :

(A2-35)
$$-\chi\frac{p'}{c^2}=-\frac{e^{-\lambda}}{r}(\nu'+\lambda')\frac{\nu'}{2}=\chi\left(\rho+\frac{p}{c^2}\right)\frac{\nu'}{2}$$

With :

(A2-37)
$$\frac{p'}{c^2}=-\frac{\nu'}{2}\left(\rho+\frac{p}{c^2}\right)$$

Finally we get the so-called « TOV equation » (Tolmann-Oppenheimer-Volkoff):

(A2-38)
$$\frac{p'}{c^2}=-\frac{m+4\pi G p r^3/c^4}{r(r-2m)}\left(\rho+\frac{p}{c^2}\right)$$

Through Newtonian approximation ($p \ll \rho c^2$ $2m \ll r$) this becomes;
(A2-39)

$$\boxed{p'=-\frac{\rho m c^2}{r^2}=-\frac{G M \rho}{r^2}}$$

In spherical symmetry, the gravitational field prevailing at a distance $r < r_s$ (inside the star of assumed constant density) is equal to the field that would be created by the mass $M(r)$ contained in a sphere of radius rs, concentrated at the center.
Although it's terribly tedious, it's essential to repeat, line by line, all these (here, classical) calculations with the aim of extending them to the calculation of the inner metric describing negative species. When we do this, we'll see that, without this precaution concerning the tensor, we'd end up with the same contraction.
Continuing the calculation, we'll now explain the complete calculation of the interior metric $g_{\mu\nu}$.
As usual, we pose: pose:

(A2-40) $$\hat{R} = \sqrt{\frac{3c^2}{8\pi G\rho}}$$

We pose :

(A2-41) $$m_{(r)} = \frac{4\pi G\,\rho r^3}{3c^2}$$

This will immediately give us one of the terms of the metric:

(A2-42) $$e^{-\lambda} = 1 - \frac{2m_{(r)}}{r} = 1 - \frac{8\pi G\rho r^2}{3c^2} \equiv 1 - \frac{r^2}{\hat{R}^2}$$

And so our inner metric is written:

(A2-43) $$ds^2 = e^{\nu}\, dx^{\circ 2} - \frac{dr^2}{1 - \frac{r^2}{\hat{R}^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right)$$

We have to build $\nu_{(r)}$. The density is constant. The we have :

(A2-44) $$\nu' = -\frac{2p'}{\rho c^2 + p} \quad \Rightarrow \quad \nu' = -\frac{2\left(\rho c^2 + p\right)'}{\rho c^2 + p} = -2\mathrm{Log}(\rho c^2 + p)'$$

(A2-46) $$-\frac{\nu}{2} = \mathrm{Log}(\rho c^2 + p) + cte \quad \Rightarrow \quad De^{-\frac{\nu}{2}} = \frac{8\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right) = -\chi\left(\rho + \frac{p}{c^2}\right)$$

Using (A1-19) :

(A2-47) $$-\frac{\nu' + \lambda'}{r} e^{-\lambda} = \chi\left(\rho + \frac{p}{c^2}\right) = -De^{-\frac{\nu}{2}} \quad \Rightarrow \quad rDe^{-\frac{\nu}{2}} = \nu' e^{-\lambda} + \lambda' e^{-\lambda} = \nu' e^{-\lambda} - \left(e^{-\lambda}\right)'$$

(A2-49) $$rDe^{-\frac{\nu}{2}} = \nu'\left(1 - \frac{r^2}{\hat{R}^2}\right) - \left(1 - \frac{r^2}{\hat{R}^2}\right)' = \nu'\left(1 - \frac{r^2}{\hat{R}^2}\right) + \frac{2r}{\hat{R}^2}$$

We pose $e^{\frac{\nu}{2}} \equiv \gamma(r) \quad \Rightarrow \quad \gamma' = \frac{\nu'}{2} e^{\frac{\nu}{2}}$

(A2-50) $$rD = \nu' e^{\frac{\nu}{2}}\left(1 - \frac{r^2}{\hat{R}^2}\right) + \frac{2r}{\hat{R}^2} e^{\frac{\nu}{2}} = 2\gamma'\left(1 - \frac{r^2}{\hat{R}^2}\right) + \frac{2r}{\hat{R}^2}\gamma$$

A peculiar solution of the equation is : $\gamma_p = \frac{\hat{R}^2 D}{2}$

We need to find a general solution to the homogeneous equation :

(A2-51) $$u'\left(1 - \frac{r^2}{\hat{R}^2}\right) + \frac{r}{\hat{R}^2} u = 0 \quad \Rightarrow \quad u = B\left(1 - \frac{r^2}{\hat{R}^2}\right)^{1/2}$$

Whence :

(A2-52) $$\gamma \equiv e^{\frac{\nu}{2}} = \frac{\hat{R}^2 D}{2} - B\left(1 - \frac{r^2}{\hat{R}^2}\right)^{1/2}$$

(A2-53) $$g_{00} = e^{\nu} = \left[A - B\left(1 - \frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^2$$

where :

(A2-54) $$\frac{\hat{R}^2 D}{2} = A \Rightarrow D = 2\frac{A}{\hat{R}^2} = \frac{2\rho}{3}\frac{8\pi G}{c^2}A = -\chi\frac{2\rho}{3}A$$

Now let's express that pressure is zero at the sphere's surface:

(A2-55) $$De^{-\frac{\nu}{2}} = -\chi\left(\rho + \frac{p}{c^2}\right) = -\chi\frac{2\rho}{3}A\left[\frac{\hat{R}^2 D}{2} - B\left(1 - \frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^{-1}$$

(A2-58) $$\rho + \frac{p}{c^2} = \frac{2\rho}{3}\frac{A}{A - B\left(1 - \frac{r^2}{\hat{R}^2}\right)^{1/2}}$$

At $r = r_s$ we get $p = 0$

(A2-59) $$1 = \frac{2}{3}\frac{A}{A - B\left(1 - \frac{r_s^2}{\hat{R}^2}\right)^{1/2}} \quad \Rightarrow \quad A = 3B\left(1 - \frac{r_s^2}{\hat{R}^2}\right)^{1/2}$$

All that remains is to determine B, which we'll do by imposing that the inner and outer metrics connect on the surface of the sphere. This translates into:

(A2-60) $$g_{00}^{int}(r_s) = e^{\nu(r_s)} = \left[A - B\left(1 - \frac{r_s^2}{\hat{R}^2}\right)^{1/2}\right]^2 = g_{00}^{ext}(r_s) = \left(1 - \frac{2GM}{r_s c^2}\right)$$

(A2-61) $$B^2\left[3\left(1 - \frac{r_s^2}{\hat{R}^2}\right)^{1/2} - \left(1 - \frac{r_s^2}{\hat{R}^2}\right)^{1/2}\right]^2 = \left(1 - \frac{2GM}{r_0 c^2}\right)$$

(A2-62) $$4B^2\left(1 - \frac{r_s^2}{\hat{R}^2}\right) = \left(1 - \frac{2GM}{r_s c^2}\right)$$

(A2-63)
$$4B^2\left(1-\frac{8\pi G\rho\, r_s^2}{3c^2}\right)=\left(1-\frac{8\pi G}{3c^2}\rho\, r_s^2\right)\Rightarrow B=\frac{1}{2}$$

(A2-64)
$$A=\frac{3}{2}\left(1-\frac{r_s^2}{\hat{R}^2}\right)^{1/2}$$

(A2-65)
$$g_{00}^{int}(r)=\left[\frac{3}{2}\left(1-\frac{r_s^2}{\hat{R}^2}\right)^{1/2}-\frac{1}{2}\left(1-\frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^2$$

Hence the interior metric:

(A2-66)
$$\boxed{ds^2=\left[\frac{3}{2}\left(1-\frac{r_s^2}{\hat{R}^2}\right)^{1/2}-\frac{1}{2}\left(1-\frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^2 dx^{\circ 2}-\frac{dr^2}{1-\frac{r^2}{\hat{R}^2}}-r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)}$$

We will now deploy the same calculation scheme, but this time adapting it to the metric describing the negative-mass species, which is then the solution to equation :

(A2-67)
$$\overline{G}_\mu^\nu \equiv \overline{R}_\mu^\nu-\frac{1}{2}\overline{g}_\mu^\nu\,\overline{R}=-\chi\frac{\sqrt{-g}}{\sqrt{-\overline{g}}}\hat{T}_\mu^\nu\equiv-\chi\,\overline{K}_\mu^\nu$$

The ratio of determinants can be written :

(A2-68)
$$\frac{\sqrt{-g}}{\sqrt{-\overline{g}}}=\frac{\sqrt{-\det(g_{\mu\nu})}}{\sqrt{-\det(\overline{g}_{\mu\nu})}}=\frac{\sqrt{e^\nu e^\lambda r^4\sin^2\theta}}{\sqrt{e^{\overline{\nu}}e^{\overline{\lambda}}r^4\sin^2\theta}}=e^{\frac{\nu}{2}}e^{\frac{\lambda}{2}}e^{-\frac{\overline{\nu}}{2}}e^{-\frac{\overline{\lambda}}{2}}\equiv k_D$$

k_D will be taken to be little different from 1, as we're still in the Newtonian approximation. This time, we calculate the effect of the presence of positive masses on the geometry $\overline{g}_{\mu\nu}$ of the negative sector. Remember that the choice of this tensor $\overline{K}_{\mu\nu}$, is entirely up to you, insofar as it can be derived from a Lagrangian derivation. As we have seen, we opt for :
(A2-69)

$$\hat{T}_\mu^\nu=\begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & \frac{p}{c^2} & 0 & 0\\ 0 & 0 & \frac{p}{c^2} & 0\\ 0 & 0 & 0 & \frac{p}{c^2}\end{pmatrix}$$

This assumption does not affect the model as a whole, since in the Newtonian approximation, pressure terms are always negligible. This limits the scope of the model to the Newtonian approximation. But this covers all known observables.
We will show that this option no longer leads to the inconsistency pointed out by Damour in his paper.
We once again decline the construction of the first member from a metric which this time is :

(A2-70) $$d\bar{s}^2 = e^{\bar{\nu}} dx^{\circ 2} - e^{\bar{\lambda}} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

The first members of the equations are the same, simply replacing (ν , λ) par $(\bar{\nu} , \bar{\lambda})$.

(A2-71) $$e^{-\bar{\lambda}} \left(\frac{1}{r^2} - \frac{\bar{\lambda}'}{r} \right) - \frac{1}{r^2} = -\chi \rho$$

(A2-72) $$e^{-\bar{\lambda}} \left(\frac{1}{r^2} + \frac{\bar{\nu}'}{r} \right) - \frac{1}{r^2} = -\chi \frac{p}{c^2}$$

(A2-73) $$e^{-\bar{\lambda}} \left[\frac{\bar{\nu}''}{2} - \frac{\bar{\nu}'\bar{\lambda}'}{4} + \frac{\bar{\nu}'^2}{4} + \frac{\bar{\nu}' - \bar{\lambda}'}{2r} \right] = -\chi \frac{p}{c^2}$$

(A2-74) $$- \frac{\bar{\nu}' + \bar{\lambda}'}{r} e^{-\bar{\lambda}} = - \chi \left(\rho - \frac{p}{c^2} \right)$$

(A2-75) $$\frac{e^{\bar{\lambda}}}{r^2} = \frac{1}{r^2} - \frac{\bar{\nu}'^2}{4} + \frac{\bar{\nu}'\bar{\lambda}'}{4} + \frac{\bar{\nu}' + \bar{\lambda}'}{2r} - \frac{\bar{\nu}''}{2}$$

We pose

(A2-76) $$e^{-\bar{\lambda}} \equiv 1 - \frac{2\bar{m}}{r} \text{ soit } 2\bar{m} = r\left(1 - e^{-\bar{\lambda}}\right)$$

(A2-77) $$2\bar{m}' = \left(1 - e^{-\bar{\lambda}}\right) + r\bar{\lambda}' e^{-\bar{\lambda}} \quad \Rightarrow \quad -\frac{2\bar{m}'}{r^2} = -\frac{1}{r^2} + e^{-\bar{\lambda}} \left(\frac{1}{r^2} - \frac{\bar{\lambda}'}{r} \right)$$

$$\bar{m}' = -4\pi r^2 \frac{G}{c^2} \rho \quad \Rightarrow \quad \bar{m}_{(r)} = \int_0^r \bar{m}'_{(r)} dr = -\frac{4}{3}\pi r^3 \rho \frac{G}{c^2} = -m$$

Then :

(A2-78) $$\bar{m}_{(r)} = - m_{(r)}$$

(A2-79)

$$\bar{\nu}' = 2 \frac{-m + 4\pi G p r^3 / c^4}{r\left(r + 2m\right)}$$

To eliminate $\bar{\nu}''$ we dérive (A2-72)

(A2-80)
$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-\bar{\lambda}'e^{-\bar{\lambda}}\left(\frac{1}{r^2}+\frac{\bar{\nu}'}{r}\right)+e^{-\bar{\lambda}}\left(\frac{-2}{r^3}+\frac{\bar{\nu}''}{r}-\frac{\bar{\nu}'}{r^2}\right)$$

$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-e^{-\bar{\lambda}}\left(\frac{\bar{\lambda}'}{r^2}+\frac{\bar{\lambda}'\bar{\nu}'}{r}+\frac{2}{r^3}-\frac{\bar{\nu}''}{r}+\frac{\bar{\nu}'}{r^2}\right)$$

$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-2\frac{e^{-\bar{\lambda}}}{r}\left(\frac{\bar{\lambda}'}{2r}+\frac{\bar{\lambda}'\bar{\nu}'}{2}+\frac{1}{r^2}-\frac{\bar{\nu}''}{2}+\frac{\bar{\nu}'}{2r}\right)$$

$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-2\frac{e^{-\bar{\lambda}}}{r}\left(\frac{1}{r^2}-\frac{\bar{\nu}'^2}{4}+\frac{\bar{\lambda}'\bar{\nu}'}{4}+\frac{\bar{\lambda}'+\bar{\nu}'}{2r}-\frac{\bar{\nu}''}{2}+\frac{\bar{\nu}'^2}{4}+\frac{\bar{\lambda}'\bar{\nu}'}{4}\right)$$

We get :

(A2-81)
$$-\chi\frac{p'}{c^2}=\frac{2}{r^3}-2\frac{e^{-\bar{\lambda}}}{r}\left(\frac{e^{\bar{\lambda}}}{r^2}+\frac{\bar{\nu}'^2}{4}+\frac{\bar{\lambda}'\bar{\nu}'}{4}\right)=-2\frac{e^{-\bar{\lambda}}}{r}\left(\frac{\bar{\nu}'^2}{4}+\frac{\bar{\lambda}'\bar{\nu}'}{4}\right)$$

(A2-82)
$$-\chi\frac{p'}{c^2}=-\frac{e^{-\bar{\lambda}}\bar{\nu}'}{2r}\left(\bar{\nu}'+\bar{\lambda}'\right)$$

Using (A2-74)

(A2-83) :
$$-\chi\frac{p'}{c^2}=-\frac{\left(\bar{\nu}'+\bar{\lambda}'\right)}{r}e^{-\bar{\lambda}}\frac{\bar{\nu}'}{2}=-\chi\left(\rho-\frac{p}{c^2}\right)\frac{\bar{\nu}'}{2}$$

Finally :

(A2-84) :
$$\boxed{\frac{p'}{c^2}=-\frac{m-4\pi Gpr^3/c^4}{r\left(r+2m\right)}\left(\rho-\frac{p}{c^2}\right)}$$

Compare this with what emerged from the analysis for positive masses, i.e. the equation (A2-39) :

$$\boxed{\frac{p'}{c^2}=-\frac{m+4\pi Gpr^3/c^4}{r\left(r-2m\right)}\left(\rho+\frac{p}{c^2}\right)}$$

We've framed these two results because that's exactly what you wanted to show.
These differential equations are not identical, unless you use the Newtonian approximation, in which case they lead to the same result:

(A2-85) :
$$p'=-\frac{m\,\rho\,c^2}{r^2}$$

As T.Damour observes in 2022, with a three-year delay, the physical and mathematical incoherence of the model disappears. One might object that this limits solutions to those that fit this Newtonian approximation. But in cosmology, what more can you ask for?
Better a model that provides computational results limited to the conditions of the Newtonian approximation (i.e. to all observationally available data) than an extremely ambitious model

(Damour and Kogan 2002) that promises us non-linear solutions but, in the end, offers no possible confrontation with observations.
As before, we're going to finalize the calculation of the interior metric of the negative species. We won't omit any calculation intermediary to make sure that an error (it happens quickly) doesn't creep into the process.

(A2-86) $$\bar{\nu}' = \frac{2p'}{\left(\rho c^2 - p\right)}$$

To express the internal metric:

(A2-87) $$e^{-\bar{\lambda}} = 1 - \frac{2\bar{m}}{r} = 1 + \frac{r^2}{\hat{R}^2}$$

ρ is constant, then :

(A2-88) $$\bar{\nu}' = \frac{-2p'}{\left(-\rho c^2 + p\right)} = -2\frac{\left(\rho c^2 - p\right)'}{\left(\rho c^2 - p\right)} = -2\,\mathrm{Log}(\rho c^2 - p)'$$

(A2-89) $$-\frac{\bar{\nu}}{2} = \mathrm{Log}(\rho c^2 - p)' + \mathrm{cte}$$

We pose :

(A2-90) $$\underset{\cdot}{D} e^{-\frac{\bar{\nu}}{2}} = -\chi\left(\rho - \frac{p}{c^2}\right)$$

(A2-91) $$\bar{D} e^{-\frac{\bar{\nu}}{2}} = -\chi\left(\rho - \frac{p}{c^2}\right) = -\frac{\bar{\nu}' + \bar{\lambda}'}{r} e^{-\bar{\lambda}}$$

(A2-92) $$-r\bar{D} e^{-\frac{\bar{\nu}}{2}} = \bar{\nu}' e^{-\bar{\lambda}} - \left(e^{-\bar{\lambda}}\right)'$$

(A2-93) $$-r\bar{D} e^{-\frac{\bar{\nu}}{2}} = \bar{\nu}'\left(1 + \frac{r^2}{\hat{R}^2}\right) - \left(1 + \frac{r^2}{\hat{R}^2}\right)' = \bar{\nu}'\left(1 + \frac{r^2}{\hat{R}^2}\right) - \frac{2r}{\hat{R}^2}$$

On pose :

(A2-94) $$e^{\frac{\bar{\nu}}{2}} \equiv \bar{\gamma}(r) \quad \rightarrow \quad \bar{\gamma}' = \frac{\bar{\nu}'}{2} e^{\frac{\bar{\nu}}{2}}$$

We get :

(A2-95) $$-r\bar{D} = 2\,\frac{\bar{\nu}'}{2} e^{\frac{\bar{\nu}}{2}}\left(1 + \frac{r^2}{\hat{R}^2}\right) - \frac{2r}{\hat{R}^2} e^{\frac{\bar{\nu}}{2}} = 2\bar{\gamma}'\left(1 + \frac{r^2}{\hat{R}^2}\right) - \frac{2r}{\hat{R}^2}\bar{\gamma}$$

To express the interior metric, aparticular solution of this differential equation is :

(A2-96) $$\bar{\gamma}_p = \frac{\hat{R}^2\,\bar{D}}{2}$$

We need to find the general solution to the homogeneous equation:

(A2-97) $$u'\left(1+\frac{r^2}{\hat{R}^2}\right)-\frac{r}{\hat{R}^2}u=0$$

That is :

(A2-98) $$u=\overline{B}\left(1+\frac{r^2}{\hat{R}^2}\right)^{1/2}$$

The the general solution becomes :

(A2-99) $$\overline{\gamma}\equiv e^{\frac{\overline{\nu}}{2}}=\frac{\hat{R}^2\overline{D}}{2}+\overline{B}\left(1+\frac{r^2}{\hat{R}^2}\right)^{1/2}$$

Let's build the elements of the metric : $\overline{g}_{\mu\nu}$:

(A2-100) $$\overline{g}_{00}=e^{\overline{\nu}}=\left[\overline{A}+\overline{B}\left(1+\frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^2$$

Where :

(A2-101) $$\frac{\hat{R}^2\overline{D}}{2}\equiv\overline{A}\quad\Rightarrow\quad\overline{D}=2\frac{\overline{A}}{\hat{R}^2}=2\,\frac{8\pi G\rho}{3c^2}\,\overline{A}=-\chi\frac{2\rho}{3}\,\overline{A}$$

We have :

(A2-102) $$\overline{D}e^{-\frac{\overline{\nu}}{2}}=-\chi(\rho-\frac{p}{c^2})=-\chi\frac{2\rho}{3}\overline{A}\;e^{-\frac{\overline{\nu}}{2}}=-\chi\,\frac{2\rho}{3}\frac{\overline{A}}{\overline{A}+\overline{B}\left(1+\frac{r^2}{\hat{R}^2}\right)^{1/2}}$$

(A2-103) $$(\rho-\frac{p}{c^2})=\frac{2\rho}{3}\frac{\overline{A}}{\overline{A}+\overline{B}\left(1+\frac{r^2}{\hat{R}^2}\right)^{1/2}}$$

Pressure is expressed as zero at the surface of the sphere.

(A2-104) $$\overline{A}=-3\,\overline{B}\left(1+\frac{r_s^2}{\hat{R}^2}\right)^{1/2}$$

To determine B, we'll make sure there's a continuous connection between the inner and outer metrics, by $r = r_s$

We have :

(A2-105) $$\overline{g}_{11}^{int}=-e^{\overline{\lambda}}=-\left(1+\frac{r^2}{\hat{R}^2}\right)^{-1}$$

(A2-106) $$\overline{g}_{00}^{int}(r_0)=e^{\overline{\nu}(r_s)}=\left[\overline{A}+\overline{B}\left(1+\frac{r_s^2}{\hat{R}^2}\right)^{1/2}\right]^2=\overline{g}_{00}^{ext}(r_s)=\left(1+\frac{r_s^2}{\hat{R}^2}\right)$$

(A2-107) $$\left[-3\,\overline{B}\left(1+\frac{r_0^2}{\hat{R}^2}\right)^{1/2}+\overline{B}\left(1+\frac{r_s^2}{\hat{R}^2}\right)^{1/2}\right]^2=\left(1+\frac{r_0^2}{\hat{R}^2}\right)=4\,\overline{B}^2\left(1+\frac{r_s^2}{\hat{R}^2}\right)$$

(A2-108)
$$\hat{B} = \frac{1}{2}$$

(A2-109)
$$\overline{A} = -\frac{3}{2}\left(1+\frac{r_s^2}{\hat{R}^2}\right)^{1/2}$$

(A2-110)
$$\overline{g}_{00}^{int}(r) = e^{\overline{\nu}} = \left[-\frac{3}{2}\left(1+\frac{r_s^2}{\hat{R}^2}\right)^{1/2} + \frac{1}{2}\left(1+\frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^2$$

Whence the final expression of the internal metric $\overline{g}_{\mu\nu}$
(A2-111)

$$d\overline{s}^2 = \left[\frac{3}{2}\left(1+\frac{r_s^2}{\hat{R}^2}\right)^{1/2} - \frac{1}{2}\left(1+\frac{r^2}{\hat{R}^2}\right)^{1/2}\right]^2 dx^{\circ 2} - \frac{dr^2}{1+\frac{r^2}{\hat{R}^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

That links to the external metric :

(A2-112)
$$d\underset{\cdot}{\overline{s}}^2 = \left(1+\frac{2GM}{c^2 r}\right)c^2dt^2 - \frac{dr^2}{1+\frac{2GM}{c^2 r}} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Under linearized form :

(A2-113)
$$d\overline{s}^2 = \left(1+\frac{3}{2}\frac{r_s^2}{\hat{R}^2} - \frac{1}{2}\frac{r^2}{\hat{R}^2}\right)dx^{\circ 2} - \left(1-\frac{r^2}{\hat{R}^2}\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

(A2-114)
$$d\underset{\cdot}{\overline{s}}^2 = \left(1+\frac{2GM}{c^2 r}\right)c^2dt^2 - \left(1-\frac{2GM}{c^2 r}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

This completes the reproduction of the detailed calculations, as I sent them to T.Damour in 2019. If he had agreed to a simple meeting, or if he had been aware of these calculations at that time, and not in 2022, three years later, we would have avoided the current unpleasant situation, six years after his first criticisms of the Janus model were put on line, supported by the dispatch to my home of a registered letter with acknowledgement of receipt and by the dispatch of his article, as stated in his letter "to anyone interested in the Janus model".
If we now want to consider the "Dipole Repeller" situation, interpreted as the presence of a conglomerate of negative mass at the center of the great voids of the very large-scale structure of the universe, we need to move on to the :

(A2-115)
$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \chi\, K_{\mu\nu}$$
$$\overline{R}_{\mu\nu} - \frac{1}{2} \overline{R}\, \overline{g}_{\mu\nu} = -\chi\, \overline{T}_{\mu\nu}$$

Metrics always have the form:

(A2-116) $$ds^2 = e^{\nu} dx^{\circ 2} - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2$$

(A2-117) $$d\bar{s}^2 = e^{\bar{\nu}} dx^{\circ 2} - e^{\bar{\lambda}} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2$$

We'll repeat all the previous calculations, starting with the components of the Ricci tensor $and\ s\ on\}$. At the end, we arrive at the expression for the metric of the positive mass population. For the external metric, we have :

(A2-118)

$$ds^2 = \left(1 + \frac{G\,|M|}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{1 + \frac{G\,|M|}{c^2 r}} - r^2 (d\theta^2 + sin^2\theta d\varphi^2)$$

For external metric, with:

(A2-119) $$\widehat{R^2} = \frac{3\,\bar{c}^2}{8\,\pi\,G\,|\bar{\rho}|}$$

$$ds^2 = \left[\frac{3}{2}\left(1 + \frac{{r_s}^2}{\widehat{R^2}}\right)^{1/2} - \frac{3}{2}\left(1 + \frac{r^2}{\widehat{R^2}}\right)^{1/2}\right]^2 - \frac{dr^2}{1 + \frac{r^2}{\widehat{R^2}}} - -r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

These two metrics reflect a repulsion, with respect to the control masses m, by this negative $\bar{M}$which corresponds to observations: galaxies flee this "dipole repeller." With respect to positive-energy photons, those that lend themselves to observation, this results in "negative lensing," which has the effect of attenuating the brightness of objects located in the background. Since we are in the Newtonian approximation, we can take the linearized form of the solutions. When it comes to positive masses, no one has ventured to calculate the geodesic trajectories of particles passing through them, which could only be considered for neutrinos. On the other hand, positive-energy photons interact with negative masses only through "antigravity" (inverse gravitational lensing effect or negative lensing) and therefore pass through these masses without hindrance. The development of the solution itself shows that the deflection effect is equivalent to that which would be produced by a negative, equivalent mass $\bar{M}(r)$, equal to that contained within a radius r and concentrated at the origin. Thus, the attenuation effect is zero for photons that pass through the negative mass, very close to its center. On the contrary, it will be maximal for photons whose trajectories are tangent to the outer boundary of this negative mass. Over time, a brightness map of the background sources will be established, in the region of the dipole repeller. It is predicted that the region of greatest attenuation will give a ring image.