On a generalized scalar product: reality of the energies and the brachistochrone problem

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Abstract

We highlight a generalized scalar product, including both the standard inner product and the PT-symmetric quantum mechanical one. Our only working hypothesis is to deal with real energies. We then consider the brachistochrone problem by computing the minimal transition time between two normalized and orthogonal states with respect to this generalized scalar product.

Keywords : Quantum mechanics, discrete symmetries, time evolution

1 Introduction

When one reads a book dealing with quantum mechanics, the first pages usually point out that this physics is based on the introduction of a positive definite Hermitian sesquilinear form (or scalar product) ensuring not only a positive norm but also real energies. If one introduces the matrix calculus, this results in a Hermitian Hamiltonian. And, in fact, this is *sufficient* to ensure that the energies are real.

In their 1998 paper [1], Bender and Boettcher have shown that there is an array of Hamiltonians which are not self-adjoint (with respect to the usual scalar product of quantum mechanics) and thus not realized through Hermitian matrices but which are nevertheless associated to real energies.

The reason they proposed this paradigm shift is actually twofold. Firstly, if quantum mechanics is based on a number of postulates (conservation of probabilities etc), only one of them has its origin in a mathematical constraint: it is the axiom of the self-adjoint Hamiltonian. Secondly, as already mentioned, this axiom is a *sufficient*, but not *necessary*, condition to ensure the real character of energies.

These non-Hermitian Hamiltonians having a real spectrum belong to a class called PT-symmetric Hamiltonians. In this denomination, P refers to spatial reflexion $x \to -x$ while the time reversal operator T is such that $t \to -t$. This last operator is traditionally chosen as the complex conjugation K. In this case, and consequently in Bender's version (for a review, see [2]), it is anti-unitary.
Bender showed that the PT-symmetric Hamiltonians are in fact CPT-symmetric ones in the sense that they are self-adjoint according to a new inner-product defined by (for more details, see [2])

$$\langle \psi_1 | \psi_2 \rangle \equiv \psi_1^{CPT} \, \psi_2 \tag{1}$$

where C is a linear operator such that

$$C^2 = I, \ , [C, PT] = [C, H] = 0 \tag{2}$$

The first relation in (2) implies that the eigenvalues of the C operator are simply ± 1. Consequently, this operator can be seen as a kind of charge conjugation.
Recently, attention has been paid [3] to the fact that T could be a unitary operator instead of being anti-unitary without losing coherence in the theory. The purpose of this paper is in particular to put in evidence the implications of such a (different) choice on inner products and energies.

In Section 2, we introduce a generalized sesquilinear form. However, we restrict the discussion to the case of Hamiltonians having two energy levels and thus to a two-dimensional subspace of the full Hilbert space. The reason is twofold: to be able to compare easily the different approaches and to prepare the ground for the study of the brachistochrone problem. It has indeed been shown in [4] that one needs only to work in the two-dimensional subspace spanned by the initial state $|\psi_I\rangle$ and the final state $|\psi_F\rangle$ to have the complete information on the minimal time transition between these two states.
In Section 3, we highlight the eigenvalues and eigenvectors of a complex Hamiltonian and the conditions to ensure that the energies are real. Imposing the eigenstates to be orthonormal will definitely fix the scalar product. We then specify three particular cases which allow us to find the standard scalar product (section 3.1), the one of CPT-symmetric quantum mechanics (section 3.2) and, finally, a possibility associated to a choice of unitary T (section 3.3).

We consider the brachistochrone problem in Section 4 by particularizing ourselves to the three types of quantum mechanics mentioned above (Sections 4.1, 4.2 and 4.3, respectively).
We finally conclude in Section 5.

2 A generalized scalar product

When dealing with two-dimensional spaces, the standard scalar product reads

$$\langle\psi_1|\psi_2\rangle = a_1^* a_2 + b_1^* b_2 \; ; \; |\psi_1\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \; ; \; |\psi_2\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{3}$$

where the notation $*$ stands for a complex conjugation. Obviously, this leads to a positive norm.
We can generalize this inner product through

$$\langle\psi_1|\psi_2\rangle = C_1 \; a_1^* a_2 + C_2 \; b_1^* a_2 + C_3 \; a_1^* b_2 + C_4 \; b_1^* b_2 \tag{4}$$

where C_1, C_2, C_3, C_4 are coefficients to be determined in the following.
We only impose on the norm to be real, which entails

$$C_1^* = C_1, C_4^* = C_4, C_3^* = C_2 \tag{5}$$

This constraint ensures that

$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^* \tag{6}$$

as well as the fact that two states differing by a phase are characterized by the same norm.
The adjoint $X^\dagger$ of an operator X is defined as usual by

$$\langle X\psi_1|\psi_2\rangle = \langle\psi_1|X^\dagger\psi_2\rangle \tag{7}$$

If X is realized through

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{8}$$

(A, B, C, D being complex numbers), then $X^\dagger$ is given by

$$X^\dagger = \frac{1}{C_1 C_4 - C_2 C_3} \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \tag{9}$$

with

$$D_1 = C_1C_4\ A^* - C_1C_3\ B^* + C_2C_4\ C^* - C_2C_3\ D^*$$
$$D_2 = C_3C_4\ (A^* - D^*) - C_3^2\ B^* + C_4^2\ C^*$$
$$D_3 = -C_1C_2\ (A^* - D^*) + C_1^2\ B^* - C_2^2\ C^*$$
$$D_4 = -C_2C_3\ A^* + C_1C_3\ B^* - C_2C_4\ C^* + C_1C_4\ D^*$$

3 Eigenvalues and eigenvectors of the Hamiltonian

We consider the most general Hamiltonian expressed as

$$H = \begin{pmatrix} r\ e^{i\theta} & s\ e^{i\phi} \\ v\ e^{i\beta} & u\ e^{i\lambda} \end{pmatrix} \tag{10}$$

Here, r, s, v, u belong to $\mathbb{R}^+$ while $\theta, \phi, \beta, \lambda$ are real numbers.
The corresponding eigenvalues are

$$E_\pm = \frac{re^{i\theta} + ue^{i\lambda} \pm \sqrt{(re^{i\theta} - ue^{i\lambda})^2 + 4sve^{i(\beta+\phi)}}}{2} \equiv \frac{1}{2}(re^{i\theta} + ue^{i\lambda} \pm w) \tag{11}$$

Despite the fact that the Hamiltonian is complex (and is Hermitian iff θ and λ are equal to 0 or π, $v = s$ and $\beta = -\phi$) these energies can be real if we impose

$$r\ \sin\theta = -u\ \sin\lambda \tag{12a}$$
$$r^2\ \sin 2\theta + u^2\ \sin 2\lambda + 2sv\ \sin(\beta + \phi) = 0 \tag{12b}$$
$$w^2 = (r\ \cos\theta - u\ \cos\lambda)^2 - 4r^2 \sin^2\theta + 4sv\ \cos(\beta + \phi) \geq 0 \tag{12c}$$

The eigenstates are given by

$$|\epsilon_+\rangle = y_+ \begin{pmatrix} \frac{e^{-i\beta}}{2v}(re^{i\theta} - ue^{i\lambda} + w) \\ 1 \end{pmatrix}, |\epsilon_-\rangle = y_- \begin{pmatrix} \frac{e^{-i\beta}}{2v}(re^{i\theta} - ue^{i\lambda} - w) \\ 1 \end{pmatrix} \tag{13}$$

The coefficients $y_\pm$ have to be fixed so that the relations

$$\langle\epsilon_\pm|\epsilon_\pm\rangle = 1,\ \langle\epsilon_\pm|\epsilon_\mp\rangle = 0 \tag{14}$$

are satisfied. We obtain

$$|y_{\pm}|^2 = \frac{2v\sqrt{sv\ \cos\beta\ \cos\phi - r^2\ \sin^2\theta}}{w^2\ \cos\beta \pm w(\cos\beta(r\ \cos\theta - u\ \cos\lambda) + 2r\ \sin\theta\ \sin\beta)} \tag{15}$$

The requests (14) also have consequences on the scalar product which is now determined by

$$C_3 = C_2 e^{-2i\beta} - 2iC_1 e^{-i\beta}\frac{r}{v}\ \sin\theta \tag{16a}$$

$$C_4 = C_1\frac{s}{v}e^{-i(\beta+\phi)} - C_2\frac{1}{v}e^{-i\beta}(r\cos\theta - u\ \cos\lambda) \tag{16b}$$

There are still two parameters to fix. One can choose to do this by requiring that the scalar product be a CPT-symmetry and that $C^2 = I$ (following the approach developed by Bender [2]). As P is realized [2] through the Pauli matrix σ_1 ,we must have

$$C = \begin{pmatrix} C_2 & C_1 \\ C_4 & C_3 \end{pmatrix} \tag{17}$$

and its square gives the identity back if

$$C_3 = -C_2\ ;\ C_1C_4 + C_2^2 = 1 \tag{18}$$

The first of these two conditions gives

$$C_2 = iC_1\frac{r\sin\theta}{v\cos\beta} \tag{19}$$

while the second of these two conditions leads to (taking account of Eq. (12b))

$$C_1 = \frac{v\ \cos\beta}{\sqrt{sv\ \cos\beta\ \cos\phi - r^2\ \sin^2\theta}} \tag{20}$$

The generalized scalar product thus finally reads

$$\langle\psi_1|\psi_2\rangle = \frac{(v\cos\beta\ a_1^*a_2 + ir\sin\theta(b_1^*a_2 - a_1^*b_2) + s\cos\phi\ b_1^*b_2)}{\sqrt{sv\ \cos\beta\ \cos\phi - r^2\ \sin^2\theta}} \tag{21}$$

Let's now look at three specific cases.

3.1 The standard quantum mechanics

This case corresponds to

$$\theta = \lambda = 0, \pi,\ v = s,\ \beta = -\phi \tag{22}$$

The constraints (12) are automatically satisfied and

$$w^2 = (r-u)^2 + 4s^2 \tag{23}$$

We recover in (21) the usual inner product (3).

3.2 The PT-symmetric quantum mechanics

As already mentioned, Bender [2] chose

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; T = K \tag{24}$$

(where K refers to the complex conjugation). Obviously, we have

$$P^2 = T^2 = I \tag{25}$$

and the PT-symmetric quantum mechanics is defined by requiring

$$PT\ H\ TP = H \tag{26}$$

The most general Hamiltonian satisfying this constraint is given by

$$H = \begin{pmatrix} re^{i\theta} & se^{i\phi} \\ se^{-i\phi} & re^{-i\theta} \end{pmatrix},\ r, s \in \mathbb{R}^+,\ \theta, \phi \in \mathbb{R} \tag{27}$$

Notice that the Bender Hamiltonian corresponds to this one with $\phi = 0$.
The operator (27) is found in the general formalism with

$$u = r,\ \lambda = -\theta,\ v = s,\ \beta = -\phi \tag{28}$$

The constraints (12) of reality of the energies come down to

$$s^2 - r^2 \sin^2\theta \geq 0 \tag{29}$$

This request is fulfilled if

1. $\sin\theta \in [-\frac{s}{r}, \frac{s}{r}]$ if $r \geq s$
2. $r \leq s$

We also get

$$w^2 = 4(s^2 - r^2\sin^2\theta) \tag{30}$$

and the energies are given by

$$E_\pm = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta} \tag{31}$$

The inner product (21) reduces to

$$\langle\psi_1|\psi_2\rangle = \frac{s\cos\phi(a_1^* a_2 + b_1^* b_2) + ir\sin\theta(b_1^* a_2 - a_1^* b_2)}{\sqrt{s^2\cos^2\phi - r^2\sin^2\theta}} \tag{32}$$

3.3 The Tu-symmetric quantum mechanics

Recently, a new version of quantum mechanics has been proposed [3]. It is based on a different choice in what concerns the time reversal operator. Indeed, we know since Wigner's work [5] that a symmetry operator is either unitary or anti-unitary. By opposition to the continuous symmetry operators which have to be unitary, the nature of the discrete symmetry operators is left to the choice of the physicists. In the context of non relativistic quantum mechanics, the time reversal operator T is (almost) unanimously identified with the complex conjugation K. This is the choice performed by Bender, for instance.

Let us thus turn ourselves to the other possibility, knowing that it is compatible with the conservation of the quantum mechanics laws [3]. If T is unitary, it can be realized with a matrix namely [3]

$$Tu = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{33}$$

It has then to anticommute with H, which gives rise to

$$H = \begin{pmatrix} re^{i\theta} & se^{i\phi} \\ se^{-i\phi} & -re^{i\theta} \end{pmatrix} \; ; \; r, s \in \mathbb{R}^+, \theta, \phi \in \mathbb{R} \tag{34}$$

This Hamiltonian is not Hermitian (unless if $\theta = \phi = 0$) and, under the Bender PT-symmetry, it is transformed like

$$PT\ H\ T^{-1}P^{-1} = \begin{pmatrix} -re^{-i\theta} & se^{-i\phi} \\ se^{-i\phi} & re^{-i\theta} \end{pmatrix} \tag{35}$$

It is thus not invariant unless $\theta = \frac{\pi}{2}, \phi = 0$.
It corresponds to the general case if we fix the parameters as follows

$$u = -r, \lambda = \theta, v = s, \beta = \phi \tag{36}$$

The reality of energies can be summarized by

$$r^2 \sin 2\theta + s^2 \sin 2\phi = 0 \tag{37a}$$

$$w^2 = 4(r^2 \cos 2\theta + s^2 \cos 2\phi) \geq 0 \tag{37b}$$

We have therefore three possible cases to ensure the reality of these energies:

1. $\cos 2\theta \geq 0, \cos 2\phi \geq 0$
2. $\cos 2\theta \geq 0, \cos 2\phi \leq 0, r \geq s$
3. $\cos 2\theta \leq 0, \cos 2\phi \geq 0, r \leq s$

As an example, we could take

$$r = \sqrt{13}, s = \sqrt{18.25}, \theta = \frac{5\pi}{16}, \phi = -\frac{4\pi}{35} \tag{38}$$

giving rise to

$$w = 5.9 \tag{39}$$

We are indeed facing a new version of quantum mechanics: the energies are real while this approach satisfies neither the constraints of standard quantum mechanics nor those of the Bender version. This is what we will call the Tu-symmetry quantum mechanics in the following.
The energies are in general

$$E_{\pm} = \pm\sqrt{r^2 \cos 2\theta + s^2 \cos 2\phi} \tag{40}$$

They thus have opposite signs, a common point with those pointed out by Dirac in his equation of relativistic quantum mechanics.
Let us conclude this Section by mentioning that the scalar product underlying the Tu-symmetry quantum mechanics is still the one given in (32).

4 The brachistochrone problem

The brachistochrone problem (from the Greek 'brakhisto' meaning 'shortest') is the problem of finding the minimum time ($t_{min} = \tau$) for which it is possible to go from an initial state $|\psi_I\rangle$ to a final state $|\psi_F\rangle$ through

$$|\psi_F\rangle = e^{-\frac{i}{\hbar}tH}\ |\psi_I\rangle \tag{41}$$

with $E_{max} - E_{min} = w$ fixed. Here, we consider only two states of energy, so that

$$E_{max} = E_+; E_{min} = E_- \tag{42}$$

and

$$w = \sqrt{(re^{i\theta} - ue^{i\lambda})^2 + 4sv\ e^{i(\beta+\phi)}} \tag{43}$$

This problem has been achieved in [6] in the standard context

$$w = \sqrt{(r-u)^2 + 4s^2}$$

by using the variational calculus.
An alternative approach has been considered in [4]. It is based on the Anandan-Aharonov relation [7] saying that the speed of a unitary evolution is proportional to the energy uncertainty:

$$\frac{2}{\bar{h}}\Delta H \tag{44}$$

This led to

$$\tau = \frac{\bar{h}\pi}{w} \tag{45}$$

if the two states $|\psi_I\rangle$ and $|\psi_F\rangle$ are orthogonal (and normalized).

Let's see what it becomes in the context of the generalized scalar product. First, we note that if $|\psi_I\rangle$ and $|\psi_F\rangle$ are two orthogonal and normalized states according to the inner product (21), any state $|\psi\rangle$ can be expressed as

$$|\psi\rangle = c_1|\psi_I\rangle + (c_2 + ic_3)|\psi_F\rangle,\ c_1, c_2, c_3 \in \mathbb{R} \tag{46}$$

This is due to the fact that two states differing by a phase are characterized by the same norm allowing to take the coefficient in front $|\psi_I\rangle$ of real instead of complex.

Moreover, taking account of (21) and asking for $|\psi\rangle$ to be normalized with respect to this inner product, we are led to the following constraint

$$c_1^2 + c_2^2 + c_3^2 = 1 \tag{47}$$

This relation is satisfied with

$$c_1 = \cos\frac{\rho}{2}, c_2 = -\sin\frac{\rho}{2}\sin\Phi, c_3 = \sin\frac{\rho}{2}\cos\Phi \tag{48}$$

where ρ and Φ are the well-known angles characterizing the Bloch sphere. This means that, in the same way as in the case of standard quantum mechanics [4], two orthogonal states are located at the north and south poles of this sphere.
We thus choose the (normalized) initial state $|\psi_I\rangle$ as

$$|\psi_I\rangle = \frac{1}{v\cos\beta}\begin{pmatrix} \sqrt{sv\cos\beta\cos\phi - r^2\sin^2\theta} \\ 0 \end{pmatrix} \tag{49}$$

and calculate the standard deviation of (10) in the state (49). We obtain

$$(\Delta H)^2 = \frac{w^2}{4} - (\frac{1}{2}(r\cos\theta - u\cos\lambda - 2ir\sin\theta) + ir\frac{\sin\theta}{\cos\beta}e^{-i\beta})^2 \tag{50}$$

with w given in (43).

Let us now focus on the three specific cases studied in Section 3.

4.1 The standard quantum mechanics

If we apply (22) to (50), we get

$$(\Delta H)^2 = \frac{w^2}{4} - \frac{1}{4}(r-u)^2 \tag{51}$$

The minimal time is recovered for a maximal standard deviation. We thus have to take

$$r = u \tag{52}$$

in which case we find the time (45). Note that the constraint (52) leads to extremely simple eigenstates:

$$|\epsilon_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm e^{i\phi} \\ 1 \end{pmatrix} \tag{53}$$

while $w = 2s$.

4.2 The PT-symmetric quantum mechanics

If we now implement (28) in (50), we obtain

$$(\Delta H)^2 = \frac{w^2}{4} - r^2 \sin^2\theta \frac{\sin^2\phi}{\cos^2\phi} \tag{54}$$

To ask for a maximal $(\Delta H)^2$ is therefore either to cancel θ (but then we fall back on the standard quantum mechanics, an option we avoid here), or to cancel ϕ (this is what was chosen by Bender [2]). Again, $\phi = 0$ is the case with the simplification of the eigenvectors:

$$|\epsilon_\pm\rangle = \frac{1}{\sqrt{2\cos\alpha}} \begin{pmatrix} \pm e^{\pm i\alpha} \\ 1 \end{pmatrix} \tag{55}$$

if

$$\sin\alpha \equiv \frac{r}{s\cos\phi}\sin\theta,\ \cos\alpha \equiv \frac{1}{s\cos\phi}\sqrt{s^2\cos^2\phi - r^2\sin^2\theta} \tag{56}$$

The difference between E_+ and E_- simplifies to

$$w = 2\sqrt{s^2 - r^2\sin^2\theta} \tag{57}$$

while the scalar product is characterized by

$$\langle\psi_1|\psi_2\rangle = \frac{1}{\cos\alpha}\left((a_1^* a_2 + b_1^* b_2) + i\sin\alpha(b_1^* a_2 - a_1^* b_2)\right) \tag{58}$$

Notice that, once again, we have

$$(\Delta H)^2 = \frac{w^2}{4} \tag{59}$$

and, consequently, the minimal time (45). This contradicts Bender's result [8] that PT-symmetric quantum mechanics allowed to decrease the transition time between two states, even going as far as making it zero. The reason is that in the paper [8], Bender chose the standard scalar product and that this one does not allow to orthonormalize the eigenvectors of the Hamiltonian.

4.3 The Tu-symmetric quantum mechanics

Taking account of (36) in (50), we are led to

$$(\Delta H)^2 = \frac{w^2}{4} - \frac{r^2\cos^2(\theta-\phi)}{\cos^2\phi} \tag{60}$$

One more time, we recover the same maximal standard deviation (59) and thus the same minimal transition time (45) provided

$$\phi = \theta \pm \frac{\pi}{2} \tag{61}$$

This choice reduces the difference between E_+ and E_- to

$$w = 2\sqrt{(r^2 - s^2)\cos 2\theta} \tag{62}$$

while the inner product now reads

$$\langle \psi_1 | \psi_2 \rangle = \frac{1}{\sqrt{s^2 - r^2}} \left(s(a_1^* a_2 + b_1^* b_2) + ir(b_1^* a_2 - a_1^* b_2) \right) \tag{63}$$

We note, in an obvious way, that if the scalar products were identical, generally speaking, for the PT- and Tu- symmetric versions of quantum mechanics, they differ as soon as additional constraints appear in order to ensure a minimal transition time between two states.

In conclusion of this section, we note that the minimal transition time between two states is the same whatever the version of quantum mechanics considered. It is precisely given by (45). This is true for the three particular cases treated here but it is also true in all generality. When we look at equation (50), we can easily convince ourselves that the maximum standard deviation is always the same, i.e. $\frac{w}{2}$, if the difference between the two energy levels is fixed at w.
The only way to improve this time, to decrease it, would be to lift our only working hypothesis, i.e. to stop considering real energies only.
Let us mention in a few words that this result can also be found in another way by returning to the equation (41). Indeed by using

$$e^{i\Sigma \vec{\sigma}.\vec{n}} = \cos \Sigma + i \sin \Sigma \ \vec{\sigma}.\vec{n} \tag{64}$$

where $\vec{\sigma}$ are the usual Pauli matrices and $\vec{n}$ a normalized vector (according to the usual scalar product), we can write the operator

$$e^{-\frac{i}{h} tH} \tag{65}$$

as

$$e^{-\frac{i}{2h} t(r\cos\theta + u\cos\lambda)} \begin{pmatrix} T_+ & T_1 \\ T_2 & T_- \end{pmatrix} \tag{66}$$

with

$$T_+ = \cos(\frac{tw}{2\bar{h}}) - \frac{i}{w}(r\cos\theta - u\cos\lambda + 2ir\sin\theta)\sin(\frac{tw}{2\bar{h}})$$
$$T_- = \cos(\frac{tw}{2\bar{h}}) + \frac{i}{w}(r\cos\theta - u\cos\lambda + 2ir\sin\theta)\sin(\frac{tw}{2\bar{h}})$$
$$T_1 = -\frac{2i}{w}se^{i\phi}\sin(\frac{tw}{2\bar{h}})$$
$$T_2 = -\frac{2i}{w}ve^{i\beta}\sin(\frac{tw}{2\bar{h}})$$

Acting this operator on the state (49), we fix the final state as

$$|\psi_F\rangle = -\frac{i}{w}\, e^{-\frac{i}{2\bar{h}}t(r\cos\theta + u\cos\lambda)}\, \frac{\sqrt{sv\cos\beta\cos\phi - r^2\sin^2\theta}}{v\cos\beta} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{67}$$

with

$$\psi_1 = r\cos\theta - u\cos\lambda + 2ir\sin\theta$$
$$\psi_2 = 2ve^{i\beta}$$

This result takes into account the fact that the final state must be normalized and possibly orthogonal to the initial state. Such an demand also implies

$$\cos(\frac{tw}{2\bar{h}}) = 0 \tag{68}$$

or the minimum standard time (45) which appears once again, confirming our previously mentioned results.

5 Conclusion

Our goal was to show that it is possible to point out real energies if we abandon the (sufficient but not necessary) hypothesis of working with Hamiltonians realized through Hermitian matrices. This observation had already been made by Bender and Boettcher in 1998 [1] when they highlighted a quantum mechanics which gives pride of place to discrete P and T symmetries.
What we are convinced of here is that this version of quantum mechanics is one of many possible. We have pointed, in particular, to the possibility of a quantum mechanics associated with a unitary time reversal operator. Real energies do appear: they are even opposed just like those of the Dirac

equation.
Each version of quantum mechanics has two points in common: the reality of energies and the minimal time to go from one quantum state to another, identical in all versions. What differentiates them is the underlying scalar product, allowing to consider a multitude of Hamiltonians which would not have been studied in the standard version because they are not Hermitian.

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