

# SCALE INVARIANT COSMOLOGY

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## **Abstract**

An attempt is made here to extend to the microscopic domain the scale invariant character of gravitation – which amounts to consider expansion as applying to any physical scale. Surprisingly, this hypothesis does not prevent the redshift from being obtained. It leads to strong restrictions concerning the choice between the presently available cosmological models and to new considerations about the notion of time. Moreover, there is no horizon problem and resorting to inflation is not necessary.

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## 1. Introduction

Since Hubble's discovery of the recession of galaxies, obtaining the variation with cosmic time of the scale parameter (or "radius")  $R(t)$  which describes the expansion of the universe is the basic problem a cosmological theory has to deal with. The closely related question of deciding relative to what the expansion takes place, or where it stops, is much more seldom explicitly stated [1]. It is commonly admitted there is no expansion at distances smaller than the size of clusters of galaxies, so that this size, or the size of our galaxy, or of the solar system, ... can be used as a reference scale. This may seem to be sensible at present, but what about the past, when  $R(t)$  was smaller than these objects? The existence and observability of permanent reference scales for time and length intervals thus appear to be assumed. It is also commonly admitted – but this argument has been questioned [2,3] – that the scale parameter cannot be applied to all length scales in the universe, on the grounds that this would amount to getting no expansion at all, so that the universe would be scale invariant and there would be no redshift. On the other hand, if the inflation theory eliminates the causality problem for the presently observed regions of the universe, it does not in the future ... In order to solve these problems, L. Nottale [1] introduces a transition from scale independence to scale dependence in the general frame of a theory of scale relativity.

Already Laplace [1, 4] pointed out the scale invariance of Newton's theory of gravitation, concluding that "the universe reduced to the smallest imaginable space would always present the same appearance to observers" and that "the laws of nature only permit us to observe relative dimensions". Poincaré made similar observations as concerns comparing time intervals as well as lengths at two different instants (and this is, of course, the only way to proceed for time intervals): he notes "we have no direct intuition of the equality of two lapses of time" – for instance those "between noon and one o'clock and between two and three o'clock" [5]. So it should be possible to allow a general variation of all length [2] and time intervals, as long as one keeps in mind that this kind of variation cannot be physically observed by comparisons necessarily performed at the same instant: there would be no permanent observable reference scale; since this looks as an unnecessary complication, the possibility for expansion to act at any scale has seldom been considered seriously. However, Hoyle and Narlikar [6], as an alternative to expansion, already used a somewhat similar view in their variable mass theory, where all masses are supposed to increase with cosmic time, which implies decreasing wavelengths for atomic radiations and decreasing sizes for atoms. Indeed, it may be observed that, as opposed to the case of pure gravitation, where the size and the period of a Keplerian orbit can be changed arbitrarily as long as Kepler's laws are respected, without any modification of Newton's constant nor of the masses of the bodies, physical constants must be allowed to change if the expansion is considered to apply, for instance, to structures the size of which is directly linked to atomic sizes. This is clear from the expression of the Bohr radius,

$$a_0 = \frac{h^2 \epsilon_0}{\pi m_e e^2} = \frac{\hbar}{m_e \alpha c} \quad , \quad (1)$$

which involves physical constants only: supposing the absence of permanent observable

reference scales for length and time intervals implies some physical constants, among which  $c$ , can vary with cosmic time.

So, since  $c$  also appears in the expression of the metric tensor, we shall suppose it to be variable and we shall look for a function  $R(t)$  which already results from one of the classical cosmological theories: such a function should not depend on hypotheses which can be made about the reference scale relative to which  $R(t)$  is defined. This leads to a highly constrained problem which cannot be solved realistically (i.e. with non-zero pressure and density) unless more than four dimensions are introduced in the description of space-time. This idea, initially due to Einstein and recently reintroduced by P. Wesson and his co-workers [7, 8], aims at obtaining the properties of matter in our four dimensional space-time in a purely geometrical way, starting from Einstein's equations

$$R_{\mu\nu} = 0 \quad (2)$$

written for five dimensions or more with no matter-energy terms on the right-hand side to find the usual 4-dimensional equations

$$R_{\mu\nu}^{(1+3)} = -\frac{8\pi G}{c^2} S_{\mu\nu} = -\frac{8\pi G}{c^2} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda) \quad (2')$$

where  $T_{\mu\nu}$  is the energy-momentum tensor of a perfect fluid, and we shall try it together with the hypothesis of a variable light velocity. Even though introducing such a geometrization of matter through a variation of the single parameter  $c$  may appear as an oversimplifying supposition, it seems to be a natural one, since if the properties of matter are to be geometrized, the physical constants they are related to should be allowed to depend on  $R(t)$  – which, together with the curvature index, is here (see §2) the only characteristic parameter of geometry – and so on  $t$  itself.

## 2. The Einstein equations

We write Einstein's equations for an arbitrary number  $n$  of space dimensions and with a variable  $c$  in a maximally symmetric space, hence with a metric which, as concerns space, generalizes to  $n$  dimensions the metric of Robertson-Walker. Choosing such a metric with  $n$  greater than 3 is obviously surprising – usually a different scale factor is associated with space dimensions beyond the third one – and this will be discussed later (§§5 and 6).

It is to be noted that, as a consequence of our hypotheses, the geometrical space-time coordinates – defined with respect to permanent reference scales – will appear to be distinct from the physically observable ones. This is obvious for lengths: expansion applying to spatial domains of arbitrary size, physical distances here are the comoving ones; the case of time intervals will be dealt with in §§3 and 4. So as to make this distinction easier, geometrical coordinates or intervals will generally be written using Latin characters, and physical coordinates or intervals using Greek ones.

Designing by 0 the index corresponding to the chronological coordinate  $x^0$ , with  $dx^0 = c(t)dt$ , and by  $i, j = 1, 2, \dots, n$  the indices associated with the spatial coordinates, the

components of the metric tensor are:

$$g_{00} = -1 \quad (3)$$

$$g_{i0} = 0 \quad (4)$$

$$g_{ij} = R^2(t)\tilde{g}_{ij}, \quad \tilde{g}_{ij} = 0 \quad (i \neq j) \quad (5)$$

where the  $\tilde{g}_{ij}$  depend on the space coordinates only. Choosing  $x^0$  rather than  $t$  and taking into account the time dependence of  $c$  at the end of the calculations allow one to get a constant  $g_{00}$  and thus a smaller number of non-zero components for the affine connection. Starting from

$$\Gamma_{\gamma\alpha}^{\beta} = \frac{1}{2}g^{\beta\eta} [\partial_{\gamma}g_{\alpha\eta} + \partial_{\alpha}g_{\eta\gamma} - \partial_{\eta}g_{\gamma\alpha}] \quad (6)$$

(where the four indices run from 0 to  $n$ ), these components are found to be (the ' designating a derivation with respect to  $x^0$ ):

$$\Gamma_{ki}^0 = RR'\tilde{g}_{ki} \quad (7)$$

$$\Gamma_{0i}^j = \Gamma_{i0}^j = \frac{R'}{R}\delta_i^j \quad (8)$$

$$\Gamma_{ki}^j = \tilde{\Gamma}_{ki}^j \quad , \quad (9)$$

so the  $\Gamma_{ki}^j$  are time-independent.

This yields for the Ricci tensor

$$R_{\alpha\beta} = \partial_{\beta}\Gamma_{\eta\alpha}^{\eta} - \partial_{\eta}\Gamma_{\alpha\beta}^{\eta} + \Gamma_{\alpha\eta}^{\lambda}\Gamma_{\beta\lambda}^{\eta} - \Gamma_{\eta\lambda}^{\alpha}\Gamma_{\alpha\beta}^{\lambda} \quad , \quad (10)$$

the non-vanishing components

$$R_{00} = n\frac{R''}{R} \quad (11)$$

$$R_{ij} = \tilde{R}_{ij} - (RR'' + (n-1)R'^2)\tilde{g}_{ij} \quad (12)$$

where [9],  $k = 0, \pm 1$  being the curvature index,

$$\tilde{R}_{ij} = -(n-1)k\tilde{g}_{ij} \quad , \quad (13)$$

hence

$$R_{ij} = -[RR'' + (n-1)R'^2 + (n-1)k]\tilde{g}_{ij} \quad . \quad (14)$$

Taking into account the hypothesis of a variable  $c$  and introducing the time variable  $t$  gives:

$$R' = \frac{dR}{dx^0} \quad (15)$$

and

$$R'' = \frac{d^2R}{dx^{02}} = \frac{1}{c^2}\frac{d^2R}{dt^2} - \frac{1}{c^3}\frac{dc}{dt}\frac{dR}{dt} \quad , \quad (16)$$

so that (11) and (14) become

$$R_{00} = \frac{n}{Rc^2} (R'' - \frac{c'}{c}R') \quad (17)$$

and

$$R_{ij} = -\left[\frac{R}{c^2}(R'' - \frac{c'}{c}R') + (n-1)(\frac{R'^2}{c^2} + k)\right] \tilde{g}_{ij} \quad , \quad (18)$$

where the prime now stands for the derivation relative to  $t$ .

Of course, the same result might have been obtained using  $t$  instead of  $x^0$  from the beginning, getting

$$g_{tt} = -c^2(t)$$

instead of (3), a non-zero

$$\Gamma_{tt}^t = \frac{c'}{c}$$

and

$$\Gamma_{ki}^t = \frac{RR'}{c^2} \tilde{g}_{ki}$$

instead of (7).

Applying (2) to (17) and (18) immediately entails

$$k = -1 \quad (19)$$

if the density is to be non-zero, so that the  $n$ -space curvature is negative, and

$$R'^2 = c^2 \quad , \quad (20)$$

hence

$$R' = \epsilon c \quad (\epsilon = \pm 1), \quad (21)$$

$$R'' = \epsilon c' \quad , \quad (22)$$

and

$$\frac{c'}{c} = \frac{R''}{R'} \quad . \quad (23)$$

Noting

$$R_{00}^{(F;1+3)} = \frac{3R''}{Rc^2} \quad (24)$$

and

$$R_{ij}^{(F;1+3)} = -\frac{1}{c^2}(RR'' + 2R'^2) \tilde{g}_{ij} \quad (i, j = 1, 3) \quad (24')$$

the parts of the Ricci tensor components which correspond to the Friedmann model with zero curvature and using equations (19) to (23), the time-time (17) and space-space (18) components of the Ricci tensor can be written:

$$R_{00} = R_{00}^{(F;1+3)} - \frac{3\epsilon c'}{Rc^2} \quad (25)$$

$$R_{ij} = R_{ij}^{(F;1+3)} + (2 + \epsilon \frac{Rc'}{c^2}) \tilde{g}_{ij} \quad (i, j = 1, 3) \quad . \quad (25')$$

Identifying (2) with (2') where the source term is:

$$S_{00} = \frac{1}{2}(\rho + \frac{3p}{c^2}) \quad (26)$$

$$S_{i0} = 0 \quad (27)$$

$$S_{ij} = \frac{1}{2}(\rho - \frac{p}{c^2})R^2 \tilde{g}_{ij} \quad (28)$$

gives:

$$\frac{4\pi G}{c^2}(\rho + \frac{3p}{c^2}) = -\frac{3\epsilon c'}{Rc^2} \quad (29)$$

$$\frac{4\pi G}{c^2}(\rho - \frac{p}{c^2}) = (2 + \epsilon \frac{Rc'}{c^2})\frac{1}{R^2} \quad . \quad (30)$$

Of course, for this identification to be possible, the negative curvature  $n$ -space has to contain an Euclidean three dimensional variety: an example of this situation will be given later (§ 5).

Both  $R$  and  $c$  depending on  $t$ , there must be a relation between them. Writing, with  $K$  constant,

$$\phi(R, c) = K \quad (31)$$

hence

$$\frac{\partial \phi}{\partial R} R' + \frac{\partial \phi}{\partial c} c' = 0 \quad (32)$$

and, using (21),

$$\frac{c'}{c} = -\epsilon \frac{\partial \phi / \partial R}{\partial \phi / \partial c} \quad (33)$$

(29) and (30) give:

$$\frac{\rho c^2 + 3p}{\rho c^2 - p} = \frac{3\delta}{2 - \delta}, \quad (34)$$

where

$$\delta = \frac{R}{c} \frac{\partial \phi / \partial R}{\partial \phi / \partial c} \quad (35)$$

So, for an equation of state (34) with  $\delta$  a constant,

$$\frac{dc}{dR} = -\frac{\partial \phi / \partial R}{\partial \phi / \partial c} = -\delta \frac{c}{R} \quad (36)$$

and

$$cR^\delta = K' \quad (37)$$

or, with  $\gamma = 1/\delta$ ,

$$Rc^\gamma = K'' \quad (38)$$

where  $K'$  and  $K''$  are constant.

The equation of state (34) can be given the form

$$p = \left(\frac{2}{\gamma} - 1\right) \frac{\rho c^2}{3} \quad (39)$$

and includes the two cases

$$p = 0 \quad (40)$$

and

$$p = \frac{\rho c^2}{3} \quad (40')$$

of the dust universe and of the radiation era for  $\gamma = 2$  and 1 respectively.

Whatever  $\gamma$ , (29) and (30) yield

$$\frac{8\pi G}{3}\rho = \frac{R'^2}{R^2} \quad , \quad (41)$$

so that the density of matter  $\rho$  corresponds, as results from the elimination procedure leading to (41), to the curvature term in the expressions (18) or (25') of  $R_{ij}$ . Introducing the Hubble constant

$$H = \frac{R'}{R} \quad , \quad (42)$$

(41) writes

$$\rho G = \frac{3}{8\pi} H^2 \quad (43)$$

– so the density equals the critical density. This gives for the density of energy  $\varepsilon = \rho c^2$ :

$$\varepsilon = \frac{3c^2}{8\pi G} H^2 \quad . \quad (43')$$

Of course, the pressure is a function of  $\gamma$ :

$$p = \frac{c^2}{8\pi G} \left(\frac{2}{\gamma} - 1\right) H^2 \quad . \quad (44)$$

The scale parameter  $R(t)$  is easily obtained deriving (38) with respect to time and using (23) to get

$$R'' + \frac{R'^2}{\gamma R} = 0 \quad (45)$$

from which it can be seen that  $\delta$  is but the deacceleration parameter  $q$ . The non-trivial solution of (45) which satisfies

$$R(0) = 0 \quad (46)$$

and

$$R(t_0) = R_0 \quad (46')$$

is:

$$R(t) = R_0 \left( \frac{t}{t_0} \right)^{\frac{\gamma}{\gamma+1}}, \quad (47)$$

as in the zero curvature Friedmann model for an equation of state of the type (39).

Thus for the radiation era ( $\gamma = 1$ ),

$$R(t) = R_0 \left( \frac{t}{t_0} \right)^{1/2} \quad (48)$$

and for the dust universe ( $\gamma = 2$ ),

$$R(t) = R_0 \left( \frac{t}{t_0} \right)^{2/3}. \quad (49)$$

### 3. Time and the redshift

In the same way as lengths remain proportional to the cosmological scale factor  $R(t)$ , intervals of cosmic time vary as  $t$  itself: for instance, the period  $T = l/c$  of a radiation of wavelength  $l$  varies as

$$T \approx \frac{R}{c} \approx R^{\frac{\gamma+1}{\gamma}} \approx t \quad (50)$$

and the same kind of variation will characterize the period of any circular motion, since its speed can be defined as a fraction of  $c$ , and thus varies as  $R^{-1/\gamma}$ , as also results for various examples (period of a pendulum, of planetary motions, ...) from the gauge relations described in §7: every time interval expands proportionally to cosmic time with respect to a permanent reference scale.

At a given  $t$ , a cosmic time interval  $dt$  can but be defined as proportional to  $t$  and to the number  $d\vartheta$  of periods a reference clock completes during this interval  $dt$ :

$$dt = k t d\vartheta \quad (51)$$

where  $k$  is a constant, the dimension of which should be the inverse of a time if  $\vartheta$  is to be a time also. Obviously, intervals of  $\vartheta$  only (and of the associated proper time), not of  $t$ , can be measured.

Let us now state the nature of  $\vartheta$  more precisely. The square of the (1 + 3) space-time interval with zero space curvature writes, following from now on the timelike convention:

$$ds^2 = c^2(t)dt^2 - dl^2 \quad (52)$$

where  $dl$  is the space interval, or, introducing  $R(t)$ , which we suppose to have the dimension of a length:

$$ds^2 = c^2(t)dt^2 - R^2(t)d\xi^2 \quad (52')$$

and here  $d\xi$  stands for the spatial distance in dimensionless comoving coordinates.

Computing  $c(t)$  (with  $\epsilon = 1$ ) from (21) and (47):

$$c(t) = \frac{\gamma}{\gamma + 1} \frac{R_0}{t_0} \left( \frac{t}{t_0} \right)^{-\frac{1}{\gamma+1}} \quad (53)$$

and writing in (51)

$$k = \frac{\gamma + 1}{\gamma} \frac{c_M}{R_M} \quad (54)$$

with

$$R(t) = R_M a(t) \quad (55)$$

where  $R_M$  and  $c_M$  are constant and  $a(t)$  is dimensionless, yields

$$ds^2 = a^2(t) [c_M^2 d\vartheta^2 - R_M^2 d\xi^2]$$

or

$$ds^2 = a^2(t) [c_M^2 d\vartheta^2 - d\lambda^2] \quad (56)$$

with

$$d\lambda = R_M d\xi \quad (57)$$

to give the spatial distance  $d\lambda$  the dimension of a length. Thus  $\vartheta$  can be identified with the conformal time, which gives the metric an expression of the conformal type, and  $c_M$  can be interpreted as the constant light velocity in the Minkowski space-time which is measured when using coordinates  $\vartheta$  and  $\lambda$ . It may be noted that, up to a scaling factor, a unique definition of the conformal time is obtained here for the dust universe as well as the for the radiation one – which does not happen in the classical theory.

So as to study the formation of the redshift, let us consider, as in [10], two events occurring at the same point in space and separated by a time interval corresponding to (56):

$$\Delta s_1 = c_M a(t_1) \Delta \vartheta \quad , \quad (58)$$

where this time interval is small as compared to time  $t_1$ . If these two events consist in the emission of two light signals which will be perceived at another point in space, the time interval between the two instants of reception will be the same in conformal time as in (58), that is,  $\Delta \vartheta$ . This results directly from (56): light propagation is described by

$$d\lambda = \pm c_M d\vartheta \quad (59)$$

so that, for a propagation along the radial coordinate  $\chi$ ,

$$d\lambda^2 = R_M^2 d\chi^2 \quad , \quad (60)$$

$$d\vartheta = \pm \frac{R_M}{c_M} d\chi \quad (61)$$

hence

$$\vartheta = \pm \frac{R_M}{c_M} \chi + \mathcal{C} \quad , \quad (62)$$

where  $\mathcal{C}$  is constant: the conformal time needed to go from the emission point to the reception one depends on the difference of the radial coordinates of these points only and not on the instant of emission, so both signals will arrive at the (same) reception point at two instants separated by the same conformal time interval  $\Delta\vartheta$  as at their emission. To this  $\Delta\vartheta$  corresponds at the reception time  $t_2$  an interval

$$\Delta s_2 = c_M a(t_2) \Delta\vartheta \quad , \quad (63)$$

hence:

$$\frac{\Delta s_2}{\Delta s_1} = \frac{a(t_2)}{a(t_1)} \quad , \quad (64)$$

or, after (55),

$$\frac{\Delta s_2}{\Delta s_1} = \frac{R(t_2)}{R(t_1)} \quad . \quad (65)$$

This conclusion is valid whatever the type of coordinates used, either  $(t, l)$  or  $(\vartheta, \lambda)$ .

The remainder of the argumentation relies entirely on the definition of proper time

$$d\tau = \frac{1}{c} ds \quad (66)$$

in general relativity:

a) with coordinates  $(\vartheta, \lambda)$ ,  $c$  is constant and equals  $c_M$ , and proper time  $d\tau_\vartheta$  is defined by:

$$ds = c_M d\tau_\vartheta \quad , \quad (67)$$

so that

$$\frac{\Delta\tau_{\vartheta_2}}{\Delta\tau_{\vartheta_1}} = \frac{R(t_2)}{R(t_1)} \quad ; \quad (68)$$

the redshift is obtained;

b) with coordinates  $(t, l)$ , proper time  $d\tau$  is defined by:

$$ds = c(t) d\tau \quad (69)$$

hence

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \frac{\Delta s_2 c(t_1)}{\Delta s_1 c(t_2)} = \frac{R(t_2) c(t_1)}{R(t_1) c(t_2)} \quad (70)$$

and, if  $\gamma$  has the same value at  $t_1$  and  $t_2$ ,

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \left[ \frac{R(t_2)}{R(t_1)} \right]^{\frac{\gamma+1}{\gamma}} = \frac{t_2}{t_1} \quad . \quad (71)$$

In the last relation,  $\Delta\tau_1$  is the period of the radiation emitted at  $t_1$  and  $\Delta\tau_2$  the period of this radiation when received at  $t_2$ ; now, the redshift is defined by the ratio between  $\Delta\tau_2$

and the emission period  $\Delta\tau_{e_2}$  of the same radiation at the reception place at  $t_2$  and, as results from (50),

$$\frac{\Delta\tau_{e_2}}{\Delta\tau_1} = \frac{t_2}{t_1} \quad , \quad (72)$$

so that

$$\frac{\Delta\tau_2}{\Delta\tau_{e_2}} = 1 \quad : \quad (73)$$

there would be no redshift if cosmic time intervals were accessible to measurement. In the present model they are not, and the redshift is obtained with conformal time intervals (68). This is a logical consequence of the absence of an observable permanent reference scale for time:  $t$  can be regarded as a purely geometrical variable which might be measured from outside the universe, but not inside it (and, by definition, there is no outside). The same can be said about the associated space variables  $l$ , and the actual physical variables are  $\vartheta$  and  $\lambda$ , physical time being the proper time associated with coordinate time  $\vartheta$  through

$$d\tau_\vartheta = a(t)d\vartheta \quad . \quad (74)$$

Of course, at a given  $t$ , intervals  $\Delta\vartheta$  of conformal time only can be measured. As concerns cosmic time and space variables, only relations established without using non-measurable quantities (such as in (72)) are physically meaningful. For example, horizon calculations can be performed with either type of variables:

a) in geometrical coordinates, they involve ratios of quantities taken at the same time only, the event and the particle horizons being defined at time  $t_1$  as

$$\varrho_e = \int_{t_1}^{\infty} \frac{c(t)dt}{R(t)} \quad (75)$$

and

$$\varrho_p = \int_0^{t_1} \frac{c(t)dt}{R(t)} \quad ; \quad (76)$$

now, from (47) and (53),

$$\frac{c(t)}{R(t)} \propto \frac{1}{t} \quad (77)$$

whatever  $\gamma$ : there is no particle nor event horizon, and no need for inflation.

b) the same result is obvious in conformal coordinates  $(\vartheta, \lambda)$  (here the lower limit of integration in  $\varrho_p$  should be replaced by  $-\infty$ , since conformal time extends to infinity, as will be seen hereafter).

#### 4. Relation between times $t$ and $\vartheta$

Integrating (51) between  $t_0$  and  $t$ , one gets

$$\vartheta(t_0, t) = \frac{1}{k} \int_{t_0}^t \frac{dt}{t} = \frac{1}{k} \ln \frac{t}{t_0} \quad (78)$$

for the conformal time interval corresponding to the interval  $(t_0, t)$  of cosmic time. This implies, introducing a third instant  $t_1$ :

$$\vartheta(t_0, t_1) = \frac{1}{k} \ln \frac{t_1}{t_0} \quad (79)$$

and

$$\vartheta(t_1, t) = \frac{1}{k} \ln \frac{t}{t_1} \quad , \quad (80)$$

hence

$$\vartheta(t_0, t) = \vartheta(t_0, t_1) + \vartheta(t_1, t) \quad : \quad (81)$$

conformal time is additive.

A cosmic time interval can be defined as an ordered pair  $(t_1, t_2)$  to be associated with the ratio  $t_2/t_1$  (or even  $R(t_2)/R(t_1)$ ) – as in (71) for instance:

$$\phi(t_1, t_2) = \frac{t_2}{t_1} \quad . \quad (82)$$

This defines on cosmic time intervals a multiplicative group law

$$\phi(t_0, t_2) = \phi(t_0, t_1)\phi(t_1, t_2) \quad (83)$$

isomorphic to the additive one specified by (78) and (81). And indeed, a theorem from group theory states that every one - parameter connected differentiable group is isomorphic to an additive one and that the additive parameter is unique [11, 12]: so  $\vartheta$  only is additive, and it can be identified with the linear time  $\theta$  introduced by Misner [13] as

$$\theta \sim \ln \frac{R(t)}{R(t_0)} \quad . \quad (84)$$

Thus, the three possible time notions (cosmic, conformal and linear) of Lévy - Leblond [11] reduce here to two ones only: cosmic time  $t$ , which is uniform, but not additive, and conformal time  $\vartheta$ , additive and, owing to (51) and the uniformity of  $t$ , non-uniform – whereas classical time is endowed with both properties. As already seen, conformal time (together with its associated proper time) can be measured and thus appears to coincide with physical time, whereas, by lack of observability of its reference scale, cosmic time  $t$  cannot. So one may wonder why not drop  $t$ : however,  $\vartheta$  is not uniform and, if time is to be geometrized, the existence of one chronological variable endowed with this property should be granted. As results from (50, 51),  $t$  shows up as the dilation parameter of intervals of cosmic time, in the same way as  $R(t)$  is the dilation parameter of intervals of length, so that not only space, but space-time itself, is in expansion. It may be noted that this interpretation of  $t$  already seems to be possible for Kepler's third law: if the larger axis of an orbit is multiplied by a factor  $R$ , the period must be multiplied by a factor  $t$  such that

$t^2/R^3$  is constant, hence the possibility of interpreting the relation  $R \sim t^{2/3}$  as a relation between the scale parameters for space and time dimensions. Indeed, the third law expresses the invariance of Kepler's problem in the inhomogeneous dilation generated by the infinitesimal operator

$$X = 3t \frac{\partial}{\partial t} + 2x^i \frac{\partial}{\partial x^i} \quad (85)$$

(N. H. Ibragimov [14]).

One notion of time only being classically used in physics, it seems interesting to approximate the relation between  $\vartheta$  and  $t$  by a linear function, that is to replace the function  $\vartheta(t)$  by its tangent in the neighbourhood of an instant  $t$ , thus shifting from  $t$  to  $\vartheta$  through a change of scale and origin which does not modify the equations of physics.

Substituting  $t_1 + \Delta t$  for  $t$  in (78), one gets:

$$\vartheta = \frac{1}{k} \ln \frac{t_1 + \Delta t}{t_0} \quad , \quad (86)$$

hence

$$\vartheta \simeq \frac{1}{k} \left( \ln \frac{t_1}{t_0} + \frac{\Delta t}{t_1} \right) \quad (87)$$

or

$$\vartheta \simeq \frac{1}{k} \left( \ln \frac{t_1}{t_0} + \frac{t}{t_1} - 1 \right) \quad . \quad (87')$$

Now, inverting (78),

$$t = t_0 \exp(k\vartheta) \quad , \quad (88)$$

gives in (47)

$$R(t) = R(t_0) \exp\left(\frac{\gamma}{\gamma+1} k\vartheta\right) \quad ; \quad (89)$$

this expression can be approximated using (87) by

$$R(t) \simeq R(t_1) \left( 1 + \frac{\gamma}{\gamma+1} \frac{\Delta t}{t_1} \right) \quad (90)$$

(which might have been obtained directly from (47)): if  $t_1$  is large, (for instance corresponding to present time, so that  $t_1$  is the "age of the universe"), this approximation is satisfactory even for large values of  $\Delta t$  as soon as they are small relative to  $t_1$ ; if  $t_1$  is small, in the neighbourhood of 0, the same linear approximation for  $\vartheta$ , in the form (87'), yields

$$R(t) \simeq R(t_1) \exp\left[\frac{\gamma}{\gamma+1} \left(\frac{t}{t_1} - 1\right)\right] \quad (91)$$

or

$$R(t) \sim \mathcal{C} \exp\left(\frac{\gamma}{\gamma+1} \frac{t}{t_1}\right) \quad , \quad (92)$$

a result which is valid for  $t$  in the vicinity of  $t_1$  and which gives, when extended to larger values of  $t$ , the same variation of  $R$  with cosmic time as in inflation theory (take for instance  $t_1 = 10^{-n}$  s in (92)).

As a last remark, the relation between cosmic and conformal times might help understand – if this is confirmed – why the oldest stars seem to be older than the universe itself. It may be noted that, defining  $t_M$  by

$$t_M = \frac{\gamma}{\gamma+1} \frac{R_M}{c_M} \quad , \quad (93)$$

(51) can be given the form

$$dt = \frac{t}{t_M} d\vartheta \quad (94)$$

so that  $dt = d\vartheta$  at  $t = t_M$ : the cosmic and the conformal times are in coincidence (have the same scale) for  $t = t_M$ . Equation (79) for example becomes with this notation

$$\vartheta(t_0, t_M) = t_M \ln \frac{t_M}{t_0} \quad . \quad (95)$$

Now, the age of a star leaving the main sequence of the H-R diagram is currently estimated from the time it has spent on this sequence, which is the longest one in its lifespan. This time is determined by dividing the total nuclear energy available on the main sequence by the amount of energy used per time unit, and of course is computed as a conformal time interval. The relation between this interval and the cosmic times  $t_M$  and  $t_0$  which respectively represent the age of the universe and the time the star entered the main sequence is given by (95) for a star which leaves the main sequence presently; it can be rewritten as

$$t_0 = t_M \exp(-\vartheta(t_0, t_M)/t_M) \quad . \quad (96)$$

Applying (96) to the results of Pierce et al [15], which imply  $t_M \simeq 7.3 \cdot 10^9$  years for  $\Omega = 1$ , gives, with  $\vartheta(t_0, t_M) \simeq 16.5 \cdot 10^9$  years for the age of the oldest star clusters

$$t_0/t_M \simeq \exp(-2.26) \simeq 0.1 \quad , \quad (97)$$

thus reducing to about  $7 \cdot 10^8$  years the cosmic time at which these were formed.

## 5. Euclidian subspaces in a constant negative curvature manifold

A well-known example of such a situation is that of 3-space in the space-time of steady-state cosmology [9]. Schrödinger [16], remarking that curvature depends on the frame, already studied such a case as the Lemaître-Robertson frame of the de Sitter universe, represented as a one-shell hyperboloid  $\mathcal{H}_1$

$$x^2 + u^2 + v^2 + y^2 - z^2 = \mathcal{R}^2 \quad (98)$$

embedded in a five dimensional space with a (1,4) Lorentzian metric

$$d\sigma^2 = -dx^2 - du^2 - dv^2 - dy^2 + dz^2 \quad . \quad (99)$$

What we need here – so as to have a negative curvature space and not space-time – is a two-shell hyperboloid  $\mathcal{H}_2$

$$x^2 + u^2 + v^2 + y^2 - z^2 = -\mathcal{R}^2 \quad (100)$$

embedded in the same Lorentz space, with the difference that this imbedding space is here completely fictitious, so that, in particular, no coordinate system in it has to be interpreted as including time as one of its components.

The Lemaître transformation leading to Euclidean subspaces of  $\mathcal{H}_1$  reads

$$\bar{x} = \frac{\mathcal{R}x}{y+z} \quad , \quad (101)$$

$$\bar{u} = \frac{\mathcal{R}u}{y+z} \quad , \quad (102)$$

$$\bar{v} = \frac{\mathcal{R}v}{y+z} \quad , \quad (103)$$

$$\bar{\theta} = \ln \frac{y+z}{\mathcal{R}} \quad . \quad (104)$$

With these new variables, the  $d\sigma^2$  (99) becomes

$$d\sigma^2 = -\exp(2\bar{\theta})(d\bar{x}^2 + d\bar{u}^2 + d\bar{v}^2) + \mathcal{R}^2 d\bar{\theta}^2 \quad (105)$$

as can be shown using the following relations deduced from (101 – 104):

$$x = \bar{x} \exp(\bar{\theta}) \quad , \quad u = \bar{u} \exp(\bar{\theta}) \quad , \quad v = \bar{v} \exp(\bar{\theta}) \quad , \quad (106)$$

$$y+z = \mathcal{R} \exp(\bar{\theta}) \quad , \quad (107)$$

and

$$y-z = \mathcal{R} \exp(-\bar{\theta}) - \frac{\bar{r}^2}{\mathcal{R}} \exp(\bar{\theta}) \quad (108)$$

where

$$\bar{r}^2 = \bar{x}^2 + \bar{u}^2 + \bar{v}^2$$

and  $y - z$  has been computed from

$$y - z = \frac{\mathcal{R}^2 - x^2 - u^2 - v^2}{y + z} .$$

It is clear from (105) that the subspaces  $(\bar{x}, \bar{u}, \bar{v})$  are Euclidean.

The same Lemaître transformation leads to Euclidean subspaces of  $\mathcal{H}_2$  with the only differences that now

$$y - z = \frac{-\mathcal{R}^2 - x^2 - u^2 - v^2}{y + z} = -\mathcal{R} \exp(-\bar{\theta}) - \frac{\bar{r}^2}{\mathcal{R}} \exp(\bar{\theta}) \quad (109)$$

and

$$d\sigma^2 = -\exp(2\bar{\theta})(d\bar{x}^2 + d\bar{u}^2 + d\bar{v}^2) - \mathcal{R}^2 d\bar{\theta}^2 . \quad (110)$$

The Euclidean subspaces  $(\bar{x}, \bar{u}, \bar{v})$  of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are the three dimensional generalizations of the parabolae traced on their restrictions to the dimensions  $(x, y, z)$  by planes  $y + z = \mathcal{C}$ , a constant, and it may be noted (as concerns  $\mathcal{H}_2$ ) that when  $\mathcal{R}$  varies with time (expansion),  $\bar{\theta}$  remains constant, so that these subspaces are preserved.

Now, depending on the value of  $\mathcal{C}$ , there is an infinity of such subspaces and it seems to be logical, so as to keep (as supposed from the beginning) a zero four dimensional density, to associate to each subspace with density  $\rho$  another one with negative density  $-\rho$  in a twofold structure with

$$\rho > 0 , \quad G > 0 \quad \text{on the 1}^{\text{st}} \text{ sheet}$$

and

$$\rho < 0 , \quad G < 0 \quad \text{on the 2}^{\text{nd}} \text{ sheet}$$

both of which possibilities being allowed as solutions of (43) (it may be noted that  $\rho G$  also, and not  $G$  alone, appears in the Einstein equations).  $\rho$  is an active gravitational mass density, so that two masses should attract when in the same sheet and repel when not if one keeps on identifying passive gravitational mass and inertial mass. Such a negative density sheet has already been proposed as a candidate for dark matter in first simulations of the large scale structure of the universe [17, 18].

## 6. The case of non-Euclidean subspaces

Such subspaces obviously exist (for  $k = -1$ , take for instance  $y^2$  constant in (100) and for  $k = 1$ ,  $z^2$  constant and larger than  $\mathcal{R}^2$ ). However, supposing  $k = -1$  in steps (24 – 30) implies a zero density (hence the necessity to take  $n > 3$ ), and  $k = 1$  yields  $\gamma = 1/2$  and 1 for the radiation and the matter dominated eras respectively, that is no correspondence

with previous theories. Hence the Friedmann solution with  $k = 0$  is the only one which can be found following the present procedure.

## 7. Gauge relations

Since, in geometrical coordinates,  $c$  has been assumed to vary with time, this should be the same for other physical constants also. This implies gauge relations which have been proposed in [19, 20] for the case  $\gamma = 2$  (equation (38)). Similar relations can be obtained here, without a special treatment being required for Schrödinger's equation. Assuming every mass varies as  $M$  and the Bohr radius as  $R$ , as already supposed, and using (38) yields for instance, if  $\alpha$  (which is dimensionless) remains constant,

$$\hbar \approx MR^{1-\frac{1}{\gamma}} \quad . \quad (111)$$

In the same way, starting from Newton's formula for the two-body problem and still using (38) gives for the Einstein constant

$$\frac{G}{c^2} \approx \frac{R}{M} \quad (112)$$

whatever  $\gamma$ . With no precision about the variance of masses, such relations are purely kinematical – being derived from the variances of length and time intervals. One more hypothesis has to be made to go further; for example, if  $G/c^2$  is to be a constant, this implies

$$M \approx R \quad (113)$$

hence

$$\hbar \approx R^{2-\frac{1}{\gamma}} \quad (114)$$

and

$$G \approx R^{-\frac{2}{\gamma}} \quad , \quad (115)$$

and so on, ...

None of these variations can be detected in the laboratory since, as in the case of length and time intervals, no permanent reference scale can be used for that. So, in the physical world, constants are but constants, as light velocity  $c_M$  in (56).

However, the above assumption about  $G/c^2$  is important: it allows the divergenceless character of the energy-momentum tensor to be kept in the second member of Einstein's equations (2'). Moreover, substituting (47) into (41) yields for the energy density corresponding to the equation of state (39)

$$\varepsilon = \frac{3c^2}{8\pi G} \left( \frac{\gamma}{\gamma+1} \right) t_0^{-2} \left( \frac{R}{R_0} \right)^{-2\frac{\gamma+1}{\gamma}} \quad (116)$$

so that a constant value of  $G/c^2$  implies

$$\varepsilon \propto R^{-2\frac{\gamma+1}{\gamma}} \quad (117)$$

and for the energy contents of a covolume  $V$ ,

$$E \propto R^{1-\frac{2}{\gamma}} \quad (118)$$

so that during the radiation era ( $\gamma = 1$ ),

$$\varepsilon \propto R^{-4} \quad (119)$$

and

$$E \propto R^{-1} \quad (120)$$

– hence, using the Planck formula, the same variation with  $R$  of the temperature of the radiation gas, as also results from the classical treatment – and during the matter-dominated era ( $\gamma = 2$ ),

$$\varepsilon \propto R^{-3} \quad (121)$$

hence a constant energy for the massive particle contents of  $V$ .

## 8. Discussion and conclusion

So as to express the hypothesis of non-observability of permanent reference scales for time and length intervals, we have assumed the scale parameter  $R(t)$  applies to any scale in the universe – which implies the possibility of a variation with cosmic time of the constants of physics. As many hypotheses, this one may be questioned and perhaps its best support lies in a number of advantages offered by the resulting theory:

– 1) The model obtained is unique; it represents the universe as an Euclidean subspace of a negative curvature variety; it is equivalent, as concerns its density and its time evolution, to the zero curvature Friedmann model, and this Euclidean character is in full agreement with observation. Interestingly, as results from the resolution of the system (29, 30), the density corresponds to the curvature of  $n$ -space; the derivative of  $c$  with respect to cosmic time appears in the pressure term only.

– 2) A clear distinction is made between the universe and the permanent scales with respect to which its time and space characteristics (its age and its scale parameter) are defined, so that these permanent scales are to be considered as external to the universe. As opposed to what can be expected, this does not prevent the redshift from being obtained in the resulting theory if one notes that cosmic time (and length) intervals – for which permanent reference scales have to exist as soon as one writes down expressions (3 – 5) or (52) for the metric – are no longer measurable quantities. The physical quantities, measurable from inside the universe, are from this viewpoint the proper time and the proper distances associated with conformal time  $\vartheta$  and comoving space coordinates  $\lambda$  in equation (56). The cosmological and the gravitational redshifts both originate in the variations of the  $g_{00}$  coefficient of the metric, the former with time, the latter with position in space – although in both cases, the shift owes its existence in the phenomenon (the emission of a radiation) being produced and observed as two distant events in space-time.

– 3) In physical (i.e. conformal) time, due to the logarithmic dependence of  $\vartheta$  on  $t$ , the origin of the universe is shifted to  $-\infty$ , corroborating the ideas developed by Lévy - Leblond and Misner in [11] and [13]. This removes the question of knowing what existed before the big bang and might help to solve the problem raised by the age of the oldest star clusters. It may be noted that this physical time is defined in close conformity with experiment, which contents itself with counting numbers of periods of a reference clock without being able to compare two of these periods.

– 4) As contrasts with the classical theory, time and space are dealt with in the same way: in the metric, both time and space differentials are provided with variable coefficients and, as well as space, cosmic time expands; thus a time arrow arises naturally.

– 5) The fact that intervals of  $\vartheta$  and  $\lambda$  only are physically measurable implies the impossibility of directly observing any expansion – hence there is no problem about where it stops. Nevertheless, of course, only experiment can decide whether this scale invariance holds: structures such as atoms or galaxies may well withstand expansion owing to their internal forces.

– 6) There are no horizon problems, hence no need for inflation, and this may supply possibilities for an alternative to the scalar field which is evoked to justify this theory at the microscopic level. As a geometric description of the universe, the usual theory appears to be incomplete when compared with the present one, and this might be a reason why it has to resort to inflation.

– 7) Derived from the variation of  $c$  with cosmic time  $t$  (not physical time), a set of laws of variation of the physical constants with  $t$  is proposed; in the same way as for  $c$  itself, these variations cannot be observed using physical time and length measurements, since any period, length, mass, ... varies according to the same law as the corresponding unit; so, from the physical point of view,  $c$  and the other physical constants remain constant. As a consequence of this, energy conservation is preserved with respect to cosmic time for radiation, since no redshift would show up if  $t$  were measurable. This seems to be natural, the cosmic time scale being uniform, and contrasts with what happens in the usual theory, where there is no global energy conservation [21]. It may be noted that (as in the usual theory) energy conservation also occurs for massive particles with respect to  $t$  as results from (113) and (38) with  $\gamma = 2$ ; this, of course, is true also with respect to  $\vartheta$ .

The possibility of an objection to the model might be found in the identification of terms involving a variable  $c$  with the energy-matter terms of another one (the Friedmann model) which assumes the opposite. However, this variability applies to a coefficient of the metric only (the observable light velocity in the new model is constant), and in the usual model matter is represented through the energy-momentum tensor, whereas in the new one this representation originates in the variable  $c$  metric, the presence of matter being integrated into the geometry through this variability of  $c$  and of the other physical constants, all of which characterize properties of matter or radiation. So these models differ in two hypotheses, and not only in what is assumed about  $c$  in cosmic coordinates: this is not more contradictory than accepting the possibility of doing physics either in the framework of an Euclidean geometry with complex physical laws or using a non-Euclidean geometry and simple physical laws. Moreover, it may be noted that if both sides of (2') are multiplied by  $c^2$ , so that  $R_{\mu\nu}$  is replaced everywhere by  $c^2 R_{\mu\nu}$ , the Ricci tensor  $c^2 R_{\mu\nu}^{(F;1+3)}$

(equations (24) and (24')) of the zero curvature Friedmann model (and of this one only) does not depend on  $c$ . And the classical equations of this model write:

$$2\frac{R''}{R} + \frac{R'^2}{R^2} = -\frac{8\pi G}{c^2}p \quad (122)$$

and

$$3\frac{R'^2}{R^2} = \frac{8\pi G}{c^2}\varepsilon \quad , \quad (123)$$

which can also be derived from (29) and (30) above: they involve  $G/c^2$  only when expressed in terms of the density of energy  $\varepsilon$ . With  $G/c^2$  constant as chosen in §7, there is no contradiction in identifying the classical terms with those of the present model.

It may also be objected that, writing  $c_M dt' = c(t)dt$ , the present metric is equivalent to Milne's for time  $t'$  and  $n$  space dimensions: indeed we have shown that Einstein's equations for the vacuum ((11) and (14)), instead of giving Milne's solution ( $R$  a linear function of  $t$ ), corresponding to an empty universe, can yield a matter as well as a radiation universe as soon as the  $g_{00}$  coefficient of the metric is allowed to be variable. Taking a constant  $g_{00}$  is in fact an arbitrary constraint, and can be considered as unnecessarily weakening the metric [22].

As a last remark, it may be noted that the basic hypotheses we have started from involve both general relativity and, through the expression of the Bohr radius, a fundamental result of quantum mechanics – which are both, at an elementary level, connected with an inverse square law.

### Acknowledgements

We wish to thank Drs. Guy Oliver and Ivan Brissaud and Profs. Michel Mizony and Léon Brenig for numerous fruitful discussions.

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